Measurement, Input, and Previous Estimation Contribution Metrics in Kalman Filter Estimation

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Abstract: - The Kalman filter is related to linear systems and produces iteratively the state estimation taking into account the current measurement, the previous input, and the previous estimation. Percentage metrics are proposed for the participation of the measurement, the previous input and the previous estimation in Kalman Filter estimation. The proposed metrics depend on the known measurements and inputs and the known system parameters. These metrics are introduced for time varying, time invariant and steady state Kalman filter.

Key-Words: - Kalman filter, Estimation, Prediction, Metrics, State Space Systems, Measurement.

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1 Introduction

The Kalman Filter [1], [2] plays a fundamental role in estimation theory, since it has been widely used in various applications: object detection and tracking [3], robotic applications [4], electric load estimation [5], weather forecasts [6], autonomous orbit determination [7], satellite orbit determination [8], power generation prediction [9], cases prediction of Covid-19 [10], control effectiveness estimation on airplanes [11], applications with time-correlated measurement errors [12], aircraft state estimation [13], GPS position estimation [14], multi-target localization [15], stock price prediction [16].

The discrete time Kalman Filter is associated with discrete time state space systems of the form:

$$x(k+1) = F(k+1) \cdot x(k) + B(k) \cdot u(k) + w(k) \quad (1)$$

$$z(k) = H(k) \cdot x(k) + v(k) \quad (2)$$

where $k \geq 0$ is the discrete time, $x(k)$ is the n dimensional state, $z(k)$ is the m dimensional measurement, $u(k)$ is the ℓ dimensional input, $F(k)$ is the transition matrix, $B(k)$ is the input matrix, $H(k)$ is the output matrix, $w(k)$ and $v(k)$ are the state noise and the measurement noise, respectively. It is known [2] that $\{w(k)\}$ and $\{v(k)\}$ are individually zero mean, Gaussian process with covariances $Q(k)$ and $R(k)$, respectively. The initial state $x(0)$ is a Gaussian random variable with mean x_0 and covariance P_0.

Kalman Filter produces iteratively the state estimation $x(k/k)$ and the estimation error covariance $P(k/k)$ as well as the state prediction $x(k+1/k)$ and the prediction error covariance $P(k+1/k)$ using the measurements $z(k)$ till time k and the known Kalman Filter parameters: $F(k), B(k), H(k), Q(k), R(k)$.

The time varying Kalman filter is summarized in the following conventional form:

time varying Kalman filter (conventional form)

$$K(k) = P(k/k-1) \cdot H^T(k) \cdot [H(k) \cdot P(k/k-1) \cdot H^T(k) + R(k)]^{-1}$$
$$x(k/k) = [I - K(k) \cdot H(k)] \cdot x(k/k-1) + K(k) \cdot z(k)$$
$$P(k/k) = [I - K(k) \cdot H(k)] \cdot P(k/k-1)$$
$$x(k+1/k) = F(k+1) \cdot x(k/k) + B(k) \cdot u(k)$$
$$P(k+1/k) = Q(k) + F(k+1) \cdot P(k/k) \cdot F^T(k+1)$$

for $k = 0,1,\dots$

with initial conditions

$$x(0/-1) = x_0, P(0/-1) = P_0$$

$K(k)$is the Kalman filter gain. I denotes the identity matrix. M^T denotes the transpose of matrix M. Note that the existence of the inverse of the matrices required to compute the Kalman filter gain, is ensured assuming that every covariance matrix $R(k)$ is positive definite; this has the significance that no measurement is exact.

The **time invariant Kalman filter** is derived for time invariant systems where all the Kalman filter parameters are constant: $F(k) = F, B(k) = B, H(k) = H, Q(k) = Q, R(k) = R$:

time invariant Kalman filter (conventional form)

$$K(k) = P(k/k-1) \cdot H^T \cdot [H \cdot P(k/k-1) \cdot H^T + R]^{-1}$$
$$x(k/k) = [I - K(k) \cdot H] \cdot x(k/k-1) + K(k) \cdot z(k)$$
$$P(k/k) = [I - K(k) \cdot H] \cdot P(k/k-1)$$
$$x(k+1/k) = F \cdot x(k/k) + B \cdot u(k)$$
$$P(k+1/k) = Q + F \cdot P(k/k) \cdot F^T$$
for $k = 0,1, \ldots$

with initial conditions
$$x(0/-1) = x_0, P(0/-1) = P_0$$

We rewrite the Kalman filter equations in order to underline the fact that **the state estimation depends on the current measurement, the previous state estimation and the previous input**:

time varying Kalman filter (estimation form)

$$x(k/k) = K(k) \cdot z(k) + A(k) \cdot x(k-1/k-1) + C(k-1) \cdot u(k-1)$$
$$P(k/k) = [I - K(k) \cdot H(k)] \cdot [Q(k-1) + F(k) \cdot P(k-1/k-1) \cdot F^T(k)]$$
for $k = 1,2, \ldots$

where
$$K(k) = [Q(k-1) + F(k) \cdot P(k-1/k-1) \cdot F^T(k)] \cdot H^T(k) \cdot [H(k) \cdot [Q(k-1) + F(k) \cdot P(k-1/k-1) \cdot F^T(k)] \cdot H^T(k) + R(k)]^{-1}$$
$$A(k) = [I - K(k) \cdot H(k)] \cdot F(k)$$
$$C(k-1) = [I - K(k) \cdot H(k)] \cdot B(k-1)$$

with initial conditions
$$x(0/0) = [I - K(0) \cdot H(0)] \cdot x_0 + K(0) \cdot z(0)$$
$$P(0/0) = [I - K(0) \cdot H(0)] \cdot P_0$$

where
$$K(0) = P_0 \cdot H^T(0) \cdot [H(0) \cdot P_0 \cdot H^T(0) + R(0)]^{-1}$$

It is worth to note that the Kalman filter coefficients $K(k), A(k), C(k-1)$ can be computed off-line, i.e. before the implementation of the filter. This is feasible due to the fact that the estimation error covariance can be computed iteratively off-line.

The **time invariant Kalman filter** is derived for time invariant systems where all the Kalman filter parameters are constant: $F(k) = F, B(k) = B, H(k) = H, Q(k) = Q, R(k) = R$:

time invariant Kalman filter (estimation form)

$$x(k/k) = K(k) \cdot z(k) + A(k) \cdot x(k-1/k-1) + C(k-1) \cdot u(k-1)$$
$$P(k/k) = [I - K(k) \cdot H] \cdot [Q + F \cdot P(k-1/k-1) \cdot F^T]$$
for $k = 1,2, \ldots$

where
$$K(k) = [Q + F \cdot P(k-1/k-1) \cdot F^T] \cdot H^T \cdot [H \cdot [Q + F \cdot P(k-1/k-1) \cdot F^T] \cdot H^T + R]^{-1}$$
$$A(k) = [I - K(k) \cdot H] \cdot F$$
$$C(k-1) = [I - K(k) \cdot H] \cdot B$$

with initial conditions
$$x(0/0) = [I - K(0) \cdot H] \cdot x_0 + K(0) \cdot z(0)$$
$$P(0/0) = [I - K(0) \cdot H] \cdot P_0$$

where
$$K(0) = P_0 \cdot H^T \cdot [H \cdot P_0 \cdot H^T + R]^{-1}$$

For time invariant systems, it is well known [2] that if the signal process model is asymptotically stable, then there exists a unique steady state value P_p of the prediction error covariance matrix. In this case, the resulting steady state Kalman filter takes the following form:

steady state Kalman filter

$$x(k/k) = K \cdot z(k) + A \cdot x(k-1/k-1) + C \cdot u(k-1)$$
for $k = 1,2, \ldots$

where
$$A = [I - K \cdot H] \cdot F$$
$$C = [I - K \cdot H] \cdot B$$

with initial condition
$$x(0/0) = [I - K \cdot H] \cdot x_0 + K \cdot z(0)$$

Note that the steady state Kalman filter coefficients K, A, C are computed off-line, by first solving the corresponding discrete time **Riccati equation** [2]:

$$P_p = Q + F \cdot P_p \cdot F^T - F \cdot P_p \cdot H^T \cdot \left[H \cdot P_p \cdot H^T + R\right]^{-1} \cdot H \cdot P_p \cdot F^T \quad (3)$$

and then calculating the **steady state gain** K:

$$K = P_p \cdot H^T \cdot \left[H \cdot P_p \cdot H^T + R\right]^{-1} \quad (4)$$

This paper investigates the participation of the current measurement, the previous state estimation and the previous input in the state estimation computation. The novelty of the paper concerns a) the off-line computation of the participation of the current measurement, the previous state estimation and the previous input in the state estimation computation, b) the introduction of metrics for these participations.

2 Ratios Definition

Define the ratios concerning the participation of the current measurement, the previous state estimation and the previous input in the state estimation computation:

$$\mathrm{K(k)\cdot z(k) = \Lambda(k)\cdot x\left(\frac{k}{k}\right)} \quad (5)$$

$$\mathrm{A(k)\cdot x(k-1/k-1) = M(k)\cdot x(k/k)} \quad (6)$$

$$\mathrm{C(k-1)\cdot u(k-1) = N(k)\cdot x(k/k)} \quad (7)$$

The ratios $\Lambda(k), M(k), N(k)$ are diagonal matrices with

$$\Lambda(k) + M(k) + N(k) = I \quad (8)$$

In addition, define the auxiliary ratios:

$$z(k+1) = \xi(k)\cdot z(k) \quad (9)$$

$$u(k+1) = \zeta(k)\cdot u(k) \quad (10)$$

The ratios $\xi(k), \zeta(k)$ are diagonal matrices.
Furtheremore, define the auxiliary ratios:

$$z(k) = \gamma(k)\cdot u(k) \text{when } n \geq m \quad (11)$$

$$u(k) = \delta(k)\cdot z(k) \text{when } m \geq n \quad (12)$$

Note that
$\gamma(k) = [\Gamma(k) \quad O(k)]$, $\Gamma(k)$ $m\times m$ is diagonal matrix and $O(k)$ $m\times(n-m)$ is zero matrix and $\delta(k) = [\Delta(k) \quad O(k)]$, $\Delta(k)$ $n\times n$ is diagonal matrix and $O(k)$ $n\times(m-n)$ is zero matrix.

3 Ratios Off-Line Computation

In this section we compute off-line the ratios concerning the participation of the current measurement, the previous input and the previous state estimation in the state estimation computation.

3.1 Time Varying and Time Invariant Kalman Filter

For the general **time varying** case we present the following derivations:

1 ratio Λk
a. n ≥ m

$$\begin{aligned}
&K(k+1)\cdot z(k+1) = \Lambda(k+1)\cdot x(k+1/k+1)\\
&\Rightarrow \Lambda^{-1}(k+1)\cdot K(k+1)\cdot z(k+1)\\
&\qquad = K(k+1)\cdot z(k+1) + A(k+1)\\
&\qquad \cdot x(k/k) + C(k)\cdot u(k)\\
&\Rightarrow \Lambda^{-1}(k+1)\cdot K(k+1)\cdot \xi(k)\cdot z(k)\\
&\qquad = K(k+1)\cdot \xi(k)\cdot z(k)\\
&\qquad +A(k+1)\cdot x(k/k) + C(k)\cdot u(k)\\
&\Rightarrow \Lambda^{-1}(k+1)\cdot K(k+1)\cdot \xi(k)\cdot \gamma(k)\cdot u(k)\\
&\qquad = K(k+1)\cdot \xi(k)\cdot z(k)\\
&\qquad +A(k+1)\cdot x(k/k) + C(k)\cdot u(k)\\
&= K(k+1)\cdot \xi(k)\cdot z(k) + A(k+1)\cdot \Lambda^{-1}(k)\cdot K(k)\\
&\qquad \cdot z(k) + C(k)\cdot u(k)\\
&= K(k+1)\cdot \xi(k)\cdot \gamma(k)\cdot u(k) + A(k+1)\cdot \Lambda^{-1}(k)\\
&\qquad \cdot K(k)\cdot \gamma(k)\cdot u(k) + C(k)\cdot u(k)\\
&\Rightarrow \Lambda^{-1}(k+1)\cdot K(k+1)\cdot \xi(k)\cdot \gamma(k)\\
&\qquad = K(k+1)\cdot \xi(k)\cdot \gamma(k)\\
&\qquad +A(k+1)\cdot \Lambda^{-1}(k)\cdot K(k)\cdot \gamma(k)\\
&\qquad +C(k)
\end{aligned}$$

Then

$$\begin{aligned}
\Lambda^{-1}(k+1) = I &+ A(k+1)\cdot \Lambda^{-1}(k)\cdot K(k)\\
&\cdot \gamma(k)\big(K(k+1)\cdot \xi(k)\cdot \gamma(k)\big)^{-1}\\
&+C(k)\cdot \big(K(k+1)\cdot \xi(k)\cdot \gamma(k)\big)^{-1}
\end{aligned}$$

b. m ≥ n

$$\begin{aligned}
&K(k+1)\cdot z(k+1) = \Lambda(k+1)\cdot x(k+1/k+1)\\
&\Rightarrow \Lambda^{-1}(k+1)\cdot K(k+1)\cdot z(k+1)\\
&\qquad = K(k+1)\cdot z(k+1) + A(k+1)\\
&\qquad \cdot x(k/k) + C(k)\cdot u(k)\\
&= K(k+1)\cdot z(k+1) + A(k+1)\cdot \Lambda^{-1}(k)\cdot K\cdot z(k)\\
&\qquad + C(k)\cdot u(k)\\
&= K\cdot z(k+1) + A(k+1)\cdot \Lambda^{-1}(k)\cdot K\cdot \xi^{-1}(k)\\
&\qquad \cdot z(k+1) + C(k)\cdot \delta(k)\cdot z(k)\\
&\Rightarrow \Lambda^{-1}(k+1)\cdot K(k+1)\\
&\qquad = K(k+1) + A(k+1)\cdot \Lambda^{-1}(k)\cdot K\\
&\qquad \cdot \xi^{-1}(k) + C(k)\cdot \delta(k)\cdot \xi^{-1}(k)\\
&\qquad \cdot z(k+1)\\
&\Rightarrow \Lambda^{-1}(k+1)\cdot K(k+1)\cdot H(k+1)\\
&\qquad = K(k+1)\cdot H(k+1) + A(k+1)\\
&\qquad \cdot \Lambda^{-1}(k)\cdot K\cdot \xi^{-1}(k)\cdot H(k+1)\\
&\qquad +C(k)\cdot \delta(k)\cdot \xi^{-1}(k)\cdot H(k+1)
\end{aligned}$$

Thus

$$\begin{aligned}
\Lambda^{-1}(k+1) = I &+ A(k+1)\cdot \Lambda^{-1}(k)\cdot K(k+1)\\
&\cdot \xi^{-1}(k)\cdot H(k+1)\\
&\cdot \big(K(k+1)\cdot H(k+1)\big)^{-1}\\
&+C(k)\cdot \delta(k)\cdot \xi^{-1}(k)\cdot H(k+1)\\
&\cdot (K(k+1)\cdot H(k+1))^{-1}
\end{aligned}$$

2 ratio Nk
a. n ≥ m

$$\begin{aligned}
&C(k+1)\cdot u(k+1) = N(k+1)\cdot x(k+1/k+1)\\
&\Rightarrow N^{-1}(k+1)\cdot C(k+1)\cdot u(k+1)\\
&\qquad = x(k+1/k+1)\\
&= K\cdot z(k+1) + A(k+1)\cdot x(k/k) + C(k)\cdot u(k)\\
&= K\cdot \gamma(k+1)\cdot u(k+1) + A(k+1)\cdot N^{-1}(k)\\
&\qquad \cdot C(k)\cdot u(k) + C(k)\cdot u(k)\\
&= K\cdot \gamma(k+1)\cdot u(k+1) + A(k+1)\cdot N^{-1}(k)\\
&\qquad \cdot C(k)\cdot \zeta^{-1}(k)\cdot u(k+1)\\
&+C(k)\cdot \zeta^{-1}(k)\cdot u(k+1)
\end{aligned}$$

$$\Rightarrow N^{-1}(k+1)\cdot C(k+1) = K(k+1)\cdot\gamma(k+1) + A(k+1)\cdot N^{-1}(k)\cdot C(k)\cdot\zeta^{-1}(k) + C(k)\cdot\zeta^{-1}(k)$$

Thus

$$N^{-1}(k+1) = K(k+1)\cdot\gamma(k+1)\cdot C^{-1}(k+1) + A(k+1)\cdot N^{-1}(k)\cdot C(k)\cdot\zeta^{-1}(k)\cdot C^{-1}(k+1)$$
$$C(k)\cdot\zeta^{-1}(k)\cdot C^{-1}(k+1)$$

b. m ≥ n

$$C(k+1)\cdot u(k+1) = N(k+1)\cdot x(k+1/k+1)$$
$$\Rightarrow N^{-1}(k+1)\cdot C(k+1)\cdot u(k+1) = x(k+1/k+1)$$
$$\Rightarrow N^{-1}(k+1)\cdot C(k+1)\cdot\delta(k+1)\cdot z(k+1) = x(k+1/k+1)$$
$$= K(k+1)\cdot z(k+1) + A(k+1)\cdot x(k/k) + C(k)\cdot u(k)$$
$$= K(k+1)\cdot z(k+1) + A(k+1)\cdot N^{-1}(k)\cdot C(k)\cdot u(k) + C(k)\cdot u(k)$$
$$= K(k+1)\cdot z(k+1) + A(k+1)\cdot N^{-1}(k)\cdot C(k)\cdot\zeta^{-1}(k)\cdot u(k+1)$$
$$+C(k)\cdot\zeta^{-1}(k)\cdot u(k+1)$$
$$= K(k+1)\cdot z(k+1) + A(k+1)\cdot N^{-1}(k)\cdot C(k)\cdot\zeta^{-1}(k)\cdot\delta(k+1)\cdot z(k+1)$$
$$+C(k)\cdot\zeta^{-1}(k)\cdot\delta(k+1)\cdot z(k+1)$$
$$\Rightarrow N^{-1}(k+1)\cdot C(k+1)\cdot\delta(k+1)$$
$$= K(k+1) + A(k+1)\cdot N^{-1}(k)\cdot C(k)\cdot\zeta^{-1}(k)\cdot\delta(k+1) + C(k)\cdot\zeta^{-1}(k)\cdot\delta(k+1)$$
$$\Rightarrow N^{-1}(k+1)\cdot C(k+1)\cdot\delta(k+1)\cdot H(k+1)$$
$$= K(k+1)\cdot H(k+1) + A(k+1)\cdot N^{-1}(k)\cdot C(k)\cdot\zeta^{-1}(k)\cdot\delta(k+1)\cdot H(k+1)$$
$$+C(k)\cdot\zeta^{-1}(k)\cdot\delta(k+1)\cdot H(k+1)$$

Thus

$$N^{-1}(k+1) = K(k+1)\cdot H(k+1)\cdot(C(k+1)\cdot\delta(k+1)\cdot H(k+1))^{-1}$$
$$+A(k+1)\cdot N^{-1}(k)\cdot C(k)\cdot\zeta^{-1}(k)\cdot\delta(k+1)\cdot H(k+1)\cdot\left(C(k+1)\cdot\delta(k+1)\cdot H(k+1)\right)^{-1}$$
$$+C(k)\cdot\zeta^{-1}(k)\cdot\delta(k+1)\cdot H(k+1)\cdot\left(C(k+1)\cdot\delta(k+1)\cdot H(k+1)\right)^{-1}$$

3 ratio Mk

a. n ≥ m

$$A(k)\cdot x(k/k) = M(k+1)\cdot x(k+1/k+1)$$
$$\Rightarrow M^{-1}(k+1)\cdot A(k)\cdot x(k/k) = x(k+1/k+1)$$
$$\Rightarrow M^{-1}(k+1)\cdot A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot z(k) = x(k+1/k+1)$$
$$\Rightarrow M^{-1}(k+1)\cdot A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot\gamma(k)\cdot u(k) = K(k+1)\cdot z(k+1) + A(k)\cdot x(k/k) + C(k)\cdot u(k)$$
$$= K(k+1)\cdot z(k+1) + A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot z(k) + C(k)\cdot u(k)$$
$$= K(k+1)\cdot\gamma(k+1)\cdot u(k+1) + A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot\gamma(k)\cdot u(k) + C(k)\cdot u(k)$$
$$= K(k+1)\cdot\gamma(k+1)\cdot\zeta(k)\cdot u(k) + A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot\gamma(k)\cdot u(k) + C(k)\cdot u(k)$$
$$\Rightarrow M^{-1}(k+1)\cdot A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot\gamma(k) = K(k+1)\cdot\gamma(k+1)\cdot\zeta(k) + A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot\gamma(k) + C(k)$$

Thus

$$M^{-1}(k+1) = K(k+1)\cdot\gamma(k+1)\cdot\zeta(k)\cdot\left(A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot\gamma(k)\right)^{-1} + I$$
$$+C(k)\cdot\left(A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot\gamma(k)\right)^{-1}$$

b. m ≥ n

$$A(k)\cdot x(k/k) = M(k+1)\cdot x(k+1/k+1)$$
$$\Rightarrow M^{-1}(k+1)\cdot A(k)\cdot x(k/k) = x(k+1/k+1)$$
$$\Rightarrow M^{-1}(k+1)\cdot A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot z(k) = x(k+1/k+1)$$
$$\Rightarrow M^{-1}(k+1)\cdot A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot\xi^{-1}(k)\cdot z(k+1) = K(k+1)\cdot z(k+1) + A(k)\cdot x(k/k) + C(k)\cdot u(k)$$
$$= K(k+1)\cdot z(k+1) + A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot z(k) + C(k)\cdot u(k)$$
$$= K(k+1)\cdot z(k+1) + A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot\xi^{-1}(k)\cdot z(k+1)$$
$$+C(k)\cdot\delta(k)\cdot z(k)$$
$$= K(k+1)\cdot z(k+1) + A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot\xi^{-1}(k)\cdot z(k+1)$$
$$+C(k)\cdot\delta(k)\cdot\xi^{-1}(k)\cdot z(k+1)$$
$$\Rightarrow M^{-1}(k+1)\cdot A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot\xi^{-1}(k)$$
$$= K(k+1) + A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot\xi^{-1}(k) + C(k)\cdot\delta(k)\cdot\xi^{-1}(k)$$
$$\Rightarrow M^{-1}(k+1)\cdot A(k)\cdot\Lambda^{-1}(k)\cdot K(k) = K(k+1)\cdot\xi(k) + A(k)\cdot\Lambda^{-1}(k)\cdot K(k) + C(k)\cdot\delta(k)$$
$$\Rightarrow M^{-1}(k+1)\cdot A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot H(k)$$
$$= K(k+1)\cdot\xi(k)\cdot H(k) + A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot H(k) + C(k)\cdot\delta(k)\cdot H(k)$$

Thus

$$M^{-1}(k+1) = K(k+1)\cdot\xi(k)\cdot H(k)\cdot(A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot H(k))^{-1} + I$$
$$+C(k)\cdot\delta(k)\cdot H(k)\cdot(A(k)\cdot\Lambda^{-1}(k)\cdot K(k)\cdot H(k))^{-1}$$

For the **time invariant** case, the above derivations hold with constant Kalman filter parameters.

It is clear that in the time varying as well as in the time invariant cases, the ratios $\Lambda(k), M(k), N(k)$ depend on the measurements, the inputs and the known system parameters. Thus, it is evident that we are able to know the contribution of the measurement, the input and the previous estimation in the estimation computation.

3.2 Steady State Kalman Filter

For the steady state case we present the following derivations:

1 ratio Λk

a. n ≥ m

$$K \cdot z(k+1) = \Lambda(k+1) \cdot x(k+1/k+1)$$
$$\Rightarrow \Lambda^{-1}(k+1) \cdot K \cdot z(k+1) = K \cdot z(k+1) + A \cdot x(k/k) + C \cdot u(k)$$
$$\Rightarrow \Lambda^{-1}(k+1) \cdot K \cdot \xi(k) \cdot z(k) = K \cdot \xi(k) \cdot z(k) + A \cdot x(k/k) + C \cdot u(k)$$
$$\Rightarrow \Lambda^{-1}(k+1) \cdot K \cdot \xi(k) \cdot \gamma(k) \cdot u(k) = K \cdot \xi(k) \cdot z(k) + A \cdot x(k/k) + C \cdot u(k)$$
$$= K \cdot \xi(k) \cdot z(k) + A \cdot \Lambda^{-1}(k) \cdot K \cdot z(k) + C \cdot u(k)$$
$$= K \cdot \xi(k) \cdot \gamma(k) \cdot u(k) + A \cdot \Lambda^{-1}(k) \cdot K \cdot \gamma(k) \cdot u(k) + C \cdot u(k)$$
$$\Rightarrow \Lambda^{-1}(k+1) \cdot K \cdot \xi(k) \cdot \gamma(k) = K \cdot \xi(k) \cdot \gamma(k) + A \cdot \Lambda^{-1}(k) \cdot K \cdot \gamma(k) + C$$

Then

$$\Lambda^{-1}(k+1) = I + A \cdot \Lambda^{-1}(k) \cdot K \cdot \gamma(k)\big(K \cdot \xi(k) \cdot \gamma(k)\big)^{-1} + C \cdot \big(K \cdot \xi(k) \cdot \gamma(k)\big)^{-1}$$

b. m ≥ n

$$K \cdot z(k+1) = \Lambda(k+1) \cdot x(k+1/k+1)$$
$$\Rightarrow \Lambda^{-1}(k+1) \cdot K \cdot z(k+1) = K \cdot z(k+1) + A \cdot x(k/k) + C \cdot u(k)$$
$$= K \cdot z(k+1) + A \cdot \Lambda^{-1}(k) \cdot K \cdot z(k) + C \cdot u(k)$$
$$= K \cdot z(k+1) + A \cdot \Lambda^{-1}(k) \cdot K \cdot \xi^{-1}(k) \cdot z(k+1) + C \cdot \delta(k) \cdot z(k)$$
$$\Rightarrow \Lambda^{-1}(k+1) \cdot K = K + A \cdot \Lambda^{-1}(k) \cdot K \cdot \xi^{-1}(k) + C \cdot \delta(k) \cdot \xi^{-1}(k) \cdot z(k+1)$$
$$\Rightarrow \Lambda^{-1}(k+1) \cdot K \cdot H = K \cdot H + A \cdot \Lambda^{-1}(k) \cdot K \cdot \xi^{-1}(k) \cdot H + C \cdot \delta(k) \cdot \xi^{-1}(k) \cdot H$$

Thus

$$\Lambda^{-1}(k+1) = I + A \cdot \Lambda^{-1}(k) \cdot K \cdot \xi^{-1}(k) \cdot H \cdot (K \cdot H)^{-1} + C \cdot \delta(k) \cdot \xi^{-1}(k) \cdot H \cdot (K \cdot H)^{-1}$$

2 ratio Nk

a. n ≥ m

$$C \cdot u(k+1) = N(k+1) \cdot x(k+1/k+1)$$
$$\Rightarrow N^{-1}(k+1) \cdot C \cdot u(k+1) = x(k+1/k+1)$$
$$= K \cdot z(k+1) + A \cdot x(k/k) + C \cdot u(k)$$
$$= K \cdot \gamma(k+1) \cdot u(k+1) + A \cdot N^{-1}(k) \cdot C \cdot u(k) + C \cdot u(k)$$
$$= K \cdot \gamma(k+1) \cdot u(k+1) + A \cdot N^{-1}(k) \cdot C \cdot \zeta^{-1}(k) \cdot u(k+1) + C \cdot \zeta^{-1}(k) \cdot u(k+1)$$
$$\Rightarrow N^{-1}(k+1) \cdot C = K \cdot \gamma(k+1) + A \cdot N^{-1}(k) \cdot C \cdot \zeta^{-1}(k) + C \cdot \zeta^{-1}(k)$$

Thus

$$N^{-1}(k+1) = K \cdot \gamma(k+1) \cdot C^{-1} + A \cdot N^{-1}(k) \cdot C \cdot \zeta^{-1}(k) \cdot C^{-1} + C \cdot \zeta^{-1}(k) \cdot C^{-1}$$

b. m ≥ n

$$C \cdot u(k+1) = N(k+1) \cdot x(k+1/k+1)$$
$$\Rightarrow N^{-1}(k+1) \cdot C \cdot u(k+1) = x(k+1/k+1)$$
$$\Rightarrow N^{-1}(k+1) \cdot C \cdot \delta(k+1) \cdot z(k+1) = x(k+1/k+1)$$
$$= K \cdot z(k+1) + A \cdot x(k/k) + C \cdot u(k)$$
$$= K \cdot z(k+1) + A \cdot N^{-1}(k) \cdot C \cdot u(k) + C \cdot u(k)$$
$$= K \cdot z(k+1) + A \cdot N^{-1}(k) \cdot C \cdot \zeta^{-1}(k) \cdot u(k+1) + C \cdot \zeta^{-1}(k) \cdot u(k+1)$$
$$= K \cdot z(k+1) + A \cdot N^{-1}(k) \cdot C \cdot \zeta^{-1}(k) \cdot \delta(k+1) \cdot z(k+1) + C \cdot \zeta^{-1}(k) \cdot \delta(k+1) \cdot z(k+1)$$
$$\Rightarrow N^{-1}(k+1) \cdot C \cdot \delta(k+1) = K + A \cdot N^{-1}(k) \cdot C \cdot \zeta^{-1}(k) \cdot \delta(k+1) + C \cdot \zeta^{-1}(k) \cdot \delta(k+1)$$
$$\Rightarrow N^{-1}(k+1) \cdot C \cdot \delta(k+1) \cdot H = K \cdot H + A \cdot N^{-1}(k) \cdot C \cdot \zeta^{-1}(k) \cdot \delta(k+1) \cdot H + C \cdot \zeta^{-1}(k) \cdot \delta(k+1) \cdot H$$

Thus

$$N^{-1}(k+1) = K \cdot H \cdot (C \cdot \delta(k+1) \cdot H)^{-1} + A \cdot N^{-1}(k) \cdot C \cdot \zeta^{-1}(k) \cdot \delta(k+1) \cdot H \cdot (C \cdot \delta(k+1) \cdot H)^{-1} + C \cdot \zeta^{-1}(k) \cdot \delta(k+1) \cdot H \cdot (C \cdot \delta(k+1) \cdot H)^{-1}$$

3 ratio Mk

a. n ≥ m

$$A \cdot x(k/k) = M(k+1) \cdot x(k+1/k+1)$$
$$\Rightarrow M^{-1}(k+1) \cdot A \cdot x(k/k) = x(k+1/k+1)$$

$$\Rightarrow M^{-1}(k+1)\cdot A\cdot\Lambda^{-1}(k)\cdot K\cdot z(k) = x(k+1/k+1)$$
$$\Rightarrow M^{-1}(k+1)\cdot A\cdot\Lambda^{-1}(k)\cdot K\cdot\gamma(k)\cdot u(k) = K\cdot z(k+1)+A\cdot x(k/k)+C\cdot u(k)$$
$$= K\cdot z(k+1)+A\cdot\Lambda^{-1}(k)\cdot K\cdot z(k)+C\cdot u(k)$$
$$= K\cdot\gamma(k+1)\cdot u(k+1)+A\cdot\Lambda^{-1}(k)\cdot K\cdot\gamma(k)\cdot u(k)+C\cdot u(k)$$
$$= K\cdot\gamma(k+1)\cdot\zeta(k)\cdot u(k)+A\cdot\Lambda^{-1}(k)\cdot K\cdot\gamma(k)\cdot u(k)+C\cdot u(k)$$
$$\Rightarrow M^{-1}(k+1)\cdot A\cdot\Lambda^{-1}(k)\cdot K\cdot\gamma(k) = K\cdot\gamma(k+1)\cdot\zeta(k)+A\cdot\Lambda^{-1}(k)\cdot K\cdot\gamma(k)+C$$

Thus
$$M^{-1}(k+1) = K\cdot\gamma(k+1)\cdot\zeta(k)\cdot\left(A\cdot\Lambda^{-1}(k)\cdot K\cdot\gamma(k)\right)^{-1}+I+C\cdot\left(A\cdot\Lambda^{-1}(k)\cdot K\cdot\gamma(k)\right)^{-1}$$

b. m ≥ n
$$A\cdot x(k/k) = M(k+1)\cdot x(k+1/k+1)$$
$$\Rightarrow M^{-1}(k+1)\cdot A\cdot x(k/k) = x(k+1/k+1)$$
$$\Rightarrow M^{-1}(k+1)\cdot A\cdot\Lambda^{-1}(k)\cdot K\cdot z(k) = x(k+1/k+1)$$
$$\Rightarrow M^{-1}(k+1)\cdot A\cdot\Lambda^{-1}(k)\cdot K\cdot\xi^{-1}(k)\cdot z(k+1) = K\cdot z(k+1)+A\cdot x(k/k)+C\cdot u(k)$$
$$= K\cdot z(k+1)+A\cdot\Lambda^{-1}(k)\cdot K\cdot z(k)+C\cdot u(k)$$
$$= K\cdot z(k+1)+A\cdot\Lambda^{-1}(k)\cdot K\cdot\xi^{-1}(k)\cdot z(k+1)+C\cdot\delta(k)\cdot z(k)$$
$$= K\cdot z(k+1)+A\cdot\Lambda^{-1}(k)\cdot K\cdot\xi^{-1}(k)\cdot z(k+1)+C\cdot\delta(k)\cdot\xi^{-1}(k)\cdot z(k+1)$$
$$\Rightarrow M^{-1}(k+1)\cdot A\cdot\Lambda^{-1}(k)\cdot K\cdot\xi^{-1}(k) = K+A\cdot\Lambda^{-1}(k)\cdot K\cdot\xi^{-1}(k)+C\cdot\delta(k)\cdot\xi^{-1}(k)$$
$$\Rightarrow M^{-1}(k+1)\cdot A\cdot\Lambda^{-1}(k)\cdot K = K\cdot\xi(k)+A\cdot\Lambda^{-1}(k)\cdot K+C\cdot\delta(k)$$
$$\Rightarrow M^{-1}(k+1)\cdot A\cdot\Lambda^{-1}(k)\cdot K\cdot H = K\cdot\xi(k)\cdot H+A\cdot\Lambda^{-1}(k)\cdot K\cdot H+C\cdot\delta(k)\cdot H$$

Thus
$$M^{-1}(k+1) = K\cdot\xi(k)\cdot H\cdot(A\cdot\Lambda^{-1}(k)\cdot K\cdot H)^{-1}+I+C\cdot\delta(k)\cdot H\cdot(A\cdot\Lambda^{-1}(k)\cdot K\cdot H)^{-1}$$

The steady state case appears where the system is time invariant and asymptotically stable. This is the case for example of random walk, which is presented in section 5. It is clear that in the steady state case, the ratios $\Lambda(k), M(k), N(k)$ depend on the measurements, the inputs and the known system parameters. Thus, it is evident that we are able to know the contribution of the measurement, the input and the previous estimation in the estimation computation.

3.3 No Input Case

In the case where there is no input, the discrete time state space system (1)-(2) becomes of the form:
$$x(k+1) = F(k+1)\cdot x(k)+w(k) \quad (13)$$
$$z(k) = H(k)\cdot x(k)+v(k) \quad (14)$$

Then for the **time varying** case we have:
$$\Lambda^{-1}(k+1) = I+E(k+1)$$
$$M^{-1}(k+1) = I+E^{-1}(k+1)$$

where
$$E(k+1) = A(k+1)\cdot\Lambda^{-1}(k)\cdot K(k+1)\cdot\xi^{-1}(k)\cdot H(k+1)\cdot(K(k+1)\cdot H(k+1))^{-1}$$

Of course
$$\Lambda(k)+M(k) = I$$

For the **time invariant** case, the above derivations hold with constant Kalman filter parameters.

In the **steady state case**, we have:
$$\Lambda^{-1}(k+1) = I+E(k+1)$$
$$M^{-1}(k+1) = I+E^{-1}(k+1)$$

where
$$E(k+1) = A\cdot\Lambda^{-1}(k)\cdot K\cdot\xi^{-1}(k)\cdot H\cdot(K\cdot H)^{-1}$$

Of course
$$\Lambda(k)+M(k) = I$$

3.4 Steady State Scalar Case

The scalar case appears when the state $x(k)$ is scalar ($n = 1$) as well as the measurement $z(k)$ is scalar ($m = 1$).

Assume $z(k) \neq 0$ (non zero measurements) and $u(k) \neq 0$ (non zero inputs).

In the steady state scalar case, we have:
$$\lambda(k) = \frac{K\cdot z(k)}{x(k/k)}$$
$$\mu(k) = \frac{A\cdot x(k-1/k-1)}{x(k/k)}$$
$$\nu(k) = \frac{C\cdot u(k-1)}{x(k/k)}$$
$$\lambda(1) = \frac{K\cdot z(1)}{x(1/1)}$$

$$\mu(1) = \frac{A \cdot x(0/0)}{x(1/1)}$$
$$\nu(1) = \frac{C \cdot u(0)}{x(1/1)}$$

Of course
$$\lambda(k) + \mu(k) + \nu(k) = 1$$

Then
$$\frac{1}{\lambda(k+1)} = 1 + \frac{A}{\lambda(k)} \cdot \frac{z(k)}{z(k+1)} + \frac{C}{K} \cdot \frac{u(k)}{z(k+1)}$$
$$\frac{1}{\mu(k+1)} = 1 + \frac{\lambda(k)}{A} \cdot \frac{z(k+1)}{z(k)} + \frac{\nu(k)}{A} \cdot \frac{u(k)}{u(k-1)}$$
$$\frac{1}{\nu(k+1)} = 1 + \frac{K}{C} \cdot \frac{z(k+1)}{u(k+1)} + \frac{A}{\nu(k)} \cdot \frac{u(k-1)}{u(k)}$$

3.5 No Input Steady State Scalar Case

In the no input steady state scalar case, we have
$$\lambda(k) = \frac{K \cdot z(k)}{x(k/k)}$$
$$\mu(k) = \frac{A \cdot x(k-1/k-1)}{x(k/k)}$$

Of course
$$\lambda(k) + \mu(k) = 1$$

Then
$$\frac{1}{\lambda(k+1)} = 1 + \frac{A}{\lambda(k)} \cdot \frac{z(k)}{z(k+1)}$$
$$\frac{1}{\mu(k+1)} = 1 + \frac{1-\mu(k)}{A} \cdot \frac{z(k+1)}{z(k)}$$

Setting
$$\varepsilon(k+1) = \frac{A}{\lambda(k)} \cdot \frac{z(k)}{z(k+1)}$$

we get
$$\lambda(k+1) = \frac{1}{1+\varepsilon(k+1)}$$
$$\mu(k+1) = \frac{\varepsilon(k+1)}{1+\varepsilon(k+1)}$$

Note that
$$\frac{1}{\varepsilon(k)} = \frac{\lambda(k)}{\mu(k)}$$
Proof.
$$\frac{\lambda(k+1)}{\mu(k+1)} = \frac{K \cdot z(k+1)}{A \cdot x(k/k)} = \frac{K}{x(k/k)} \cdot \frac{z(k+1)}{A} = \frac{\lambda(k)}{z(k)} \cdot \frac{z(k+1)}{A} = \frac{\lambda(k)}{A} \cdot \frac{z(k+1)}{z(k)} = \frac{1}{\varepsilon(k+1)}$$

Note that
$$\frac{1}{\varepsilon(k)} = \frac{\lambda(k)}{\mu(k)} = \frac{K \cdot z(k)}{A \cdot x(k-1/k-1)}$$

4 Metrics

In this section we define the percentages concerning the participation of the current measurement, the previous state estimation and the previous input in the state estimation computation.

At the end of the section, we propose the metrics concerning the participation of the current measurement, the previous state estimation and the previous input in the state estimation computation.

4.1 Percentages for Time Varying and Time Invariant Kalman Filter

For the general **time varying** case we define the following percentages:
$$\lambda'(k) = \frac{\|K(k) \cdot z(k)\|}{\|A(k) \cdot x(k-1/k-1)\| + \|K(k) \cdot z(k)\| + \|C(k-1) \cdot u(k-1)\|} \quad (15)$$
$$\mu'(k) = \frac{\left\|A(k) \cdot x\left(k-\frac{1}{k}-1\right)\right\|}{\left\|A(k) \cdot x\left(k-\frac{1}{k}-1\right)\right\| + \|K(k) \cdot z(k)\| + \|C(k-1) \cdot u(k-1)\|} \quad (16)$$
$$\nu'(k) = \frac{\|C(k-1) \cdot u(k-1)\|}{\left\|A(k) \cdot x\left(k-\frac{1}{k}-1\right)\right\| + \|K(k) \cdot z(k)\| + \|C(k-1) \cdot u(k-1)\|} \quad (17)$$

where $\|V\|$ denotes the norm-2 (Euclidean norm) of vector V.

Of course
$$0 < \lambda'(k) < 1$$
$$0 < \mu'(k) < 1$$
$$0 < \nu'(k) < 1$$

and
$$\lambda'(k) + \mu'(k) + \nu'(k) = 1 \quad (18)$$

We also define the following auxiliary ratios:
$$\varepsilon(k) = \frac{\|A(k) \cdot x(k-1/k-1)\|}{\|K(k) \cdot z(k)\|}$$
$$\rho(k) = \frac{\|A(k) \cdot x(k-1/k-1)\|}{\|C(k-1) \cdot u(k-1)\|}$$
$$\pi(k) = \frac{\|C(k-1) \cdot u(k-1)\|}{\|K(k) \cdot z(k)\|}$$

Then we have:
$$\varepsilon(k) = \frac{\|A(k) \cdot x(k-1/k-1)\|}{\|K(k) \cdot z(k)\|}$$
$$= \frac{\|A(k) \cdot \Lambda^{-1}(k-1) \cdot K(k-1) \cdot z(k-1)\|}{\|K(k) \cdot z(k)\|}$$

$$\rho(k) = \frac{\|A(k) \cdot x(k-1/k-1)\|}{\|C(k-1) \cdot u(k-1)\|}$$
$$= \frac{\|A(k) \cdot \Lambda^{-1}(k-1) \cdot K(k-1) \cdot z(k-1)\|}{\|C(k-1) \cdot u(k-1)\|}$$
$$\pi(k) = \frac{\|C(k-1) \cdot u(k-1)\|}{\|K(k) \cdot z(k)\|} = \frac{\varepsilon(k)}{\rho(k)}$$

and
$$\lambda'(k) = \frac{1}{\varepsilon(k) + 1 + \pi(k)}$$
$$\mu'(k) = \frac{1}{1 + \frac{1}{\varepsilon(k)} + \frac{1}{\rho(k)}}$$
$$\nu'(k) = \frac{1}{\rho(k) + \frac{1}{\pi(k)} + 1}$$

For the **time invariant** case, the above derivations hold with constant Kalman filter parameters.

4.2 Percentages for Steady state Kalman Filter

In the steady state case we define the following percentages:

$$\lambda'(k) = \frac{\|K \cdot z(k)\|}{\|A \cdot x(k-1/k-1)\| + \|K \cdot z(k)\| + \|C \cdot u(k-1)\|} \quad (19)$$

$$\mu'(k) = \frac{\|A \cdot x(k-1/k-1)\|}{\|A \cdot x(k-1/k-1)\| + \|K \cdot z(k)\| + \|C \cdot u(k-1)\|} \quad (20)$$

$$\nu'(k) = \frac{\|C \cdot u(k-1)\|}{\|A \cdot x(k-1/k-1)\| + \|K \cdot z(k)\| + \|C \cdot u(k-1)\|} \quad (21)$$

where $\|V\|$ denotes the norm-2 (Euclidean norm) of vector V.

Of course
$0 < \lambda'(k) < 1$
$0 < \mu'(k) < 1$
$0 < \nu'(k) < 1$

and
$\lambda'(k) + \mu'(k) + \nu'(k) = 1$(22)

We also define the following auxiliary ratios:
$$\varepsilon(k) = \frac{\|A \cdot x(k-1/k-1)\|}{\|K \cdot z(k)\|}$$
$$\rho(k) = \frac{\|A \cdot x(k-1/k-1)\|}{\|C \cdot u(k-1)\|}$$
$$\pi(k) = \frac{\|C \cdot u(k-1)\|}{\|K \cdot z(k)\|}$$

Then we have:
$$\varepsilon(k) = \frac{\|A \cdot x(k-1/k-1)\|}{\|K \cdot z(k)\|} = \frac{\|A \cdot \Lambda^{-1}(k-1) \cdot K \cdot z(k-1)\|}{\|K \cdot z(k)\|}$$
$$\rho(k) = \frac{\|A \cdot x(k-1/k-1)\|}{\|C \cdot u(k-1)\|} = \frac{\|A \cdot \Lambda^{-1}(k-1) \cdot K \cdot z(k-1)\|}{\|C \cdot u(k-1)\|}$$
$$\pi(k) = \frac{\|C \cdot u(k-1)\|}{\|K \cdot z(k)\|} = \frac{\varepsilon(k)}{\rho(k)}$$

and
$$\lambda'(k) = \frac{1}{\varepsilon(k) + 1 + \pi(k)}$$
$$\mu'(k) = \frac{1}{1 + \frac{1}{\varepsilon(k)} + \frac{1}{\rho(k)}}$$
$$\nu'(k) = \frac{1}{\rho(k) + \frac{1}{\pi(k)} + 1}$$

4.3 No Input Steady State Scalar Case

For the no input steady state scalar case we define the following percentages:
$$\lambda'(k) = \frac{|K \cdot z(k)|}{|A \cdot x(k-1/k-1)| + |K \cdot z(k)|}$$

$$\mu'(k) = \frac{|A \cdot x(k-1/k-1)|}{|A \cdot x(k-1/k-1)| + |K \cdot z(k)|}$$

where $|c|$ denotes the absolute value of the scalar quanity c.

Of course
$0 < \lambda'(k) < 1$
$0 < \mu'(k) < 1$

and
$\lambda'(k) + \mu'(k) = 1$

Then, we derive
$$\frac{1}{\lambda'(k)} = 1 + \frac{|A \cdot x(k-1/k-1)|}{|K \cdot z(k)|} = 1 + \left|\frac{A \cdot x(k-1/k-1)}{K \cdot z(k)}\right| = 1 + |\varepsilon(k)|$$

$$\frac{1}{\mu'(k)} = 1 + \frac{|K \cdot z(k)|}{|A \cdot x(k-1/k-1)|}$$
$$= 1 + \left|\frac{K \cdot z(k)}{A \cdot x(k-1/k-1)}\right|$$
$$= 1 + \frac{1}{|\varepsilon(k)|} = \frac{1 + |\varepsilon(k)|}{|\varepsilon(k)|}$$

Furtheremore, note that

$$\frac{1}{\varepsilon(k+1)} = \frac{\lambda(k+1)}{\mu(k+1)} = \frac{\frac{1}{\mu(k+1)}}{\frac{1}{\lambda(k+1)}} = \frac{1 + \frac{1-\mu(k)}{A} \cdot \frac{z(k+1)}{z(k)}}{1 + \frac{A}{\lambda(k)} \cdot \frac{z(k)}{z(k+1)}}$$
$$= \frac{1 + \frac{\lambda(k)}{A} \cdot \frac{z(k+1)}{z(k)}}{1 + \frac{A}{\lambda(k)} \cdot \frac{z(k)}{z(k+1)}} = \frac{\frac{A \cdot z(k) + \lambda(k) \cdot z(k+1)}{A \cdot z(k)}}{\frac{A \cdot z(k) + \lambda(k) \cdot z(k+1)}{\lambda(k) \cdot z(k+1)}}$$
$$= \frac{\lambda(k)}{A} \cdot \frac{z(k+1)}{z(k)}$$

Then

$$\frac{\varepsilon(k+1)}{\varepsilon(k)} = \frac{\frac{A \cdot z(k)}{\lambda(k) \cdot z(k+1)}}{\frac{A \cdot z(k-1)}{\lambda(k-1) \cdot z(k)}}$$
$$= \frac{\lambda(k-1)}{\lambda(k)} \cdot \frac{z(k)}{z(k+1)} \cdot \frac{z(k)}{z(k-1)}$$
$$= \frac{\frac{1}{\lambda(k)}}{\frac{1}{\lambda(k-1)}} \cdot \frac{z(k)}{z(k+1)} \cdot \frac{z(k)}{z(k-1)}$$
$$= \frac{1 + \varepsilon(k)}{1 + \varepsilon(k-1)} \cdot \frac{z(k)}{z(k+1)} \cdot \frac{z(k)}{z(k-1)}$$

Thus

$$\varepsilon(k+1) = \varepsilon(k) \cdot \frac{1 + \varepsilon(k)}{1 + \varepsilon(k-1)} \cdot \frac{z(k)}{z(k+1)} \cdot \frac{z(k)}{z(k-1)}$$

The percentages are computed as:

$$\lambda'(k) = \frac{1}{1 + |\varepsilon(k)|}$$
$$\mu'(k) = \frac{|\varepsilon(k)|}{1 + |\varepsilon(k)|}$$

where

$$\varepsilon(k+1) = \varepsilon(k) \cdot \frac{1 + \varepsilon(k)}{1 + \varepsilon(k-1)} \cdot \frac{z(k)}{z(k+1)} \cdot \frac{z(k)}{z(k-1)}$$

for $k = 2,3, \ldots$

$$\varepsilon(1) = \frac{\mu(1)}{\lambda(1)} = \frac{A \cdot x(0/0)}{K \cdot z(1)}$$
$$\varepsilon(2) = \frac{\mu(2)}{\lambda(2)} = \frac{A \cdot x(1/1)}{K \cdot z(2)}$$

where

$$x(0/0) = (1 - K \cdot H) \cdot x_0 + K \cdot z(0)$$
$$x(1/1) = A \cdot x(0/0) + K \cdot z(1)$$

In the following we present another **alternitive** computation of the percetages.

$$\lambda'(k) = \frac{1}{1 + |\varepsilon(k)|}$$
$$\mu'(k) = \frac{|\varepsilon(k)|}{1 + |\varepsilon(k)|}$$
$$\varepsilon(k) = \frac{\mu(k)}{\lambda(k)}$$
$$\frac{1}{\lambda(k+1)} = 1 + \frac{A}{\lambda(k)} \cdot \frac{z(k)}{z(k+1)}$$
$$\frac{1}{\mu(k+1)} = 1 + \frac{1 - \mu(k)}{A} \cdot \frac{z(k+1)}{z(k)}$$
$$\lambda(1) = \frac{K \cdot z(1)}{x(1/1)}$$
$$\mu(1) = \frac{A \cdot x(0/0)}{x(1/1)}$$

Remark.
The percentages

$$\lambda'(k) = \frac{|K \cdot z(k)|}{|A \cdot x(k-1/k-1)| + |K \cdot z(k)|}$$
$$\mu'(k) = \frac{|A \cdot x(k-1/k-1)|}{|A \cdot x(k-1/k-1)| + |K \cdot z(k)|}$$

are defined using absolute values and lead to the percentages ratio

$$r'(k) = \frac{\lambda'(k)}{\mu'(k)} = \frac{|K \cdot z(k)|}{|A \cdot x(k-1/k-1)|}$$

Let define the percentages without using absolute values:

$$\lambda''(k) = \frac{K \cdot z(k)}{A \cdot x(k-1/k-1) + K \cdot z(k)}$$
$$\mu''(k) = \frac{A \cdot x(k-1/k-1)}{A \cdot x(k-1/k-1) + K \cdot z(k)}$$

Of course

$\lambda''(k) < 1$

$\mu''(k) < 1$

and

$\lambda''(k) + \mu''(k) = 1$

Then the percentages ratio becomes

$$r''(k) = \frac{\lambda''(k)}{\mu''(k)} = \frac{K \cdot z(k)}{A \cdot x(k-1/k-1)}$$

It is clear that

$r'(k) = |r''(k)|$

This percentage ratio characterizes the filter; the non-use of absolute values does not add anything more than the use of absolute values.
This remark can be extended in the input steady state scalar case.

4.4 Proposed Metrics

Finally, we propose the metrics concerning the participation of the current measurement, the previous state estimation and the previous input in the state estimation computation.

We propose the **mean values of the percentages** to be the metrics for the participation of the measurement, the previous estimation and the input in Kalman filter estimation. The proposed metrics depend on the known measurements and inputs and the known system parameters.

Analogous metrics can be defined for the state **prediction**. This is feasible rewriting the Kalman filter equations in order to underline the fact that **the state prediction depends on the current measurement, the previous state prediction and the previous input**.

The prediction form of the time varying Kalman filter has as follows:

time varying Kalman filter (prediction form)

$$x(k+1/k) = C(k)\cdot z(k) + A(k)\cdot x(k/k-1) + B(k)\cdot u(k)$$

$$P(k+1/k) = Q(k) + F(k+1)\cdot\left[[I-K(k)\cdot H(k)]\cdot P(k/k-1)\right]\cdot F^T(k+1)$$

for $k = 0,1,\ldots$

where

$$K(k) = P(k/k-1)\cdot H^T(k)\cdot[H(k)\cdot P(k/k-1)\cdot H^T(k) + R(k)]^{-1}$$

$$A(k) = F(k+1)\cdot[I-K(k)\cdot H(k)]$$

$$C(k) = F(k+1)\cdot K(k)$$

with initial conditions

$$x(0/-1) = x_0, P(0/-1) = P_0$$

The prediction form of the time invariant Kalman filter has as follows:

time invariant Kalman filter (prediction form)

$$x(k+1/k) = C(k)\cdot z(k) + A(k)\cdot x(k/k-1) + B\cdot u(k)$$

$$P(k+1/k) = Q + F\cdot\left[[I-K(k)\cdot H]\cdot P(k/k-1)\right]\cdot F^T$$

for $k = 0,1,\ldots$

where

$$K(k) = P(k/k-1)\cdot H^T\cdot[H\cdot P(k/k-1)\cdot H^T + R]^{-1}$$

$$A(k) = F\cdot[I-K(k)\cdot H]$$

$$C(k) = F\cdot K(k)$$

with initial conditions

$$x(0/-1) = x_0, P(0/-1) = P_0$$

The prediction form of the steady state time varying Kalman filter has as follows:

steady state Kalman filter (prediction form)

$$x(k+1/k) = C(k)\cdot z(k) + A(k)\cdot x(k/k-1) + B\cdot u(k)$$

for $k = 0,1,\ldots$

where

$$A = F\cdot[I-K\cdot H]$$

$$C = F\cdot K$$

with initial condition

$$x(0/-1) = x_0$$

5 Examples

Example 1. Random walk. A scalar system without input ($u(k) = 0$) which represents the random walk movement is assumed in this example.

The system parameters are:
$F = H = Q = R = 1$.

The steady state Kalman filter has been sumulated for 100 time lags. The steady state Kalman Filter coefficients are:
$P_p = 1.618, K = 0.618, A = 0.382$.

Figure 1 shows the percentage contributions of the current measurement and the previous state estimation in the state estimation computation.

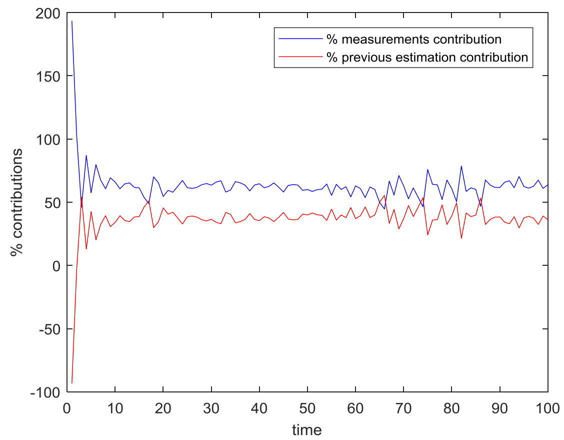

Fig. 1: Example 1. Percentage contributions
Source: created by the authors.

The **mean values of the percentages** for the participation of the measurement and the previous estimation in Kalman filter estimation are:

mean % measurements contribution = 63.547%

mean % previous estimations contribution = 36.453%

Example 2. Electric load estimation. A scalar system without input which describes the hourly electric load estimation model is assumed in this example.

The system parameters are:
$F = 1, H = 1, Q = 0.235, R = 0.655$.

The steady state Kalman filter has been simulated for a week period (24h/day x 7days = 168days). The steady state Kalman Filter coefficients are:
$P_p = 0.527, K = 0.446, A = 0.554$.

Figure 2 shows the percentage contributions of the current measurement and the previous state estimation in the state estimation computation.

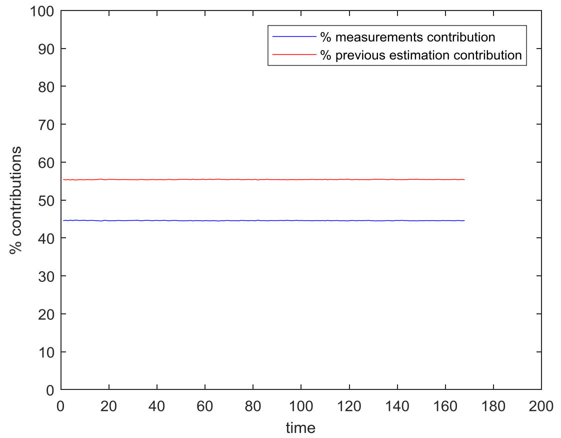

Fig. 2: Example 2. Percentage contributions
Source: created by the authors.

The **mean values of the percentages** for the participation of the measurement and the previous estimation in Kalman filter estimation are:
mean % measurements contribution = 44.587%
mean % previous estimations contribution = 55.413%

Example 3. Free falling object. A system with input which represents the movement of a free falling object is assumed in this example.

The state vector consists of the height and the velocity $(n = 2)$ of the falling object: $x(k) = \begin{bmatrix} h(k) \\ \upsilon(k) \end{bmatrix}$

The measurement is the height with noise $(m = 1)$.

The system parameters are:

$$F = \begin{bmatrix} 1 & -dt \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, u(k) = \begin{bmatrix} -\frac{1}{2} \cdot g \cdot dt^2 \\ g \cdot dt \end{bmatrix}, \; H = [1 \quad 0], Q = 0, R = 0.2, dt = 0.01s.$$

The time invariant Kalman filter has been simulted for a free falling object from a height of $h = 100m$ with zero initial velocity; then the falling time is $t = \sqrt{2h/g} = 4.515s$, where $g = 9.81m/s^2$ is the gravitational acceleration.

Figure 3 shows the percentage contributions of the current measurement, the previous state estimation and the previous input in the state estimation computation.

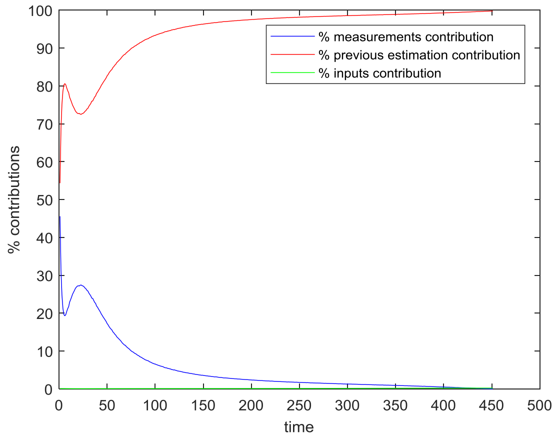

Fig. 3: Example 3. Percentage contributions
Source: created by the authors.

The **mean values of the percentages** for the participation of the measurement, the previous estimation and the input in Kalman filter estimation are:
mean % measurements contribution = 5.292%
mean % previous estimations contribution = 94.568%
mean % inputs contribution = 0.140%

6 Conclusion

Kalman filter computes iteratively the estimation using the current measurement, the previous estimation and the previous input. We introduced new percentage metrics for the participation of the current measurement, the previous estimation and the previous input in Kalman filter estimation. The proposed metrics are valid for the time varying Kalman filter, the time invariant Kalman filter and the steady state Kalman filter. The proposed metrics depend on the known measurements, inputs and system parameters. They indicate how the measurements, the inputs and the previous estimation affect the Kalman filter estimation. A subject of future research is to propose percentage metrics for the participation of the measurements and the inputs in the FIR form of the steady state Kalman filter.

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