A New Nakagami–Zero-Truncated Poisson Distribution: Properties, Estimation, and Applications

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Abstract: This paper introduces the Nakagami zero-truncated Poisson (NZTP) distribution, a new three-parameter lifetime model obtained by compounding the Nakagami distribution with the zero-truncated Poisson distribution through the maximum operator. The construction is motivated by parallel-system reliability settings, where the system lifetime is determined by the maximum of a random number of component lifetimes. The probability density function, cumulative distribution function, survival function, and hazard rate function of the proposed distribution are derived in closed form. Fundamental statistical properties are established, including raw moments, the moment generating function, quantile function, Shannon entropy, Rényi entropy, and order statistics. Parameter estimation is carried out via maximum likelihood, and the score equations are derived explicitly. A Monte Carlo simulation study is conducted to assess the finite-sample performance of the estimators under three parameter configurations and four sample sizes. The bias and RMSE decrease as the sample size increases. The practical applicability of the proposed distribution is demonstrated through an application to carbon fibre strength data, where the NZTP model fits the data as well as the gamma zero-truncated Poisson (GZTP) model.

Key-Words: -Nakagami distribution; zero-truncated Poisson; compounding; lifetime distribution; maximum likelihood estimation; parallel system; hazard rate

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1 Introduction

The Nakagami distribution, originally introduced by [1], has become an important probability model in reliability engineering, survival analysis, and wireless communication. Its growing popularity in statistical modeling is largely due to its flexibility in describing skewed and heavy-tailed lifetime data through the shape parameter m. In particular, the Nakagami distribution contains important special cases, including the one-sided Gaussian distribution when $m = 1/2$ and the Rayleigh distribution when $m = 1$. For $m > 1$, the distribution is suitable for modeling more general fading, reliability, and lifetime patterns. This flexibility has made the Nakagami distribution a useful tool in both theoretical and applied statistical research. Several extensions of this distribution have been proposed. For example, [2] proposed a mixture of the Nakagami and length-biased Nakagami distributions.

Another important development in distribution theory is the use of zero-truncated Poisson (ZTP) compounding. The ZTP distribution is particularly useful for modeling count data in situations where zero counts are not possible. In compounded lifetime models, it is often used as a latent count variable representing the number of competing or complementary causes associated with an event. The idea of introducing additional parameters through compounding is closely related to the work of [3], who proposed a geometric compounding approach for extending existing distribution families. Subsequently, [4] developed the exponential–Poisson distribution using a minimum compounding scheme, while [5] introduced the Poisson–exponential distribution using the complementary maximum operator. These studies provided a foundation

for many ZTP-based lifetime models, including extensions reviewed by [6] and further developments by [7] for Weibull baseline distributions.

More recently, [8] proposed a broader ZTP-generated framework by allowing the compounding parameter to take negative values. This unified formulation connects the minimum and maximum compounding constructions and provides meaningful interpretations in competing-risk and complementary-risk settings. These developments highlight the continuing relevance of ZTP compounding as a powerful mechanism for generating flexible lifetime distributions.

Motivated by these ideas, this paper introduces a new distribution obtained by compounding the Nakagami distribution with the zero-truncated Poisson distribution through the maximum operator. Unlike the mixture model in [2], this model describes a parallel system and has a closed-form quantile function. The resulting Nakagami zero-truncated Poisson (NZTP) distribution is designed for strictly positive lifetime data and is particularly relevant to parallel-system reliability settings, where the system lifetime is determined by the maximum of several component lifetimes. The model includes the Nakagami distribution as a special case when $\lambda \to 0$. Its hazard rate is increasing for $m \geq 1/2$ and bathtub-shaped for $m < 1/2$.

2 The Nakagami Zero-Truncated Poisson Distribution

Let $X_1, X_2, \ldots, X_N$ be independent and identically distributed random variables following the Nakagami distribution with shape parameter $m > 0$ and scale parameter $\Omega > 0$. The probability density function and cumulative distribution function are given, respectively, by

$$\begin{aligned} g(x; m, \Omega) =& \frac{2m^m}{\Gamma(m)\Omega^m} x^{2m-1} \\ & \times \exp\left(-\frac{m}{\Omega}x^2\right), \qquad x > 0, \end{aligned} \tag{1}$$

and

$$\begin{aligned} G(x; m, \Omega) =& P\left(m, \frac{m}{\Omega}x^2\right) \\ =& \frac{\gamma\left(m, \frac{m}{\Omega}x^2\right)}{\Gamma(m)}, \qquad x > 0, \end{aligned} \tag{2}$$

where $\Gamma(m)$ is the complete gamma function, $\gamma(m, y)$ is the lower incomplete gamma function, and $P(m, y) = \gamma(m, y)/\Gamma(m)$ denotes the regularized lower incomplete gamma function.

Let N follow a zero-truncated Poisson distribution with parameter $\lambda > 0$, having probability mass function

$$P(N = n) = \frac{e^{-\lambda}\lambda^n}{n!\,(1 - e^{-\lambda})}, \quad n = 1, 2, \ldots. \tag{3}$$

The zero-truncated Poisson distribution ensures that at least one component is present in the system.

Assume that N is independent of $\{X_i\}_{i\geq 1}$. Define the system lifetime as

$$Z = \max\{X_1, X_2, \ldots, X_N\}. \tag{4}$$

The maximum operator in (4) arises naturally in parallel-system reliability, where system failure occurs only after all N components have failed. Therefore, the system lifetime is governed by the largest component lifetime.

Conditioning on $N = n$, the conditional distribution of Z is

$$P(Z \leq z \mid N = n) = [G(z; m, \Omega)]^n, \quad z > 0. \tag{5}$$

The unconditional CDF of Z is obtained by compounding (5) with the zero-truncated Poisson PMF in (3). Using the Taylor expansion $e^w = \sum_{n=0}^{\infty} w^n/n!$, the CDF of the proposed Nakagami zero-truncated Poisson distribution is

$$F_Z(z; \lambda, m, \Omega) = \frac{\exp\left\{\lambda P\left(m, \frac{m}{\Omega}z^2\right)\right\} - 1}{e^{\lambda} - 1}, \quad z > 0. \tag{6}$$

Differentiating (6) with respect to z and using $u'(z) = g(z; m, \Omega)$, the probability density function is

$$\begin{aligned} f_Z(z; \lambda, m, \Omega) =& \frac{2\lambda m^m}{(e^{\lambda} - 1)\Gamma(m)\Omega^m} z^{2m-1} \\ & \times \exp\left[-\frac{m}{\Omega}z^2 + \lambda\frac{\gamma\left(m, \frac{m}{\Omega}z^2\right)}{\Gamma(m)}\right], \end{aligned} \tag{7}$$

where $z > 0$. A random variable Z having CDF (6) and PDF (7) is said to follow the Nakagami zero-truncated Poisson distribution, denoted $Z \sim \mathrm{NZTP}(\lambda, m, \Omega)$.

Remark 1. *1. As $\lambda \to 0^+$, the zero-truncated Poisson distribution degenerates to $P(N = 1) = 1$. Hence, Z converges in distribution to the baseline Nakagami random variable.*

2. For $m = 1$ and $\Omega = 2\sigma^2$, the Nakagami distribution reduces to the Rayleigh distribution. Consequently, the NZTP family reduces to the Rayleigh zero-truncated Poisson distribution.

3. For $m = 1/2$ and $\Omega = \sigma^2$, the baseline Nakagami distribution reduces to the half-normal distribution, yielding a corresponding half-normal zero-truncated Poisson special case.

2.1 Survival and Hazard Rate Functions

The survival function of $Z \sim \mathrm{NZTP}(\lambda, m, \Omega)$ is obtained from (6):

$$S_Z(z;\lambda,m,\Omega) = \frac{e^{\lambda} - \exp\left\{\lambda \frac{\gamma\left(m, \frac{m}{\Omega} z^2\right)}{\Gamma(m)}\right\}}{e^{\lambda}-1}, \quad z > 0. \tag{8}$$

Note that $S_Z(0) = 1$ and $S_Z(z) \to 0$ as $z \to \infty$, as required for a valid lifetime distribution.

Using (7) and (8), the hazard rate function is

$$\begin{aligned} h_Z(z;\lambda,m,\Omega) = &\frac{2\lambda m^m}{\Gamma(m)\Omega^m} z^{2m-1} \\ &\times \exp\left[-\frac{m}{\Omega} z^2 + \lambda \frac{\gamma\left(m, \frac{m}{\Omega} z^2\right)}{\Gamma(m)}\right] \\ &\times \left[e^{\lambda} - \exp\left\{\lambda \frac{\gamma\left(m, \frac{m}{\Omega} z^2\right)}{\Gamma(m)}\right\}\right]^{-1} \end{aligned} \tag{9}$$

Proposition 1. *The hazard rate function* (9) *satisfies:*

1. $\lim_{z\to 0^+} h_Z = \begin{cases} 0, & m > \frac{1}{2}, \\ \frac{\lambda}{e^{\lambda}-1}\sqrt{\frac{2}{\pi\Omega}}, & m = \frac{1}{2}, \\ \infty, & 0 < m < \frac{1}{2}. \end{cases}$
2. $\lim\limits_{z\to\infty} h_Z = \infty$ *for all* $m, \lambda > 0$.

Proposition 2. *The hazard rate* (9) *is never decreasing (DFR) or upside-down bathtub-shaped (UBT). It is bathtub-shaped (BT) for* $0 < m < 1/2$ *and increasing (IFR) for* $m \geq 1/2$.

By Proposition 1, $h_Z(z) \to \infty$ as $z \to \infty$. A DFR or UBT hazard must decrease for large z, so these two shapes are not possible. For $0 < m < 1/2$, $h_Z(z) \to \infty$ also as $z \to 0^+$, so h_Z first decreases and then increases. For $m \geq 1/2$, h_Z starts at a finite value.

To find the exact shape, we take the log of (7) and differentiate:

$$\begin{aligned} \eta(z) &= -\frac{f_Z'(z)}{f_Z(z)} \\ &= -\frac{2m-1}{z} + \frac{2m}{\Omega} z - \lambda\, g(z;m,\Omega). \end{aligned} \tag{10}$$

Since $h_Z = f_Z/S_Z$, we have

$$h_Z'(z) = h_Z(z)\,[\,h_Z(z) - \eta(z)\,]. \tag{11}$$

So h_Z increases when $h_Z > \eta$ and decreases when $h_Z < \eta$.

Figure 1 (Appendix) shows the PDF, CDF, survival function and hazard rate function of the NZTP distribution for $m = 0.3, 0.5, 1.0$ and 2.0, with $\lambda = 0.6$ and $\Omega = 2.0$ fixed in all four panels. When $m = 0.3$, the PDF is decreasing and goes to infinity at $z = 0$. When $m \geq 0.5$, the PDF has one mode, and the mode moves to the right as m increases ($z \approx 0.62$, 1.19 and 1.35). For larger m, the PDF is more peaked. So the CDF rises more steeply, and the survival function stays near 1 for longer and then falls faster. The hazard rate is bathtub-shaped for $m = 0.3$ and increasing for $m \geq 0.5$, and it rises faster for larger m. So m controls the shape near zero and the type of hazard. The parameter λ also changes the shape. For example, for $m = 0.5$ the Nakagami PDF is decreasing, but the NZTP PDF has a mode at $z \approx 0.62$.

3 Statistical Properties

This section establishes fundamental structural properties of the $\mathrm{NZTP}(\lambda, m, \Omega)$ distribution, including moments, generating functions, quantile function, and entropy measures.

3.1 Moments

Proposition 3. *Let* $Z \sim \mathrm{NZTP}(\lambda, m, \Omega)$. *The* r*th raw moment is*

$$\mathrm{E}(Z^r) = \frac{1}{e^{\lambda}-1} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \mathrm{E}\left[\max_{1\leq i\leq n} X_i^r\right], \tag{12}$$

where for $n \geq 1$ *and* $r > 0$,

$$\mathrm{E}\left[\max_{1\leq i\leq n} X_i^r\right] = \int_0^{\infty} r z^{r-1} \left[1 - G(z;m,\Omega)^n\right] dz. \tag{13}$$

The Nakagami baseline moments are known in closed form [1]:

$$\mathrm{E}(X_1^r) = \left(\frac{\Omega}{m}\right)^{r/2} \frac{\Gamma(m + r/2)}{\Gamma(m)}, \qquad r > -2m. \tag{14}$$

Corollary 1. *The mean and variance of* $Z \sim \mathrm{NZTP}(\lambda, m, \Omega)$ *are*

$$\begin{aligned} \mu_Z &= \mathrm{E}(Z) \\ &= \frac{1}{e^{\lambda}-1} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \int_0^{\infty} \left[1 - G(z;m,\Omega)^n\right] dz, \end{aligned} \tag{15}$$

and

$$\sigma_Z^2 = \mathrm{Var}(Z) = \mathrm{E}(Z^2) - \mu_Z^2, \tag{16}$$

where $\mathrm{E}(Z^2)$ *is obtained from* (12) *with* $r = 2$.

3.2 Moment Generating Function

Proposition 4. *The moment generating function (MGF) of* $Z \sim \mathrm{NZTP}(\lambda, m, \Omega)$ *is*

$$M_Z(t) = 1 + \frac{t}{e^\lambda - 1} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \times \int_0^\infty e^{tz} \left[1 - G(z; m, \Omega)^n\right] dz, \qquad t \in \mathbb{R}. \tag{17}$$

The MGF exists for all real t *because the tail decays like* $\exp(-mz^2/\Omega)$.

3.3 Quantile Function

Proposition 5. *For* $u \in (0, 1)$*, the quantile function* $Q_Z(u) = F_Z^{-1}(u)$ *of the* $\mathrm{NZTP}(\lambda, m, \Omega)$ *distribution is*

$$Q_Z(u) = \sqrt{\frac{\Omega}{m} P^{-1}\left(m,\ \frac{1}{\lambda} \ln[1 + u(e^\lambda - 1)]\right)}, \tag{18}$$

where $P^{-1}(m, \cdot)$ *denotes the inverse regularized lower incomplete gamma function.*

3.4 Entropy Measures

Proposition 6. *The Shannon entropy of* $Z \sim \mathrm{NZTP}(\lambda, m, \Omega)$ *is*

$$\mathcal{H}(Z) = -\frac{\lambda}{e^\lambda - 1} \times \int_0^\infty g(z) e^{\lambda G(z)} \ln\left[\frac{\lambda g(z) e^{\lambda G(z)}}{e^\lambda - 1}\right] dz. \tag{19}$$

Proposition 7. *For* $\alpha > 0$*,* $\alpha \neq 1$*, the Rényi entropy of order* α *is*

$$\mathcal{H}_\alpha(Z) = \frac{1}{1-\alpha} \ln\left[\frac{\lambda^\alpha}{(e^\lambda - 1)^\alpha} \times \int_0^\infty g(z)^\alpha e^{\alpha\lambda G(z)} dz\right]. \tag{20}$$

3.5 Order Statistics

Let $Z_{1:k} \leq Z_{2:k} \leq \cdots \leq Z_{k:k}$ denote the order statistics of a random sample of size k from $\mathrm{NZTP}(\lambda, m, \Omega)$. The PDF of the ith order statistic is [9]

$$f_{i:k}(z) = \frac{k!}{(i-1)!\,(k-i)!} \times f_Z(z)\, F_Z(z)^{i-1} \left[1 - F_Z(z)\right]^{k-i}, \tag{21}$$

where f_Z and F_Z are given by (7) and (6), respectively.

4 Parameter Estimation

This section describes the estimation procedure for the unknown parameters of the proposed NZTP distribution using the method of maximum likelihood.

4.1 Maximum Likelihood Estimation

Let $z_1, z_2, \ldots, z_n$ be a random sample from the NZTP distribution. The log-likelihood function is

$$\ell(\boldsymbol{\theta}) = n \log\left(\frac{2\lambda m^m}{(e^\lambda - 1)\Gamma(m)\Omega^m}\right) + (2m-1)\sum_{i=1}^n \log z_i - \frac{m}{\Omega}\sum_{i=1}^n z_i^2 + \lambda \sum_{i=1}^n P\left(m, \frac{m}{\Omega} z_i^2\right). \tag{22}$$

The maximum likelihood estimator of $\boldsymbol{\theta} = (\lambda, m, \Omega)^\top$ is defined as

$$\hat{\boldsymbol{\theta}} = \arg \max_{\lambda>0,\, m>0,\, \Omega>0} \ell(\lambda, m, \Omega). \tag{23}$$

For notational convenience, define $y_i = m z_i^2/\Omega$ and $P_i = P(m, y_i)$. The score equations are:

$$U_\lambda(\boldsymbol{\theta}) = \frac{n}{\lambda} - \frac{n e^\lambda}{e^\lambda - 1} + \sum_{i=1}^n P_i = 0, \tag{24}$$

$$U_m(\boldsymbol{\theta}) = n\left(\log m + 1 - \psi(m) - \log\Omega\right) + 2\sum_{i=1}^n \log z_i - \frac{1}{\Omega}\sum_{i=1}^n z_i^2 + \lambda \sum_{i=1}^n \Big[\frac{\partial P(m, y_i)}{\partial m} + \frac{z_i^2}{\Omega} \frac{y_i^{m-1} e^{-y_i}}{\Gamma(m)}\Big] = 0, \tag{25}$$

$$U_\Omega(\boldsymbol{\theta}) = -\frac{nm}{\Omega} + \frac{m}{\Omega^2}\sum_{i=1}^n z_i^2 - \lambda \sum_{i=1}^n \frac{m z_i^2}{\Omega^2} \frac{y_i^{m-1} e^{-y_i}}{\Gamma(m)} = 0, \tag{26}$$

where $\psi(m) = d\log\Gamma(m)/dm$ is the digamma function and $\partial P(m, y_i)/\partial m$ is the derivative with respect to the first argument, with y_i fixed. Since (24)–(26) do not admit closed-form solutions, numerical optimization via Newton–Raphson, quasi-Newton, BFGS, or L-BFGS-B algorithms is required.

4.2 Properties of the MLE

4.2.1 Identifiability

For small z, we have

$$P(m, y) \approx \frac{y^m}{\Gamma(m+1)}, \tag{27}$$

and

$$e^{\lambda P} - 1 \approx \lambda P. \tag{28}$$

So (7) gives

$$F_Z(z) \approx \frac{\lambda\, m^m}{(e^\lambda - 1)\,\Gamma(m+1)\,\Omega^m}\, z^{2m}, \quad z \to 0^+. \tag{29}$$

For large z, we have

$$e^\lambda - e^{\lambda P} \approx \lambda e^\lambda (1 - P). \tag{30}$$

So (8) gives

$$\log S_Z(z) \approx -\frac{m}{\Omega}\, z^2, \qquad z \to \infty.$$

If two parameter vectors give the same CDF, the power z^{2m} gives the same m, and the tail gives the same m/Ω, so the same Ω. The constant then gives the same $\lambda/(e^\lambda - 1)$. This function is decreasing in λ, so λ is also the same. Hence the model is identifiable.

4.2.2 Existence of $\hat{\lambda}$

Differentiating (24) gives

$$\begin{aligned} \frac{\partial U_\lambda}{\partial \lambda} &= -\frac{n}{\lambda^2} + \frac{n\, e^\lambda}{(e^\lambda - 1)^2} \\ &= n\left[\frac{1}{4\sinh^2(\lambda/2)} - \frac{1}{\lambda^2}\right] < 0, \end{aligned} \tag{31}$$

because $\sinh x > x$ for $x > 0$. So, for fixed (m, Ω), U_λ is decreasing in λ. Furthermore, the limiting behavior of U_λ is given by

$$\lim_{\lambda \to 0^+} U_\lambda = \sum_{i=1}^n P_i - \frac{n}{2}, \tag{32}$$

and

$$\lim_{\lambda \to \infty} U_\lambda = \sum_{i=1}^n P_i - n < 0. \tag{33}$$

Therefore, (24) has a unique positive solution $\hat{\lambda} > 0$ when

$$\bar{P} = \frac{1}{n}\sum_{i=1}^n P_i > \frac{1}{2}. \tag{34}$$

Conversely, when $\bar{P} \le 1/2$, the maximum occurs at the boundary $\hat{\lambda} = 0$, and the model reduces to the Nakagami distribution. Although the uniqueness of the joint maximum likelihood estimate over (λ, m, Ω) has not been established analytically, multiple starting values are used in the numerical optimization to assess the stability of the resulting estimates.

4.2.3 Asymptotic Properties

Since the support $(0, \infty)$ is independent of the model parameters, the parameter space is open, and the log-density function $\log f_Z$ is three times continuously differentiable with respect to (λ, m, Ω), the MLE is consistent and asymptotically normal under the standard regularity conditions [10]. Specifically,

$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \xrightarrow{d} N_3\left(\mathbf{0}, \mathbf{I}(\boldsymbol{\theta})^{-1}\right), \tag{35}$$

where $\boldsymbol{\theta} = (\lambda, m, \Omega)^\mathsf{T}$ and $\mathbf{I}(\boldsymbol{\theta})$ denotes the Fisher information matrix for a single observation.

Accordingly, the Fisher information is estimated using the observed information matrix, defined as

$$\mathbf{J}(\hat{\boldsymbol{\theta}}) = -\left.\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\mathsf{T}}\right|_{\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}}. \tag{36}$$

The standard errors of the parameter estimates are then obtained as

$$\mathrm{SE}(\hat{\theta}_j) = \sqrt{\left[\mathbf{J}(\hat{\boldsymbol{\theta}})^{-1}\right]_{jj}}, \qquad j = 1, 2, 3. \tag{37}$$

4.3 Simulation Study

A Monte Carlo simulation study was conducted to examine the finite-sample performance of the maximum likelihood estimators of the NZTP distribution. Random samples were generated by the inverse transform method. We drew $U \sim \mathrm{Uniform}(0, 1)$ and set $Z = Q_Z(U)$ from (16). This method is exact, so no rejection step is needed. We used $B = 2000$ replications for each of the sample sizes $n = 50, 100, 250$, and 500.

Three parameter (λ, m, Ω) configurations were considered:

$$(0.5, 1.0, 1.0), \quad (1.0, 1.5, 1.0), \quad (2.0, 2.0, 1.5).$$

The MLE was found by the L-BFGS-B algorithm on $\log\lambda$, $\log m$ and $\log\Omega$. We used three starting values, $\lambda_0 \in \{0.3, 1, 3\}$, with $\Omega_0 = n^{-1}\sum_{i=1}^n z_i^2$ and $m_0 = \Omega_0^2/\hat{\mathrm{Var}}(z_i^2)$, and kept the result with the largest log-likelihood. In some replications, $\hat{\lambda}$ went to 0. Conv. is the proportion of replications

with $\hat{\lambda} > 0$, and the other columns use only these replications. CP is the proportion of 95% Wald intervals $\hat{\theta}_j \pm 1.96\,\mathrm{SE}(\hat{\theta}_j)$ that contain the true value. Let $\hat{\theta}_1, \ldots, \hat{\theta}_{B^*}$ be the estimates of a parameter θ from the B^* replications with $\hat{\lambda} > 0$. In Table 1 (Appendix), Table 2 (Appendix), and Table 3 (Appendix), Mean is $\bar{\theta} = B^{*-1} \sum_b \hat{\theta}_b$, Bias is $\bar{\theta} - \theta$, SD is the standard deviation of the $\hat{\theta}_b$, and RMSE is $\{B^{*-1} \sum_b (\hat{\theta}_b - \theta)^2\}^{1/2}$.

4.4 Results

Table 1 (Appendix), Table 2 (Appendix), and Table 3 (Appendix) present the simulation results for the three parameter configurations. This agrees with the consistency of the MLEs under standard regularity conditions [10], but it is numerical evidence only, not a proof. For all three parameters, the bias and RMSE decrease as n increases. The estimates of m and Ω are good. Their bias is small, and their CP is close to 0.95 in most cases. The estimate of λ is less accurate. It has a positive bias and a large RMSE for small n, and its CP is above 0.95 because its standard error is large. In Table 3 (Appendix), the CP of m is below 0.95 (0.88 and 0.87 for $n = 250$ and 500), so the normal approximation is not yet good at these sample sizes. The proportion of replications with $\hat{\lambda} > 0$ increases with n and with λ. When λ is small, many samples have $\bar{P} \leq 1/2$, so $\hat{\lambda} \to 0$. So λ is the hardest parameter to estimate, and large samples are needed.

5 Application to Carbon Fibre Strength Data

The practical usefulness of the proposed NZTP distribution is illustrated using a carbon fibre strength dataset consisting of 69 positive observations measured in GPa. The proposed NZTP distribution is compared with the gamma zero-truncated Poisson (GZTP) distribution. The fitted models are compared using the Akaike information criterion (AIC), Bayesian information criterion (BIC), and Kolmogorov–Smirnov (K–S) statistic. The GZTP model is the gamma distribution (shape α, rate β) compounded with the ZTP distribution through the minimum. We also fit the Nakagami model, which is the limit of NZTP as $\lambda \to 0$.

Table 4 (Appendix) gives the results. For the NZTP model, $\bar{P} = 0.652 > 1/2$, so $\hat{\lambda}$ exists. The SE of $\hat{\lambda}$ is large (2.410), so λ is not well estimated from 69 observations. The AIC and BIC of NZTP and GZTP differ by only 0.14, so the two models fit about equally well. The Nakagami model has the smallest AIC and BIC. The likelihood ratio test of $\lambda \to 0$ gives a statistic of 0.88 and $p = 0.17$, using a 50:50 mixture of χ^2_0 and χ^2_1 because $\lambda = 0$ is on the boundary [11]. So these data do not need the extra parameter λ.

The usual K–S p-values are too large, because the parameters are estimated from the same data. So we computed p-values by a parametric bootstrap with $B = 1000$ samples. We also report the Anderson–Darling (A^*) and Cramér–von Mises (W^*) statistics. The bootstrap p-values are still large (0.886 or more) for all three models. This is plausible, because with $n = 69$ these tests have low power and all three models fit the data well. So the tests show that the fits are adequate, but they cannot rank the models.

6 Conclusion

This paper introduced the Nakagami zero-truncated Poisson (NZTP) distribution as a new three-parameter lifetime model for positive reliability data. The distribution was constructed by compounding the Nakagami distribution with the zero-truncated Poisson distribution through the maximum operator, motivated by parallel-system reliability settings. The probability density function, cumulative distribution function, survival function, and hazard rate function were derived in closed form. The hazard rate is increasing for $m \geq 1/2$ and bathtub-shaped for $m < 1/2$.

Fundamental statistical properties were established, including raw moments, the moment generating function, quantile function, Shannon entropy, Rényi entropy, and order statistics. Parameter estimation was carried out via maximum likelihood, and the score equations were derived explicitly. The bias and RMSE decrease as the sample size increases. The real-data application to carbon fibre strength data showed that the NZTP model fits the data as well as the GZTP model. For these data, the Nakagami model is also adequate, so a larger data set is needed to show the benefit of λ.

Future work may consider Bayesian estimation, censored data modelling, regression extensions incorporating covariates, and bivariate extensions of the proposed model.

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Conflicts of Interest

The authors declare that they have no conflicts of interest. All equations, figures, tables and text are new.

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APPENDIX

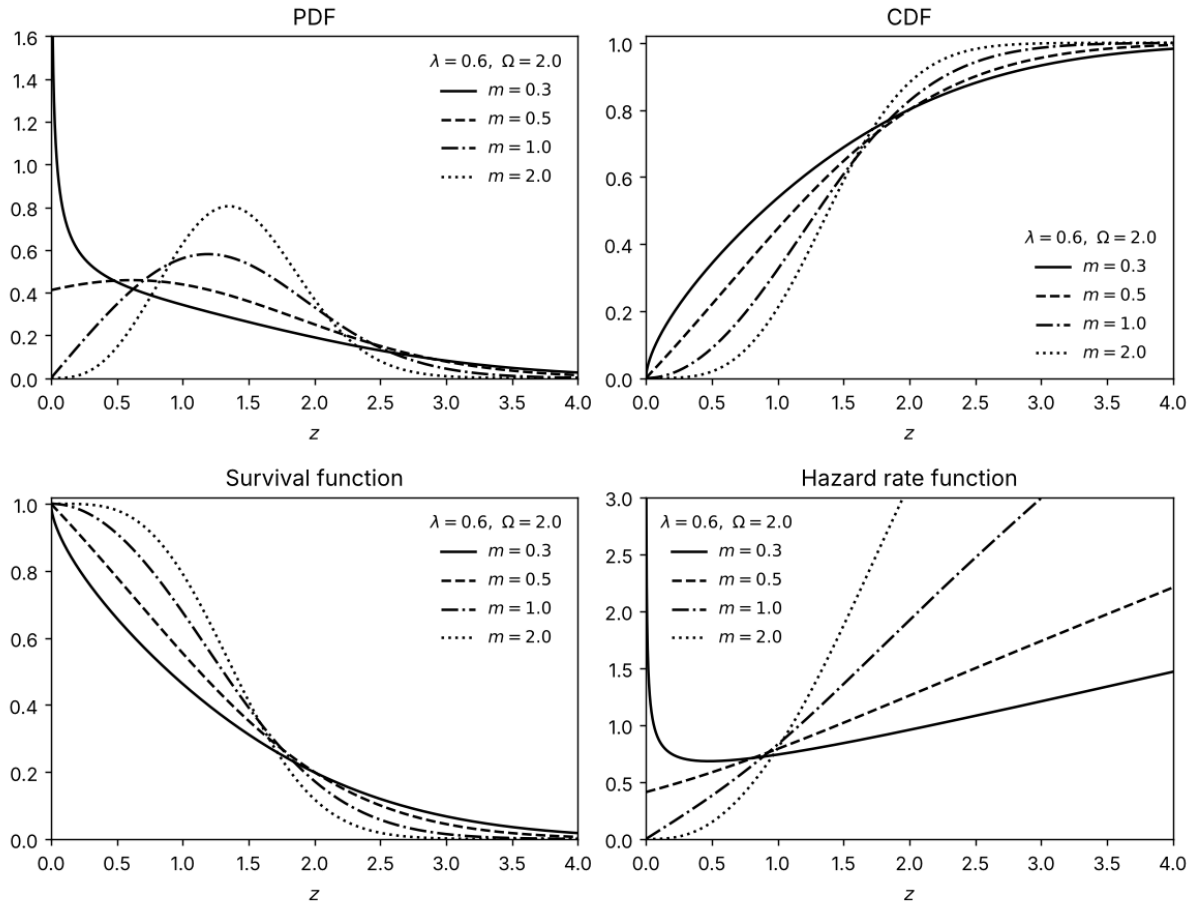


Fig. 1: PDF (top-left), CDF (top-right), survival function (bottom-left) and hazard rate function (bottom-right) of the NZTP distribution for $m = 0.3, 0.5, 1.0, 2.0$, with $\lambda = 0.6$ and $\Omega = 2.0$ fixed in all panels.

Table 1. Simulation results for $(\lambda, m, \Omega) = (0.5, 1.0, 1.0)$ based on $B = 2000$ replications. Conv. is the proportion of replications with $\hat{\lambda} > 0$.

n	Par.	Mean	Bias	SD	RMSE	CP	Conv.
50	λ	1.625	1.125	1.721	2.055	0.999	0.604
	m	0.875	-0.125	0.283	0.309	0.949	
	Ω	0.826	-0.174	0.237	0.294	0.910	
100	λ	1.428	0.928	1.581	1.833	0.992	0.648
	m	0.878	-0.122	0.248	0.276	0.927	
	Ω	0.851	-0.149	0.221	0.267	0.902	
250	λ	1.017	0.517	1.133	1.245	0.985	0.723
	m	0.929	-0.071	0.179	0.193	0.959	
	Ω	0.913	-0.087	0.168	0.189	0.948	
500	λ	0.724	0.224	0.584	0.626	0.992	0.815
	m	0.962	-0.038	0.108	0.114	0.983	
	Ω	0.958	-0.042	0.115	0.123	0.972	

Table 2. Simulation results for $(\lambda, m, \Omega) = (1.0, 1.5, 1.0)$ based on $B = 2000$ replications. Conv. is the proportion of replications with $\hat{\lambda} > 0$.

n	Par.	Mean	Bias	SD	RMSE	CP	Conv.
50	λ	1.880	0.880	3.550	3.656	0.997	0.675
	m	1.395	-0.105	0.469	0.481	0.948	
	Ω	0.903	-0.097	0.229	0.248	0.950	
100	λ	1.771	0.771	1.680	1.848	0.997	0.734
	m	1.370	-0.130	0.416	0.436	0.926	
	Ω	0.907	-0.093	0.229	0.247	0.920	
250	λ	1.396	0.396	1.267	1.327	0.983	0.861
	m	1.417	-0.083	0.315	0.325	0.948	
	Ω	0.951	-0.049	0.181	0.187	0.947	
500	λ	1.171	0.171	0.960	0.975	0.977	0.931
	m	1.467	-0.033	0.250	0.252	0.959	
	Ω	0.982	-0.018	0.148	0.149	0.967	

Table 3. Simulation results for $(\lambda, m, \Omega) = (2.0, 2.0, 1.5)$ based on $B = 2000$ replications. Conv. is the proportion of replications with $\hat{\lambda} > 0$.

n	Par.	Mean	Bias	SD	RMSE	CP	Conv.
50	λ	2.578	0.578	4.807	4.840	1.000	0.771
	m	2.108	0.108	0.806	0.813	0.947	
	Ω	1.486	-0.014	0.367	0.367	0.981	
100	λ	2.285	0.285	1.824	1.845	0.999	0.845
	m	2.030	0.030	0.661	0.661	0.933	
	Ω	1.493	-0.007	0.341	0.341	0.957	
250	λ	2.160	0.160	1.396	1.405	0.959	0.944
	m	2.012	0.012	0.550	0.550	0.883	
	Ω	1.503	0.003	0.297	0.297	0.930	
500	λ	2.137	0.137	1.181	1.189	0.924	0.986
	m	1.992	-0.008	0.467	0.467	0.869	
	Ω	1.495	-0.005	0.255	0.255	0.902	

Table 4. Parameter estimates and goodness-of-fit statistics for the carbon fibre strength data ($n = 69$). For K–S, A^* and W^*, the values in parentheses are parametric bootstrap p-values ($B = 1000$; for GZTP, 770 of the 1000 bootstrap fits converged).

Model	$\hat{\lambda}$	$\hat{m}$	$\hat{\Omega}$	$\hat{\alpha}$	$\hat{\beta}$	AIC	BIC	K–S	A^*	W^*
NZTP	1.941	4.825	5.035	–	–	103.582	110.284	0.0361	0.126	0.014
	(2.410)	(2.375)	(1.389)					(0.998)	(0.980)	(0.992)
GZTP	30.264	–	–	9.543	1.798	103.722	110.424	0.0383	0.138	0.015
	(58.369)			(3.708)	(1.608)			(0.997)	(0.960)	(0.986)
Nakagami	–	6.223	6.251	–	–	102.463	106.931	0.0454	0.207	0.026
		(1.032)	(0.302)					(0.977)	(0.886)	(0.912)