An Algebraic Unification of Astrolabe-HyperG Algebra with Lorentz Geometry via Stereographic Projection and Hessian Duality

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Abstract: This paper introduces the Astrolabe-HyperG algebra $\mathfrak{ahg}(2,1)$, a four-dimensional reductive Lie algebra unifying stereographic projection with Lorentzian geometry. The hyper G-matrix decomposition $A^{-T} = D_1BD_2$, with diagonal calibrations D_1, D_2 and B normalizable to $SL(2,\mathbb{R})$ up to reflection, is interpreted via the astrolabe's rete, alidade, and tympan. The algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$; its semisimple part is $\mathfrak{so}(2,1)$, but the full algebra is not. This paper introduces the Astrolabe-HyperG algebra $\mathfrak{ahg}(2,1)$, a four-dimensional reductive Lie algebra unifying stereographic projection with Lorentzian geometry. The hyper G-matrix decomposition $A^{-T} = D_1BD_2$, with diagonal calibrations D_1, D_2 and B normalizable to $SL(2,\mathbb{R})$ up to reflection, is interpreted via the astrolabe's rete, alidade, and tympan. The algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$; its semisimple part is $\mathfrak{so}(2,1)$, but the full algebra is not. A lemma links the bordered Hessian of constrained optimization to this decomposition. The Fan product of dual matrices extends the structure, while conic spectral decomposition yields spacelike, timelike, and lightlike cases under the duality between conics and dual vectors. Applications to geometric optics and planar robotics illustrate the theory. Three conjectures and the Astrolabe-HyperG Principle are stated. Applications to geometric optics and planar robotics illustrate the theory. Three conjectures and the Astrolabe-HyperG Principle are stated.

Key-Words: Astrolabe, stereographic projection, hyper G-matrix, dual numbers, Lorentzian geometry, Lie algebra $\mathfrak{so}(2,1)$, Study mapping, ruled surfaces, Fan product.
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1 Introduction

The astrolabe, invented in ancient Greece, embodies the geometric principle of stereographic projection mapping the celestial sphere onto a plane. In modern algebraic terms, this projection can be represented by hyper G-matrices satisfying $A^{-T} = D_1BD_2$, where D_1, D_2 are diagonal scaling matrices and B is normalizable to $SL(2,\mathbb{R})$ up to a fixed reflection. The geometric foundations of such transformations trace back to seminal work on line geometry and dual numbers as shown in, [1], which established the connection between spatial kinematics and algebraic structures.

This work demonstrates that this decomposition reflects a deep algebraic structure: the Astrolabe-HyperG algebra $\mathfrak{ahg}(2,1)$. The theory of Lie groups and Lie algebras, particularly the structure of $\mathfrak{so}(2,1)$ and its representations, is extensively treated in the comprehensive text in, [2]. The semisimple part of $\mathfrak{ahg}(2,1)$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,1)$, where the isomorphism is the adjoint representation $\mathrm{ad} : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(\mathfrak{sl}(2,\mathbb{R}))$ composed with the standard identification $\mathfrak{gl}(\mathfrak{sl}(2,\mathbb{R})) \cong \mathfrak{gl}(3,\mathbb{R})$ and the Killing form of signature $(2,1)$; the full algebra is four-dimensional and reductive, and is therefore *not* isomorphic to $\mathfrak{so}(2,1)$. This distinction shows that stereographic projection, Lagrange multiplier optimization, and dual hyperbolic motion share the same semisimple algebraic core.

The interpretation of conic sections through conditional extremal problems, which forms a crucial bridge between the Hessian matrix and the hyper G-matrix decomposition, was developed in, [3]. This approach reveals that the bordered Hessian matrix from constrained optimization naturally admits the decomposition $\mathcal{H}_{2\times 2} = D_1BD_2$, where D_1 and D_2 are determined by the second derivatives of the Lagrangian. This is stated precisely in Lemma 3.1 (see in particular (23) and (25)) and is structurally related to—but distinct from—the hyper G-matrix decomposition $A^{-T} = D_1BD_2$ of Definition 2.1 (equation (4)): the two share the same $SL(2,\mathbb{R})$ factor but act on different objects.

Dual numbers and dual matrices provide the algebraic foundation for representing spatial motions. The work in, [4] presented a comprehensive treatment of dual numbers, vectors, and matrices in instantaneous spatial kinematics, establishing the conventional framework for handling screws and wrenches. The work in, [5] further developed the application of Lie groups to space kinematics, providing a rigorous algebraic foundation for motion analysis.

Recently, the work in, [6] introduced the Fan product of dual matrices, providing a new algebraic framework for dual number matrices. The Fan product, defined entrywise with a sign change for off-diagonal elements (see (29) below), offers novel properties including a distinct identity matrix I_F and specialized notions of orthogonality and invertibility. We show that this Fan product structure naturally extends the Astrolabe-HyperG algebra, enabling new applications in kinematic analysis. Geometric algebra provides an alternative unified language for spatial computations, as demonstrated in, [7], offering complementary insights into the algebraic structures underlying stereographic projection and Lorentzian geometry.

From a quantitative optimization perspective, the work in, [8] developed a penalty theory for constrained symmetric eigenvalue problems, establishing exact characterizations of the extremal eigenvalues of the reduced Hessian via full-space penalized eigenvalue problems. Their Projected Hessian Quantification Theorem extends the classical Projected Hessian Lemma, which originates from Finsler's theorem, by providing explicit error expansions and a Hellmann–Feynman sensitivity relation along the eigenvalue penalty path. This quantitative perspective is complementary to the structural decomposition established in Lemma 3.1: whereas the work in, [8] quantifies how the extremal eigenvalues of the reduced Hessian depend on the penalty parameter, our decomposition $\mathcal{H}_{2\times 2} = D_1 B D_2$ in (23) reveals the underlying $SL(2,\mathbb{R})$ algebraic structure that governs the signature and Lorentzian causal character of the constrained extremum.

In a broader measure-theoretic setting, the work in, [9] developed a Kantorovich–Rubinstein duality theory for the Hessian, extending classical optimal transport duality to functionals involving second derivatives. Their framework characterizes dual pairs of measures and establishes a general variational principle for Hessian-dependent problems. This duality perspective is conceptually aligned with the algebraic duality underlying the hyper G-matrix decomposition $A^{-T} = D_1 B D_2$ in, (4): while the work in [9] operates at the level of measures and convex functionals, our decomposition captures the finite-dimensional algebraic skeleton of the same duality, with the bordered Hessian $\mathcal{H}_{2\times 2}$ in (23) playing the role of the primal object and the $SL(2,\mathbb{R})$ factor B encoding the dual signature information.

From a global geometric perspective, Hessian structures also impose strong topological constraints on the underlying manifolds. The work in, [10] recently studied the topology of compact, orientable 3-dimensional Hessian manifolds, with particular emphasis on Koszul-type structures, and derived consequences for their Betti numbers and Thurston geometries. While the present work operates at the level of finite-dimensional algebraic decompositions rather than global manifold topology, both perspectives share the underlying theme that second-order data—encoded by a Hessian—carries rigid geometric information.

Hessian-based geometric frameworks have also been developed in applied and complex-systems settings. The work in, [11] introduced the Omega metric $\omega = \nabla^2\Omega$ as a Hessian Riemannian structure for finite-depth structural variation, linking local structural increments to global geodesics, curvature, and contraction properties. Although the present work is concerned with the algebraic decomposition of the bordered Hessian in constrained optimization rather than with a Riemannian metric on a state space, both approaches illustrate the general principle that second-order data can generate rigid geometric structure. This reinforces the broader motivation for the Hessian–HyperG correspondence developed in Section 3.

Throughout this work, the *dual hyperbolic unit sphere* is the set defined by

$$\mathbb{H}_0^2 = \{\hat{v} = v + \varepsilon v^* \in \mathbf{\Delta}_1^3 : \langle v, v\rangle = -1,\ \langle v, v^*\rangle = 0\} \tag{1}$$

which serves as the dual model of the hyperbolic plane and is the natural arena for the mapping in, [1].

In geometric optics, the hyper G-matrix decomposition $A^{-T} = D_1 B D_2$ (equation (4)) admits a physical interpretation: D_2 represents the transformation from object space to canonical coordinates, B represents the intrinsic optical transformation (lens or mirror), and D_1 represents the transformation from canonical coordinates to image space. The Lagrange multiplier λ corresponds to the reduced optical path length, and the dual hyperbolic angle φ corresponds to the angle of incidence or refraction. A concrete ray-transfer example is given in Subsection 3.1 (see (52)–(56)).

In robotics and kinematics, the Fan product provides a new algebraic tool for analyzing kinematic singularities. For a planar robot arm with joint angles

θ_i, the Fan product of dual transformation matrices yields $\hat{T}_{\text{total}} = \hat{T}_1 \star \hat{T}_2 \star \cdots \star \hat{T}_n$, where each $\hat{T}_i$ is a dual matrix representing a joint transformation. Singular configurations of the classical Jacobian correspond to degeneracies of the dual transformation family. A planar two-revolute-joint arm is treated explicitly in Subsection 3.1 (see (58) and (62)).

In computational geometry, the duality enables efficient computation through the following steps: first, compute the extremal spectrum via the eigenvalue problem using the quadratic form matrix Q (equation (34) below); second, map the eigenvalues to dual hyperbolic angles φ_i via $\lambda_i = \pm\cosh\varphi_i$; third, compute timelike lines L_i via the mapping in, [1]; finally, transform back to critical points using the inverse transformation in, [3]. This approach avoids solving quartic equations directly, providing a more efficient algebraic method.

The remainder of this paper is organized as follows. Section 2 introduces hyper G-matrices and their relationship to the astrolabe's stereographic projection. We define the Astrolabe-HyperG algebra $\mathfrak{ahg}(2,1)$, prove its structure theorems, and present an explicit faithful matrix representation. Section 3 unifies three structures: the Hessian matrix from constrained optimization (Lemma 3.1), the Fan product of dual matrices (Definition 3.4), and the standard spectral decomposition of conics (Proposition 3.9). We show how the bordered Hessian decomposes as $\mathcal{H}_{2\times 2} = D_1 B D_2$ (equation (23)), how the Fan product respects this decomposition, and how the sign of the eigenvalues of the quadratic form Q determines the Lorentzian causal character of the associated dual vectors. Subsection 3.1 develops explicit examples in geometric optics and robotics. Section 4 concludes the paper, states three conjectures, and presents the Astrolabe-HyperG Principle. The physical interpretation of the hyper G-matrix decomposition is summarized in Table 1.

2 Hyper G-Matrices, Astrolabe, and the Algebra $\mathfrak{ahg}(2,1)$

This section establishes the fundamental algebraic structures underlying the astrolabe's stereographic projection. We first recall the stereographic projection mapping and its Jacobian, which motivates the definition of hyper G-matrices. These matrices satisfy the decomposition $A^{-T} = D_1 B D_2$ (equation (4)) with D_1, D_2 diagonal and B normalizable to $SL(2,\mathbb{R})$ up to a fixed reflection. We interpret this decomposition physically in terms of the astrolabe's components (see Table 1): the rete (D_1), alidade (B), and tympan (D_2). Subsequently, we define the Astrolabe-HyperG algebra $\mathfrak{ahg}(2,1)$ as a four-dimensional reductive algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$. The main result of this section is Theorem 2.5. A faithful matrix representation ρ of $\mathfrak{ahg}(2,1)$ into 2×2 complex matrices is provided (Definition 2.7, equation (15)), and Proposition 2.10 shows that every hyper G-matrix of scalar-plus-$SL(2,\mathbb{R})$ type arises as the exponential of some $X \in \mathfrak{ahg}(2,1)$ under this representation (equation (19)).

The astrolabe's stereographic projection (2) from the north pole $(0,0,1)$ onto the plane $z=0$ is:

$$\pi(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right). \tag{2}$$

The Jacobian matrix (3) at (x,y,z) is:

$$J = \frac{1}{(1-z)^2}\begin{pmatrix} 1-z & 0 & x \\ 0 & 1-z & y \end{pmatrix}. \tag{3}$$

Definition 2.1. *A hyper G-matrix is a 2×2 real matrix A satisfying:*

$$A^{-T} = D_1 B D_2, \tag{4}$$

where $D_1 = \operatorname{diag}(\alpha_1, \alpha_2)$, $D_2 = \operatorname{diag}(\beta_1, \beta_2)$ with $\alpha_i, \beta_i > 0$, and B is normalizable to $SL(2,\mathbb{R})$ up to a fixed reflection. Equivalently, $B \in SL^{\pm}(2,\mathbb{R})$, where

$$SL^{\pm}(2,\mathbb{R}) = \{M \in GL(2,\mathbb{R}) : \det M = \pm 1\}. \tag{5}$$

If $\det B > 0$, one normalizes B to $SL(2,\mathbb{R})$ by scaling by $(\det B)^{-1/2}$. If $\det B < 0$, then $\operatorname{diag}(1,-1)(\det B)^{-1/2} B$ lies in $SL(2,\mathbb{R})$. The sign of $\det B$ will be tied in Lemma 3.1 (see (26)) to the signature of the reduced Hessian.

For the stereographic projection, restricting to the tangent plane at (x_0, y_0, z_0) gives $A = \frac{1}{1-z_0} I_2$, so $A^{-T} = (1-z_0) I_2 = D_1 I_2 D_2$ with $D_1 = (1-z_0) I_2$, $D_2 = I_2$.

Table 1: Physical interpretation of the hyper G-matrix decomposition (4) in the astrolabe.

Matrix	Physical Component	Mathematical Role
D_1	Rete (star map)	Calibration / Scaling
B	Alidade (sighting rule)	Intrinsic projection ($SL(2,\mathbb{R})$ up to reflection)
D_2	Tympan (plate with coordinates)	Normalization
A^{-T}	Overall transformation	Dual action

Source: Created by the author.

Definition 2.2. *The Astrolabe-HyperG algebra $\mathfrak{ahg}(2,1)$ is the real Lie algebra with basis $\{H, E_+, E_-, Z\}$ and commutation relations:*

$$\begin{aligned} [H, E_\pm] &= \pm 2E_\pm, \qquad [E_+, E_-] = H, \\ [Z, H] &= 0, \qquad [Z, E_\pm] = 0. \end{aligned} \tag{6}$$

The element

$$K = \tfrac{1}{2}H + Z \tag{7}$$

satisfies $[K, H] = 0$ *and* $[K, E_\pm] = \pm E_\pm$. *Thus* K *differs from* $H/2$ *by the central element* Z. *The algebra is four-dimensional, its center is the one-dimensional subspace* $\mathbb{R}Z$ *spanned by* Z, *and its derived subalgebra is*

$$\mathfrak{h} = \langle H, E_+, E_- \rangle \cong \mathfrak{sl}(2, \mathbb{R}). \tag{8}$$

Remark 2.3. *The subalgebra generated by* H, E_+, E_- *is isomorphic to* $\mathfrak{sl}(2, \mathbb{R})$ *via the standard identification (note that the* $\mathfrak{sl}(2, \mathbb{R})$ *commutation relations conventionally written* $[h, e_\pm] = \pm 2e_\pm$, $[e_+, e_-] = h$ *are exactly* (6)*). The element* Z *is central. The element* $K = H/2 + Z$ *(see* (7)*) acts on* $\mathfrak{h}$ *as the inner derivation* $\mathrm{ad}(H/2)$, *so it is* not *an outer derivation. The full algebra satisfies*

$$\mathfrak{ahg}(2, 1) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}Z \tag{9}$$

as a Lie algebra direct sum (with the one-dimensional abelian factor spanned by Z*). Since* $\mathbb{R}Z \cong \mathbb{R}$ *as a vector space, one may equivalently write* $\mathfrak{ahg}(2, 1) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$. *The algebra is reductive and not solvable. Its semisimple part is* $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(2, 1)$, *the isomorphism being given by the adjoint representation of* $\mathfrak{sl}(2, \mathbb{R})$ *onto* $\mathfrak{so}(\mathfrak{sl}(2, \mathbb{R}), \kappa)$, *where* κ *is the Killing form of signature* $(2, 1)$. *The full algebra is not isomorphic to* $\mathfrak{so}(2, 1)$ *because* $\dim \mathfrak{ahg}(2, 1) = 4$, *whereas* $\dim \mathfrak{so}(2, 1) = 3$.

Proposition 2.4. $\mathfrak{ahg}(2, 1)$ *is a 4-dimensional reductive Lie algebra. It is not isomorphic to* $\mathfrak{so}(2, 1)$ *(dimension 3) or to* $\mathfrak{sl}(2, \mathbb{R})$ *(dimension 3). Its derived subalgebra is 3-dimensional and isomorphic to* $\mathfrak{sl}(2, \mathbb{R})$.

Proof. By Definition 2.2, $\mathfrak{ahg}(2, 1)$ has basis $\{H, E_+, E_-, Z\}$, so its dimension is exactly 4. The commutation relations (6) show that Z is central and that $\mathfrak{h} = \langle H, E_+, E_- \rangle$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Hence

$$\mathfrak{ahg}(2, 1) = \mathfrak{h} \oplus \mathbb{R}Z \tag{10}$$

as a Lie algebra direct sum (10), where the abelian factor is spanned by Z. Since $\mathfrak{sl}(2, \mathbb{R})$ is semisimple and the complement is abelian, the algebra is reductive. The derived subalgebra is $\mathfrak{h}$, of dimension 3. Therefore no isomorphism with $\mathfrak{so}(2, 1)$ or $\mathfrak{sl}(2, \mathbb{R})$ can exist. □

Theorem 2.5. $\mathfrak{ahg}(2, 1)$ *is a 4-dimensional Lie algebra with the following structure:*

i. The subalgebra $\mathfrak{h} = \langle H, E_+, E_- \rangle$ *is isomorphic to* $\mathfrak{sl}(2, \mathbb{R})$, *with commutation relations* (6).

ii. The element Z *is central, and*

$$\mathfrak{ahg}(2, 1) = \mathfrak{h} \oplus \mathbb{R}Z \tag{11}$$

as a Lie algebra direct sum, with the abelian factor spanned by Z.

iii. The element $K = H/2 + Z$ *(equation* (7)*) is not an outer derivation; its action on* $\mathfrak{h}$ *equals the inner derivation* $\mathrm{ad}(H/2)$.

iv. The derived subalgebra (12) *is*

$$\mathfrak{ahg}(2, 1)^{(1)} = [\mathfrak{ahg}(2, 1), \mathfrak{ahg}(2, 1)] = \mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{R}). \tag{12}$$

v. The center is the one-dimensional subspace $\mathbb{R}Z$ *spanned by* Z.

vi. Since $\mathfrak{h}$ *is perfect, the derived series stabilizes at* $\mathfrak{h}$*; therefore* $\mathfrak{ahg}(2, 1)$ *is* ***not solvable****. It is reductive, and its semisimple part is* $\mathfrak{h} \cong \mathfrak{so}(2, 1)$.

Thus $\mathfrak{ahg}(2, 1)$ *is a non-solvable 4-dimensional reductive Lie algebra, isomorphic to the direct sum* $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}Z \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$ *(see* (11)*).*

Proof. Claims (i) and (ii) follow directly from Definition 2.2: the elements H, E_+, E_- satisfy the $\mathfrak{sl}(2, \mathbb{R})$ commutation relations (6), and Z commutes with all generators (equation (6)). For (iii), observe that

$$\mathrm{ad}(K)|_{\mathfrak{h}} = \mathrm{ad}(H/2)|_{\mathfrak{h}}, \tag{13}$$

because Z is central, as expressed in (13). Since $H/2 \in \mathfrak{h}$, this is an inner derivation. For (iv), every bracket among H, E_+, E_-, Z lies in $\mathfrak{h}$:

$$[E_+, E_-] = H, \ [H, E_\pm] = \pm 2E_\pm, \ [Z, \cdot] = 0, \tag{14}$$

as shown in (14). Conversely, H and $E_\pm$ are themselves brackets, so the derived subalgebra equals $\mathfrak{h}$. For (v), if $aH + bE_+ + cE_- + dZ$ is central, then bracketing with E_+ and E_- forces $a = b = c = 0$, leaving the one-dimensional span of Z; hence the center is $\mathbb{R}Z$. For (vi), since $\mathfrak{h} \cong \mathfrak{sl}(2, \mathbb{R})$ is perfect, the derived series stabilizes at $\mathfrak{h} \neq 0$, so the algebra is not solvable. The direct-sum decomposition (11) into semisimple and abelian parts shows that it is reductive. □

Remark 2.6. *The algebra* $\mathfrak{ahg}(2, 1)$ *is a genuine 4-dimensional reductive Lie algebra. It is isomorphic to the direct sum* $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$ *(see* (9)*). In particular, the semisimple part is* $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(2, 1)$, *but the full algebra is not isomorphic to* $\mathfrak{so}(2, 1)$.

Definition 2.7. *Define a representation* $\rho : \mathfrak{ahg}(2, 1) \to \mathrm{Mat}_{2\times 2}(\mathbb{C})$ *by:*

$$\rho(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho(E_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$\rho(E_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \rho(Z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{15}$$

Consequently,

$$\rho(K) = \rho(H/2 + Z) = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \qquad (16)$$

The choice $\rho(Z) = I_2$ *in* (15) *is essentially forced by the algebraic structure: since* Z *is central in* $\mathfrak{ahg}(2,1)$ *and the representation is irreducible, Schur's lemma implies that* $\rho(Z)$ *must be a non-zero scalar matrix. Thus* $\rho(Z) = \lambda I_2$ *with* $\lambda \in \mathbb{C} \setminus \{0\}$*, and we choose the normalization* $\lambda = 1$ *for convenience. In particular, any scalar multiple of* I_2 *is admissible; the choice* $\lambda = 1$ *makes* $\rho(K)$ *in* (16) *diagonal with entries* $3/2$ *and* $1/2$*.*

Remark 2.8. *The representation* (15) *satisfies all commutation relations* (6)*:*

$$[\rho(K), \rho(H)] = 0, \qquad [\rho(K), \rho(E_\pm)] = \pm\rho(E_\pm), \qquad (17)$$

as stated in (17)*. It is faithful. Indeed, if*

$$a\rho(H) + b\rho(E_+) + c\rho(E_-) + d\rho(Z) = 0, \qquad (18)$$

as in (18)*, then the off-diagonal entries give* $b = c = 0$*, the diagonal difference gives* $a = 0$*, and the remaining scalar gives* $d = 0$*. Hence the kernel is trivial.*

Remark 2.9. *This representation is irreducible: the only subspaces invariant under all* $\rho(H)$*,* $\rho(E_+)$*,* $\rho(E_-)$*,* $\rho(Z)$ *are* $\{0\}$ *and* $\mathbb{C}^2$*. Indeed,* $\rho(E_+)$ *maps* e_2 *to* e_1*, and* $\rho(E_-)$ *maps* e_1 *to* e_2*, so the two basis vectors are linked. The central element* $\rho(Z)$ *is scalar and does not affect irreducibility.*

Proposition 2.10. *Let* $\rho : \mathfrak{ahg}(2,1) \to \mathfrak{gl}(2,\mathbb{C})$ *be the faithful representation of Definition 2.7 (equation* (16)*). Then:*

i. For any $X \in \mathfrak{ahg}(2,1)$*, the matrix* $A = \exp(\rho(X))$ *is a hyper G-matrix in the sense of Definition 2.1 (equation* (4)*).*

ii. Conversely, every hyper G-matrix of the form $A = e^c B$*, with* $c \in \mathbb{R}$*,* $e^c > 0$*, and* $B \in \exp(\rho(\mathfrak{sl}(2,\mathbb{R})))$*, arises as* $A = \exp(\rho(X))$ *for some* $X \in \mathfrak{ahg}(2,1)$*.*

Note that the exponential map $\exp : \mathfrak{sl}(2,\mathbb{R}) \to SL(2,\mathbb{R})$ *is not surjective, so not every element of* $SL(2,\mathbb{R})$ *is of the form* $\exp(\rho(X_{\mathfrak{sl}}))$*. The converse (ii) therefore characterizes a proper subclass of hyper G-matrices, namely those whose* B*-factor lies in the image of the exponential. A more general version using the Cartan or KAK decomposition could remove this restriction, but we restrict to the exponential case for simplicity.*

Proof. Write $X = X_{\mathfrak{sl}} + cZ$, where $X_{\mathfrak{sl}} \in \mathfrak{sl}(2,\mathbb{R})$ and $c \in \mathbb{R}$. Since Z is central,

$$\begin{aligned} \exp(\rho(X)) &= \exp(\rho(X_{\mathfrak{sl}}))\exp(c\rho(Z)) \\ &= e^c \exp(\rho(X_{\mathfrak{sl}})). \end{aligned} \qquad (19)$$

Set $B = \exp(\rho(X_{\mathfrak{sl}}))$. Because $X_{\mathfrak{sl}} \in \mathfrak{sl}(2,\mathbb{R})$, we have $B \in SL(2,\mathbb{R})$. Thus $A = e^c B$, and

$$A^{-T} = e^{-c}B^{-T} = (e^{-c}I_2)B^{-T}I_2, \qquad (20)$$

which, by (20), is of the form (4) with $D_1 = e^{-c}I_2$, $D_2 = I_2$, and $B^{-T} \in SL(2,\mathbb{R})$. This proves (i). The converse follows by reversing the same computation: if $A = e^c B$ with $B = \exp(\rho(X_{\mathfrak{sl}}))$, then $A = \exp(\rho(X_{\mathfrak{sl}} + cZ))$. This proves (ii). □

Remark 2.11. *The proposition is unconditional in the reductive direct-sum model. The earlier need for a Baker–Campbell–Hausdorff convergence hypothesis is eliminated because the central element* Z *commutes with all of* $\mathfrak{ahg}(2,1)$ *(see* (6)*).*

3 Unification of Hessian Geometry, Fan Product, and Spectral Decomposition

This section develops the core mathematical framework that unifies three structures: the Hessian matrix from constrained optimization, the Fan product of dual matrices, and the standard spectral decomposition of conics. We first establish the Hessian-HyperG connection, which shows that the bordered Hessian matrix of a constrained optimization problem decomposes as $\mathcal{H}_{2\times 2} = D_1 B D_2$ (equation (23)) with B normalizable to $SL(2,\mathbb{R})$ up to a fixed reflection (see (25)). This decomposition reveals that the second-order conditions for constrained extrema are encoded in the same algebraic structure that governs the astrolabe's stereographic projection (compare (4) and (23)). Next, we extend the Astrolabe-HyperG algebra by incorporating the Fan product of dual matrices, following in, [6]. Finally, we recall the standard spectral decomposition of conics, which shows that every conic decomposes into positive, negative, and null eigenspaces of its quadratic form matrix (equation (34)). Under the duality in, [1], [3], these components correspond to spacelike, timelike, and lightlike dual vectors, respectively, and the associated one-parameter families of extremals sweep out ruled surfaces in $\mathbb{R}^3_1$ via the mapping in, [1].

Lemma 3.1. *Let*

$$\mathcal{L}(x,y,\lambda) = (x-a)^2 + (y-b)^2 + \lambda(Ax^2 + \beta xy + Cy^2 + Dx + Ey + F) \qquad (21)$$

be the Lagrangian for minimizing the distance to (a, b) *subject to the conic constraint*

$$Ax^2 + \beta xy + Cy^2 + Dx + Ey + F = 0. \quad (22)$$

At a critical point satisfying the stationarity conditions, the 2×2 *principal submatrix of the bordered Hessian (corresponding to the* x, y *variables) is*

$$\mathcal{H}_{2\times 2} = \begin{pmatrix} 2+2\lambda A & \lambda\beta \\ \lambda\beta & 2+2\lambda C \end{pmatrix}. \quad (23)$$

Assume $2 + 2\lambda A > 0$ *and* $2 + 2\lambda C > 0$*. Put*

$$D_1 = D_2 = \begin{pmatrix} \sqrt{2+2\lambda A} & 0 \\ 0 & \sqrt{2+2\lambda C} \end{pmatrix}. \quad (24)$$

Then

$$B = D_1^{-1}\mathcal{H}_{2\times 2}D_2^{-1} = \begin{pmatrix} 1 & \frac{\lambda\beta}{\sqrt{(2+2\lambda A)(2+2\lambda C)}} \\ \frac{\lambda\beta}{\sqrt{(2+2\lambda A)(2+2\lambda C)}} & 1 \end{pmatrix}. \quad (25)$$

Its determinant is exactly

$$\begin{aligned} \det(B) &= 1 - \frac{\lambda^2\beta^2}{(2+2\lambda A)(2+2\lambda C)} \\ &= \frac{4(1+\lambda A)(1+\lambda C) - \lambda^2\beta^2}{4(1+\lambda A)(1+\lambda C)}. \end{aligned} \quad (26)$$

If $\det(B) > 0$*, set* $s = \sqrt{\det(B)}$*; then* $s^{-1}B \in SL(2, \mathbb{R})$*. If* $\det(B) < 0$*, then* $s^{-1}B$ *has determinant* -1*, and* $\mathrm{diag}(1, -1)s^{-1}B \in SL(2, \mathbb{R})$*. Thus* B *is normalizable to* $SL(2, \mathbb{R})$ *up to the fixed reflection* $\mathrm{diag}(1, -1)$*. The sign of* $\det(B)$ *in* (26) *equals the sign of* $\det(\mathcal{H}_{2\times 2})$*, which is the signature of the reduced Hessian.*

Proof. The decomposition $\mathcal{H}_{2\times 2} = D_1 B D_2$ follows by direct multiplication using (24) and (25). The determinant formula (26) is obtained by direct computation from (25). If $\det(B) > 0$, scaling by s^{-1} yields determinant 1. If $\det(B) < 0$, scaling by s^{-1} yields determinant -1, and multiplying by $\mathrm{diag}(1, -1)$ changes the sign of the determinant without changing its absolute value, yielding an element of $SL(2, \mathbb{R})$. Finally,

$$\det(\mathcal{H}_{2\times 2}) = (2+2\lambda A)(2+2\lambda C) - \lambda^2\beta^2, \quad (27)$$

as given in (27), and $\det(B) = \det(\mathcal{H}_{2\times 2})/[(2 + 2\lambda A)(2+2\lambda C)]$ (see (26)), so the signs agree because the denominator is positive. □

Remark 3.2. *The assumptions* $2 + 2\lambda A > 0$ *and* $2 + 2\lambda C > 0$ *are not always satisfied: for instance, if* $A < 0$ *and* λ *is large positive, or if* $A = 0$*, the diagonal factors* D_1*,* D_2 *in* (24) *cannot be taken to be real positive diagonal matrices, and the decomposition must be formulated with complex or sign-adjusted factors. The lemma is therefore a statement about the* definite regime *of the constrained optimization problem. In the indefinite or degenerate regime, the same algebraic structure persists but with* D_1, D_2 *valued in* $\mathbb{C}^*$ *or* $\mathbb{R}^*\backslash\{0\}$*; the causal character of the associated dual vector is then encoded by the sign pattern of the eigenvalues of* $\mathcal{H}_{2\times 2}$ *in* (23) *rather than by the positivity of the diagonal entries of* D_1, D_2*.*

Remark 3.3. *The Hessian-HyperG lemma is not a nominal analogy. The* same $SL(2, \mathbb{R})$*-valued factor* B *that governs the astrolabe's stereographic projection (see* (4)*) also encodes the second-order signature of the constrained extremum. Specifically,* $\det(B)$ *in* (26) *is positive exactly when the reduced Hessian* (23) *is definite, and negative when it is indefinite. Under the identification* $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(2, 1)$ *(see Remark 2.3), this sign correspondence is the spacelike/timelike distinction.*

Definition 3.4. *For* $\hat{U} = U + \varepsilon U^* \in \mathbb{D}^{m\times n}$ *and* $\hat{V} = V + \varepsilon V^* \in \mathbb{D}^{m\times n}$*, the Fan product* $\star$ *is defined entrywise as:*

$$\hat{U} \star \hat{V} = (U \star V) + \varepsilon(U^* \star V + U \star V^*), \quad (28)$$

where for real matrices $P = [p_{ij}]$*,* $Q = [q_{ij}]$*, the Fan product is:*

$$(P \star Q)_{ij} = \begin{cases} p_{ii}q_{ii}, & i = j, \\ -p_{ij}q_{ij}, & i \neq j. \end{cases} \quad (29)$$

Definition 3.5. *The F-identity matrix* $I_F \in \mathbb{R}^{m\times m}$ *is defined by:*

$$(I_F)_{ij} = \begin{cases} 1, & i = j, \\ -1, & i \neq j. \end{cases} \quad (30)$$

The dual F-identity is $\hat{I}_F = I_F + \varepsilon 0$*.*

Theorem 3.6. *Let* $\hat{P}, \hat{Q} \in \mathbb{D}^{2\times 2}$ *be regular dual matrices under the Fan product* (28)–(29) *in the sense defined in [6]. Then*

$$(\hat{P} \star \hat{Q})^{-1} = \hat{P}^{-1} \star \hat{Q}^{-1}. \quad (31)$$

This is given in, [6, Theorem 4.7].

Proof. The identity (31) is given in, [6, Theorem 4.7]. The hyper G-matrix decomposition (4) is preserved under the Fan product because $\hat{D}_1$ and $\hat{D}_2$ are diagonal and the off-diagonal sign change in (29) does not affect the determinant of a 2×2 matrix. Hence the normalized B-factor remains in $SL(2, \mathbb{R})$ up to the same fixed reflection. □

Definition 3.7. *A dual matrix $\hat{A} = A + \varepsilon A^*$ is called **F-orthogonal** if:*

$$\hat{A} \star \hat{A}^T = \hat{I}_F. \tag{32}$$

By the definition in [6, Definition 5.4], this condition is equivalent to $a_{ij} \cdot a_{ji} = 1$ for all i, j for the real part, with analogous conditions for the dual part.

Definition 3.8. *A dual matrix $\hat{A} = A + \varepsilon A^*$ is called **F-nilpotent** under the Fan product if there exists $k \in \mathbb{N}$ such that*

$$(\hat{A})^{\star k} = 0 + \varepsilon 0, \tag{33}$$

where $\star k$ denotes the k-fold Fan product. By [6, Corollary 5.11], the only F-nilpotent matrix is the zero matrix.

Proposition 3.9. *Let*

$$Q = \begin{pmatrix} A & \beta/2 \\ \beta/2 & C \end{pmatrix} \tag{34}$$

be the symmetric matrix associated with the quadratic part of a conic. Then the eigenspaces of Q decompose as

$$\mathbb{R}^2 = E_+(Q) \oplus E_-(Q) \oplus E_0(Q), \tag{35}$$

where E_+, E_-, E_0 are the eigenspaces corresponding to positive, negative, and zero eigenvalues, respectively. Under the duality in, [1], [3]:

i. *$E_+(Q)$ consists of spacelike directions (positive-definite quadratic form),*
ii. *$E_-(Q)$ consists of timelike directions (negative-definite quadratic form),*
iii. *$E_0(Q)$ consists of lightlike (null) directions, present if and only if $\det Q = 0$.*

The conic $\mathscr{C}$ is classified as follows:

- ***elliptic type** if and only if $\det Q > 0$ (equivalently, both eigenvalues are nonzero and have the same sign). In this case, the sign of $\operatorname{tr} Q$ distinguishes two subcases:*
  - *$\operatorname{tr} Q > 0$: the quadratic form is positive definite, giving a real ellipse (non-empty);*
  - *$\operatorname{tr} Q < 0$: the quadratic form is negative definite, giving the empty set.*
- ***hyperbolic type** if and only if $\det Q < 0$ (equivalently, the two eigenvalues have opposite signs);*
- ***parabolic type** if and only if $\det Q = 0$ (equivalently, exactly one eigenvalue vanishes).*

The decomposition (35) *is standard; the correspondence with causal character is obtained via the duality in, [1], [3].*

Proof. Since Q in (34) is real and symmetric, the spectral theorem gives an orthonormal eigen-decomposition $Q = R^T \Lambda R$, with $R \in O(2)$ and $\Lambda = \operatorname{diag}(\mu_1, \mu_2)$. After the orthogonal change of coordinates $(x, y)^T = R^T (u, v)^T$, the quadratic form diagonalizes:

$$Q(x, y) = \mu_1 u^2 + \mu_2 v^2. \tag{36}$$

If both eigenvalues have the same sign ($\det Q > 0$), the quadratic form is definite and the conic is of elliptic type. Its trace $\operatorname{tr} Q = \mu_1 + \mu_2$ has the same sign as both eigenvalues; if $\operatorname{tr} Q > 0$ we get a real ellipse, if $\operatorname{tr} Q < 0$ we get the empty set. If the eigenvalues have opposite signs ($\det Q < 0$), the quadratic form is indefinite and the conic is a hyperbola. If exactly one eigenvalue vanishes ($\det Q = 0$), the conic is a parabola or a degenerate pair of lines. The identification with spacelike/timelike/lightlike dual vectors follows by identifying the sign of the induced quadratic form (36) with the Lorentzian causal character under the duality in, [1], [3]. □

Remark 3.10. *Under the mapping in [1], a one-parameter family of dual points on the dual hyperbolic unit sphere $\mathbb{H}_0^2$ (defined in* (1)*) corresponds to a one-parameter family of lines in $\mathbb{R}_1^3$, whose union is a ruled surface. The extremal family obtained from $E_+(Q)$ or $E_-(Q)$ (see* (35)*) under the duality in, [1], [3] therefore sweeps out a ruled surface in $\mathbb{R}_1^3$. When the quadratic form is positive on $E_+(Q)$ (spacelike), the ruled surface is foliated by spacelike lines; when negative on $E_-(Q)$ (timelike), by timelike lines. This provides the geometric interpretation of the spectral components in terms of ruled surfaces and connects the algebraic decomposition to classical line geometry in, [1].*

Example 3.11. *Consider the non-degenerate conic defined by the equation*

$$x^2 - 2xy - 3y^2 + 4x - 4y = 0. \tag{37}$$

In the standard form $Ax^2 + \beta xy + Cy^2 + Dx + Ey + F = 0$ (see (22)*), we have*

$$\begin{aligned} A = 1, \quad \beta = -2, \quad C = -3, \\ D = 4, \quad E = -4, \quad F = 0. \end{aligned} \tag{38}$$

The quadratic form matrix is (compare (34)*)*

$$Q = \begin{pmatrix} A & \beta/2 \\ \beta/2 & C \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix}. \quad (39)$$

Spectral decomposition: The characteristic polynomial of Q in (39) *is*

$$\det(Q - \mu I) = \mu^2 + 2\mu - 4 = 0, \quad (40)$$

so the eigenvalues are

$$\begin{aligned} \mu_1 &= -1 + \sqrt{5} \approx 1.236 > 0, \\ \mu_2 &= -1 - \sqrt{5} \approx -3.236 < 0. \end{aligned} \quad (41)$$

Since $\det Q = \mu_1\mu_2 = -4 < 0$ *(see* (41)*), the conic* (37) *is of hyperbolic type.*
Eigenvectors: For $\mu_1 = -1 + \sqrt{5}$ in (41)*, the equation $(Q - \mu_1 I)v = 0$ gives, from the first row,*

$$\begin{aligned} (1 - \mu_1)v_1 - v_2 = 0 \implies v_2 &= (1 - \mu_1)v_1 \\ &= (2 - \sqrt{5})v_1. \end{aligned} \quad (42)$$

Hence

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 - \sqrt{5} \end{pmatrix} \quad \textit{(positive eigenspace, spacelike)}. \quad (43)$$

Similarly, for $\mu_2 = -1 - \sqrt{5}$,

$$v_2 = (1 - \mu_2)v_1 = (2 + \sqrt{5})v_1, \quad (44)$$

so

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 + \sqrt{5} \end{pmatrix} \quad \textit{(negative eigenspace, timelike)}. \quad (45)$$

Decomposition: The spectral decomposition of $\mathbb{R}^2$ associated with Q is (compare (35)*)*

$$\mathbb{R}^2 = E_+(Q) \oplus E_-(Q), \quad (46)$$

with $E_+(Q) = \operatorname{span}\{\mathbf{v}_1\}$ and $E_-(Q) = \operatorname{span}\{\mathbf{v}_2\}$ (see (43) *and* (45)*). The null eigenspace $E_0(Q)$ is trivial because $\det Q \neq 0$. Under the mapping in, [1], $E_+(Q)$ corresponds to spacelike dual vectors and $E_-(Q)$ to timelike dual vectors.*
Ruled surface interpretation: The extremal congruence associated with $E_-(Q)$ sweeps out a timelike ruled surface in $\mathbb{R}^3_1$, while the congruence associated with $E_+(Q)$ sweeps out a spacelike ruled surface, in the sense of Remark 3.10.
Affine-invariant data: $\Delta = \det Q = -4 < 0$, $\operatorname{rank} Q = 2$, and the rank of the augmented matrix is 3. These invariants classify the conic (37) *as a hyperbola up to affine transformation. As shown in Corollary 3.12, the trace $A + C = -2$ is* not *an affine invariant.*

Corollary 3.12. *The sign and the vanishing/non-vanishing of $\Delta = \det Q = AC - \beta^2/4$ (where Q is as in* (34)*) are affine invariants. The quantity $\delta = A + C = \operatorname{tr} Q$ is* not *an affine invariant. The affine classification of conics is determined by the rank and signature of Q together with the rank of the augmented matrix.*

Proof. Under an affine change of coordinates $x = Mu + \mathbf{t}$ with $M \in GL(2, \mathbb{R})$, the quadratic form transforms as

$$\begin{aligned} Q(x) &= (Mu + \mathbf{t})^T Q (Mu + \mathbf{t}) \\ &= u^T (M^T Q M) u + (\text{linear terms}) + \text{const.} \end{aligned} \quad (47)$$

Hence the quadratic form matrix transforms by congruence as shown in (47):

$$Q' = M^T Q M. \quad (48)$$

Therefore

$$\det Q' = (\det M)^2 \det Q, \quad (49)$$

so the sign and the vanishing/non-vanishing of $\det Q$ are affine invariant. The rank of Q is also preserved under congruence by an invertible matrix. The trace, however, transforms as

$$\operatorname{tr} Q' = \operatorname{tr}(M^T Q M), \quad (50)$$

as given in (50), which is not equal to $\operatorname{tr} Q$ in general. For example, take $Q = \operatorname{diag}(1, 1)$ and $M = \operatorname{diag}(t, 1)$ with $t \neq 1$. Then $Q' = \operatorname{diag}(t^2, 1)$ and

$$\operatorname{tr} Q' = t^2 + 1 \neq 2 = \operatorname{tr} Q. \quad (51)$$

Thus $\delta = A + C$ is not an affine invariant. □

3.1 Applications to Geometric Optics and Robotics

We now give two explicit illustrations of the algebraic framework: a lossless two-lens optical system and a planar two-revolute-joint (2R) robot arm. Both use the same hyper G-matrix decomposition $\mathcal{H}_{2\times 2} = D_1 B D_2$ from Lemma 3.1 (see (23)) and the Fan product inverse (Theorem 3.6, equation (31)).

Optical system. Consider a paraxial optical system consisting of two thin lenses of focal lengths f_1, f_2 separated by a distance d. The ray-transfer (ABCD) matrix is

$$M = \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix}. \quad (52)$$

For a lossless system, $\det M = 1$, so $M \in SL(2, \mathbb{R})$. Direct multiplication of (52) gives

$$M = \begin{pmatrix} 1 - \dfrac{d}{f_1} & d \\ -\dfrac{1}{f_2} + \dfrac{d}{f_1 f_2} - \dfrac{1}{f_1} & 1 - \dfrac{d}{f_2} \end{pmatrix}, \quad (53)$$

as stated in (53), so its trace is

$$\operatorname{tr} M = 2 - d\left(\frac{1}{f_1} + \frac{1}{f_2}\right) = 2 - \frac{d(f_1+f_2)}{f_1 f_2}. \quad (54)$$

The classical classification of periodic optical systems is based on this trace (54): the system is *elliptic* if $|\operatorname{tr} M| < 2$, *parabolic* if $|\operatorname{tr} M| = 2$, and *hyperbolic* if $|\operatorname{tr} M| > 2$. This classification coincides with the classification of the corresponding $SL(2,\mathbb{R})$-conjugacy class: the elliptic case corresponds to rotations (conjugate to a rotation matrix, with a fixed point inside the hyperbolic plane $\mathbb{H}^2$), the parabolic case to unipotent elements (conjugate to a Jordan block, with a fixed point on the ideal boundary $\partial\mathbb{H}^2$), and the hyperbolic case to elements with two distinct real eigenvalues (with two fixed points on $\partial\mathbb{H}^2$). Under the duality in, [1], [3], these three cases correspond to spacelike, lightlike, and timelike directions on the dual hyperbolic unit sphere $\mathbb{H}_0^2$ (see (1)).

As a concrete instance, take $f_1 = f_2 = f$ and $d = 2f$ (a 4f system). Substituting into the general formula (54) gives

$$\operatorname{tr} M = 2 - \frac{2f(2f)}{f^2} = 2 - 4 = -2, \quad (55)$$

which agrees with the direct computation

$$\begin{aligned} M &= \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & 2f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2f \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (56)$$

Thus $M \in SL(2,\mathbb{R})$, $\operatorname{tr} M = -2$ (see (55)), and M is parabolic. The matrix M in (56) is conjugate in $SL(2,\mathbb{R})$ to the Jordan block $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ (up to sign), whose logarithm has a nilpotent part; this is a representative of the lightlike/unipotent class. This confirms the physical interpretation of the parabolic case as the boundary between stable (elliptic) and unstable (hyperbolic) optical systems, and shows that the Lie-algebraic classification carries direct optical meaning.

2R planar robot arm. Consider a planar arm with two revolute joints, link lengths ℓ_1, ℓ_2, and joint angles θ_1, θ_2. The end-effector position is

$$p(\theta_1,\theta_2) = \big(\ell_1\cos\theta_1 + \ell_2\cos(\theta_1+\theta_2),\ \ell_1\sin\theta_1 + \ell_2\sin(\theta_1+\theta_2)\big). \quad (57)$$

The Jacobian is

$$J(\theta_1,\theta_2) = \begin{pmatrix} -\ell_1\sin\theta_1 - \ell_2\sin(\theta_1+\theta_2) & -\ell_2\sin(\theta_1+\theta_2) \\ \ell_1\cos\theta_1 + \ell_2\cos(\theta_1+\theta_2) & \ell_2\cos(\theta_1+\theta_2) \end{pmatrix}. \quad (58)$$

and the arm is singular exactly when $\det J = 0$, i.e., $\ell_1\ell_2\sin\theta_2 = 0$, so $\theta_2 \in \{0,\pi\}$.

In the dual framework, we associate to each joint the dual rotation matrix

$$\hat{R}_i = \begin{pmatrix} \cos\hat{\theta}_i & -\sin\hat{\theta}_i \\ \sin\hat{\theta}_i & \cos\hat{\theta}_i \end{pmatrix}, \qquad \hat{\theta}_i = \theta_i + \varepsilon d_i, \quad (59)$$

as in (59), where d_i denotes the infinitesimal displacement of joint i. Since $R_i \in SO(2) \subset SL(2,\mathbb{R})$, its real part satisfies the hyper G-matrix condition (4) with $D_1 = D_2 = I$ and $B = R_i^{-T} = R_i \in SL(2,\mathbb{R})$. The dual part is naturally interpreted via the mapping in, [1], and $\hat{R}_i$ is the dual analogue of a hyper G-matrix.

The Fan product of two dual rotation matrices is (see (28))

$$\hat{R}_1 \star \hat{R}_2 = (R_1 \star R_2) + \varepsilon(R_1^* \star R_2 + R_1 \star R_2^*), \quad (60)$$

as given in (60), and by Theorem 3.6 (equation (31)) the inverse of this Fan product factorizes as $\hat{R}_1^{-1} \star \hat{R}_2^{-1}$. The real part of the Fan product is

$$R_1 \star R_2 = \begin{pmatrix} \cos\theta_1\cos\theta_2 & -\sin\theta_1\sin\theta_2 \\ -\sin\theta_1\sin\theta_2 & \cos\theta_1\cos\theta_2 \end{pmatrix}, \quad (61)$$

whose determinant is

$$\begin{aligned} \det(R_1 \star R_2) &= \cos^2\theta_1\cos^2\theta_2 - \sin^2\theta_1\sin^2\theta_2 \\ &= \cos(\theta_1+\theta_2)\cos(\theta_1-\theta_2). \end{aligned} \quad (62)$$

This determinant (62) vanishes precisely when $\theta_1 + \theta_2 \in \frac{\pi}{2}+\pi\mathbb{Z}$ or $\theta_1 - \theta_2 \in \frac{\pi}{2}+\pi\mathbb{Z}$. This is a degeneracy of the Fan product of the two dual rotations and represents a novel dual singularity distinct from the classical Jacobian singularity $\sin\theta_2 = 0$ (see (58)). Note that the rotation matrices are *not* F-orthogonal in the sense of Definition 3.7 (equation (32)): for $R \in SO(2)$, $R \star R^T = \begin{pmatrix} \cos^2\theta & \sin^2\theta \\ \sin^2\theta & \cos^2\theta \end{pmatrix} \neq I_F$, and in fact no real rotation matrix can be F-orthogonal because the F-identity (30) requires off-diagonal entries -1, which cannot be achieved by non-negative products of sines. The Fan product therefore provides a genuinely new invariant of the dual rotation family, complementary to the classical Jacobian determinant (58), and the configurations where it degenerates are dual singularities rather than classical ones.

These two examples show that the algebraic structure introduced in Sections 2–3 is not merely formal. In optics, the $SL(2,\mathbb{R})$-conjugacy classification of lossless ABCD matrices (53) reproduces the standard elliptic/parabolic/hyperbolic classification of periodic optical systems via the trace formula (54). In robotics, the Fan product of

dual rotations detects a dual degeneracy (62) that is complementary to, and distinct from, the classical Jacobian singularity (58). These findings confirm that the algebraic framework encodes genuine quantitative information in both contexts.

4 Conclusion

The Astrolabe-HyperG algebra $\mathfrak{ahg}(2,1)$ provides an algebraic unification of stereographic projection, constrained optimization, and Lorentzian geometry. The hyper G-matrix decomposition $A^{-T} = D_1 B D_2$ in (4) is fundamentally related to the structure of $SL(2,\mathbb{R})$, revealing that the astrolabe's physical components correspond to algebraic structures as shown in Table 1. The full algebra $\mathfrak{ahg}(2,1)$ is four-dimensional and is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$ (see (9)); its semisimple part is $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,1)$. The Hessian-HyperG lemma (Lemma 3.1) establishes a direct link between constrained optimization and this algebraic framework, showing that the bordered Hessian matrix $\mathcal{H}_{2\times 2}$ in (23) from Lagrange multiplier problems naturally admits the decomposition $\mathcal{H}_{2\times 2} = D_1 B D_2$ with B normalizable to $SL(2,\mathbb{R})$ up to a fixed reflection, at least in the definite regime discussed in Remark 3.2.

We have extended the theory by incorporating the Fan product of dual matrices following in, [6]. This extension introduces the F-identity matrix I_F (Definition 3.5, equation (30)) and the notions of F-orthogonal (Definition 3.7, equation (32)) and F-nilpotent (Definition 3.8, equation (33)) matrices, providing a richer algebraic structure for analyzing dual hyper G-matrices. The Fan product inverse theorem (Theorem 3.6) establishes that $(\hat{P} \star \hat{Q})^{-1} = \hat{P}^{-1} \star \hat{Q}^{-1}$ in (31) for regular dual matrices under the Fan product, a property that respects the hyper G-matrix decomposition (4).

The spectral decomposition of conics (Proposition 3.9) shows that the eigenspaces of the quadratic form Q in (34) decompose as $\mathbb{R}^2 = E_+(Q) \oplus E_-(Q) \oplus E_0(Q)$ (see (35)); the conic is classified as elliptic, hyperbolic, or parabolic according to the signs of the eigenvalues, with the trace distinguishing real ellipses from the empty set within the elliptic case. Under the duality in, [1], [3], these components correspond to spacelike, timelike, and lightlike dual vectors, respectively, as illustrated in Example 3.11 (see equations (37)–(46)), and the associated extremal congruences sweep out ruled surfaces in $\mathbb{R}^3_1$ (Remark 3.10). The affine invariants are the sign and rank of $\Delta = \det Q$ (see (49)); the trace $\delta = A + C$ is not affine invariant (Corollary 3.12, equation (51)).

Subsection 3.1 has provided two explicit illustrations: a lossless two-lens optical system (equations (52)–(56)), where the $SL(2,\mathbb{R})$-conjugacy classification of ABCD matrices coincides with the standard elliptic/parabolic/hyperbolic classification of periodic optical systems, and a planar 2R robot arm (equations (57)–(62)), where the Fan product of dual rotations detects a dual degeneracy complementary to the classical Jacobian singularity. These examples confirm that the algebraic structure is quantitative, not merely nominal.

We conclude with three conjectures, each of which opens a concrete research direction.

Conjecture 4.1. *There exists an equivalence of categories*

$$\Phi : \mathsf{ConicExt} \longrightarrow \mathsf{DualMot}, \tag{63}$$

where ConicExt *is the category whose objects are triples $(Q, \mathcal{L}, c)$ consisting of a quadratic form Q as in* (34)*, a Lagrangian $\mathcal{L}$ of the form studied in Lemma 3.1 (equation* (21)*), and a fixed Lagrange multiplier c, and whose morphisms are affine changes of coordinates respecting the Lagrangian; and* DualMot *is the category whose objects are one-parameter dual motions on $\mathbf{\Delta}^3_1$ generated by the exponential of $\mathfrak{ahg}(2,1)$, and whose morphisms are equivariant dual maps. Conjecturally, the functor Φ in* (63) *sends a conic extremal problem to the dual motion whose ruled surface is swept by the extremal family under the duality in, [1], [3]. The construction is expected to be essentially surjective at the level of objects under suitable non-degeneracy assumptions, and full at the level of morphisms when the affine group is restricted to the subgroup preserving the Lagrangian.*

Conjecture 4.2. *Let $\mathcal{E}$ be the moduli space of extremal congruences of a fixed non-degenerate conic, assumed to be a smooth even-dimensional manifold. We conjecture that $\mathcal{E}$ carries a natural symplectic form ω defined by*

$$\omega(X,Y) = \langle X, JY \rangle_{\mathbf{\Delta}^3_1}, \qquad X, Y \in T\mathcal{E}, \tag{64}$$

where $\langle \cdot,\cdot \rangle_{\mathbf{\Delta}^3_1}$ is the dual Lorentzian metric and J is the almost complex structure induced by the identification of the tangent space of $\mathcal{E}$ with a subspace of $\mathbf{\Delta}^3_1$ through the duality in, [1], [3]. The closedness of ω in (64) *would follow if J is integrable; the non-degeneracy of ω would follow if $T\mathcal{E}$ has even rank and the metric pairing is non-degenerate on $T\mathcal{E}$. The Fan product is conjectured to be compatible with ω in the sense that the Fan product exponential map is a local symplectomorphism.*

Conjecture 4.3. *For every integer $n \geq 1$ there exists a reductive Lie algebra $\mathfrak{ahg}(n,1)$ of dimension*

$$\dim \mathfrak{ahg}(n,1) = \frac{n(n+1)}{2} + 1, \tag{65}$$

with a faithful representation into $\mathfrak{sl}(n+1,\mathbb{R}) \oplus \mathbb{R}$*, whose derived subalgebra is isomorphic to* $\mathfrak{so}(n,1)$*, and whose hyper G-matrix decomposition generalizes* (4) *to* $(n+1) \times (n+1)$ *matrices. The semisimple part of* $\mathfrak{ahg}(n,1)$ *would then be the Lorentz algebra* $\mathfrak{so}(n,1)$*, and the full algebra would be the direct sum* $\mathfrak{so}(n,1) \oplus \mathbb{R}$*. For* $n = 2$ *the formula* (65) *gives* $\dim \mathfrak{ahg}(2,1) = 3 + 1 = 4$*, in agreement with Definition 2.2 (see* (11)*).*

The Astrolabe-HyperG Principle:

Every stereographic projection-based measurement system can be represented as a dual matrix transformation $A^{-T} = D_1 B D_2$ in (4), where D_1, D_2 are diagonal and B is normalizable to $SL(2,\mathbb{R})$ up to a fixed reflection. The semisimple part of this structure is $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,1)$; the full algebra is the direct sum $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$ (see (9)). The Fan product introduced in [6] provides a natural extension of this algebraic structure to dual matrices, with applications in kinematics, optics, and computational geometry.

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