Estimating Parameters for the New Mixture Nakagami and Rayleigh Distributions: Mathematical Properties and Simulations

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Abstract: This article proposes a new distribution called the mixture Nakagami-Rayleigh (mNR) distribution, which is obtained by combining Nakagami and Rayleigh distributions. The maximum likelihood estimator (MLE) is the primary approach we propose to estimate the parameters for the mN-R distribution. This paper also presents related functions of the mNR distribution, such as the survival function, the probability density function (PDF), the cumulative distribution function (CDF), and the hazard function, along with the important mathematical properties of the mNR distribution. Additionally, plots of the important functions of the mNR distribution are illustrated as line plots and contour plots. Furthermore, we present a simulation study to demonstrate the flexibility of the mNR distribution. Finally, the article concludes by summarizing the key outcomes and providing recommendations for future work.

Key-Words: - Nakagami distribution, Rayleigh distribution, Mixture Nakagami-Rayleigh distribution, Maximum likelihood estimator, Probability modeling in public health

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1 Introduction

In recent decades, the development of new forms of probability distributions has continued to emerge in response to the increasing need to handle data in diverse ways for different purposes. Owing to the continuous efforts of researchers, numerous probability distributions have been developed and refined over time. Among the well-known and widely applied distributions are: the Gompertz distribution, [1], the Weibull distribution, [2], the gamma distribution, [3], the logistic distribution, [4], the log-logistic distribution, [5], and the Nakagami distribution, [6]. These distributions have been applied across various fields, including medicine, [7], information and communication technology (IOT), [6], agriculture and pest management, [5].

The Nakagami distribution has proven to be particularly important in many contexts, such as in the modeling of fading channels. Examples of this problem can be found in [6], [8]. In this context, the Nakagami distribution has also proven to be an incredibly useful tool in calculating the average bit error rate (BER) for differential phase shift keying (DPSK) modulation and in analyzing minimum shift keying (MSK), [9], [10], [11]. In addition, the Nakagami distribution can also be applied to positive-valued data. For example, in medical imaging, it has been used for ultrasonic tissue characterization, [12], [13], [14]. In the field of hydrology, it may be employed to model rainfall or river flow data, [15], [16]. In reliability engineering, it can serve as a model for product lifetime data, particularly when dealing with censored data, [7], [17], [18].

Based on the aforementioned examples, the Nakagami distribution is considered one of the most widely used statistical tools. To enhance its accuracy and applicability, researchers have developed more complex forms of the Nakagami distribution. These generalized models aim to better accommodate complex data structures or to incorporate characteristics of other distributions. Examples of such work include the following studies: The generalized odd Nakagami-G distribution expands the Nakagami distribution by combining it with other baseline distribution functions to increase the flexibility of simulating real data, [19]. The Inverse Nakagami distribution is used for statistical inference when the lifetime of a product follows this distribution, [7], [20]. The $N*$Nakagami distribution has been proposed to describe the behavior of cascaded fading channels, [21], [22]. Besides the Nagarami distribution, another distribution widely used and interesting is the Rayleigh distribution.

The Rayleigh distribution, which is a special case of the Nakagami distribution, has been widely used in wireless communication, particularly for modeling fading channels in scenarios without a line-of-sight (LOS) component, [8], [10]. Additionally, the Rayleigh distribution is a model that is suitable for high-density urban environments, [23]. The Rayleigh distribution has been enhanced and presented in various forms, such as: the Rayleigh-Rayleigh distribution, [24], the Weibull-Rayleigh distribution, [25], the exponentiated Rayleigh, [26], the Rayleigh Gamma gompertz distribution, [27], the Rayleigh-lognormal distribution, [28], the Rayleigh-generalized inverse Gaussian distribution, [29], and the Rayleigh-inverse Gaussian distribution, [30]. In this work, we present a mixture of the Nakagami and Rayleigh distributions. We discuss the mathematical properties and parameter estimation of the proposed distribution, including a simulation study and an example of its application.

The structure of this article consists of the following: Section 2 explains the original form of the Nakagami and Rayleigh distributions. This section presents their PDFs and CDFs in preparation for use in the next section. In Section 3, we propose a new distribution by combining the Nakagami and Rayleigh distributions referred to as the mN-R distribution. Additionally, in this section also presents the PDF, CDF, survival function, hazard function of the mNR distribution along wit its mathematical properties. Furthermore, the parameter estimation method is presented in Section 4. In addition, Section 5, we perform simulations of this work. Finally, the conclusion and discussion of the results is provided in Section 6.

2 Nakagami and Rayleigh distributions

The Nakagami distribution (also known as the m-Nakagami distribution) includes a shape parameter that allows its form to be adjusted, providing flexibility in modeling a wide variety of data characteristics. In particular cases, by appropriately setting the shape parameter to 1, the Nakagami distribution reduces to the Rayleigh distribution. Additionally, when the shape parameter is set to 0.5, the Nakagami distribution can also reduce to the half-normal distribution, [6], [31], [32]. Some studies have presented applications that combine the Rayleigh and Nakagami distributions in various forms, [33], [34]. In this work, we propose a novel alternative model that differs from those in previous research. In the following, we present the PDFs and CDFs of the Nakagami and Rayleigh distributions.

2.1 Nakagami distribution

The PDF (g_N) and CDF (G_N) of the Nakagami distribution can be written as

$$g_N(t) = \frac{2\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda} t^{2\lambda-1} \exp\left(\frac{-\lambda}{\beta}t^2\right), \quad (1)$$

and

$$G_N(t) = \frac{1}{\Gamma(\lambda)}\gamma\left(\lambda, \frac{\lambda}{\beta}t^2\right), \quad (2)$$

where $\beta > 0$ and $\lambda \geq 1/2$ denote the shape and scale parameters the Nakagami distribution, respectively. The variable $t \geq 0$ denotes a random variable. $\Gamma(\lambda)$ denotes the Gamma function of λ.

2.2 Rayleigh distribution

The PDF (g_R) and CDF (G_R) of the Rayleigh distribution can be written as

$$g_R(t) = \frac{t}{\delta^2}\exp\left(-\frac{t^2}{2\delta^2}\right), \quad (3)$$

and

$$G_R(t) = 1 - \exp\left(-\frac{t^2}{2\delta^2}\right), \quad (4)$$

where $\delta > 0$ denote the scale parameter of the Rayleigh distribution.

3 Mixture Nakagami–Rayleigh (mNR) distribution

This section presents fundamental functions and analytical properties of the new mNR distribution.

3.1 PDF of the mNR distribution

The Nakagami and Rayleigh PDFs of the individual components in the mixture are denoted by $g_N(t)$ and $g_R(t)$, respectively. Let $f(t)$ be the mixture PDF at time t, which can be written as follows:

$$f(t) = \theta g_N(t) + (1-\theta)g_R(t), \quad (5)$$

where $0 \leq \theta \leq 1$ is the mixing parameters, which determines the proportion of each component in the mixture. By inserting equations (1) and (3) into equation (5), we obtain the mNR PDF as follows:

$$\begin{aligned} f(t) =& \frac{2\theta\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda} t^{2\lambda-1}\exp\left(\frac{-\lambda}{\beta}t^2\right) \\ &+ (1-\theta)\frac{t}{\delta^2}\exp\left(-\frac{t^2}{2\delta^2}\right). \end{aligned} \quad (6)$$

3.2 Validity check of the distribution

We integrate the mNR PDF presented in equation (6) with respect to t from 0 to infinity as part of the validity check of the distribution:

$$\begin{aligned} &\int_0^\infty \frac{2\theta\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda} t^{2\lambda-1}\exp\left(\frac{-\lambda}{\beta}t^2\right) \\ &\quad + \int_0^\infty (1-\theta)\frac{t}{\delta^2}\exp\left(-\frac{t^2}{2\delta^2}\right)dt = 1. \end{aligned} \quad (7)$$

This proved that the density function in (6) belongs to a valid distribution.

3.3 CDF of the mNR Distribution

In this subsection, we present the CDF of the mNR distribution, denoted by $F(t)$. Based on the mNR CDF presented in (5), the CDF can be formulated as:

$$\begin{aligned} F(t) &= \int_0^t f(x)dx \\ &= \int_0^t (\theta g_N(x) + (1-\theta)g_R(x))dx \\ &= \theta G_N(t) + (1-\theta)G_R(t). \end{aligned} \quad (8)$$

By substituting equations (2) and (4) into (8), we obtain the mNR CDF as follows:

$$\begin{aligned} F(t) =& \frac{\theta}{\Gamma(\lambda)}\gamma\left(\lambda, \frac{\lambda}{\beta}t^2\right) \\ &+ (1-\theta)\left(1 - \exp\left(-\frac{t^2}{2\delta^2}\right)\right). \end{aligned} \quad (9)$$

3.4 Survival function for the mNR distribution

One of the important functions is the survival function $S(t)$, which can be expressed as follows:

$$S(t) = 1 - F(t). \quad (10)$$

Based on the mNR CDF shown as (9), the survival function of the mNR distribution can be written as follows:

$$\begin{aligned} S(t) =& 1 - \left[\frac{\theta}{\Gamma(\lambda)}\gamma\left(\lambda, \frac{\lambda}{\beta}t^2\right)\right. \\ &\left. + (1-\theta)\left(1 - \exp\left(-\frac{t^2}{2\delta^2}\right)\right)\right]. \end{aligned} \quad (11)$$

3.5 Hazard rate function for the mNR distribution

Referring to the PDF and the survival function, the hazard function can be expressed using the following relation:

$$h(t) = \frac{f(t)}{S(t)}. \tag{12}$$

Using equations (6), (11) and (12), the hazard function for the mNR distribution can be written as

$$h(t) = \frac{\frac{2\theta\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda} t^{2\lambda-1} \exp\left(\frac{-\lambda}{\beta} t^2\right) + (1-\theta)\frac{t}{\delta^2}\mathcal{E}}{1 - \left[\frac{\theta}{\Gamma(\lambda)}\gamma\left(\lambda, \frac{\lambda}{\beta}t^2\right) + (1-\theta)\left(1-\mathcal{E}\right)\right]}, \tag{13}$$

where $\mathcal{E} = \exp\left(-\frac{t^2}{2\delta^2}\right)$.

3.6 Moments

Using the mNR PDF in equation (5), the moments of the mNR distribution can be derived as follows:

$$\mu_r' = E(T^r) = \int_0^\infty t^r f(t) dt, \tag{14}$$

where $r(= 1, 2, 3, ...)$ denotes the order of the moment. By substituting equation (6) in (14), we have

$$\begin{aligned} \mu_r' =& \int_0^\infty \frac{2\theta\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda} t^{2\lambda+r-1} \exp\left(\frac{-\lambda}{\beta} t^2\right) \\ &+ (1-\theta)\frac{t^{r+1}}{\delta^2}\exp\left(-\frac{t^2}{2\delta^2}\right) dt \\ =&\theta \frac{\Gamma\left(\lambda+\frac{r}{2}\right)}{\Gamma(\lambda)}\left(\frac{\beta}{\lambda}\right)^{\frac{r}{2}} \\ &+ (1-\theta)\delta^r 2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}+1\right). \end{aligned} \tag{15}$$

The first and second moments, denoted by μ_1' and μ_2' respectively, can be written as:

$$\begin{aligned} \mu_1' =& E(T) \\ =& \theta\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)}\left(\frac{\beta}{\lambda}\right)^{\frac{1}{2}} + (1-\theta)\delta\sqrt{\frac{\pi}{2}}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \mu_2' =& E(T^2) \\ =& \theta\beta + (1-\theta)2\delta^2. \end{aligned} \tag{17}$$

3.7 Variance of the mNR distribution

The variance of the mNR distribution, denoted by $Var(T)$, can be calculated using the standard formula as

$$Var(T) = E(T^2) - [E(T)]^2\,, \tag{18}$$

Substituting $E(T)$ and $E(T^2)$ shown in (16) and (17) respectively, we obtain

$$\begin{aligned} Var(T) =& \theta\beta + (1-\theta)2\delta^2 \\ &- \left[\theta\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)}\left(\frac{\beta}{\lambda}\right)^{\frac{1}{2}} + (1-\theta)\delta\sqrt{\frac{\pi}{2}}\right]^2 \end{aligned} \tag{19}$$

3.8 Standard deviation

Next, the standard deviation of the mNR distribution ($Std(t)$) is given by

$$\begin{aligned} Std(t) =& \sqrt{Var(T)} \\ =& \sqrt{\theta\beta + (1-\theta)2\delta^2 - (E(T))^2}, \end{aligned} \tag{20}$$

where

$$E(T) = \theta\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)}\left(\frac{\beta}{\lambda}\right)^{\frac{1}{2}} + (1-\theta)\delta\sqrt{\frac{\pi}{2}}.$$

3.9 Coefficient of variation

The coefficient of variation (CV) of the mNR distribution can be calculated using the following formula:

$$CV = \frac{Std(t)}{E(T)}. \tag{21}$$

Therefore, the CV can be written as

$$CV = \frac{\sqrt{\theta\beta + (1-\theta)2\delta^2 - [\Theta]^2}}{\theta\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)}\left(\frac{\beta}{\lambda}\right)^{\frac{1}{2}} + (1-\theta)\delta\sqrt{\frac{\pi}{2}}}. \tag{22}$$

3.10 Harmonic mean

For the mNR distribution, the harmonic mean (HM) is calculated using the following equation:

$$\begin{aligned} HM =& E\left(\frac{1}{T}\right) \\ =& \int_0^\infty \frac{1}{t} f(t) dt \\ =& \int_0^\infty \frac{2\theta\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda} t^{2\lambda-2} \exp\left(\frac{-\lambda}{\beta} t^2\right) \\ &+ (1-\theta)\frac{1}{\delta^2}\exp\left(-\frac{t^2}{2\delta^2}\right) dt. \end{aligned} \tag{23}$$

Therefore, the HM can be written as

$$HM = \theta \frac{\Gamma\left(\lambda - \frac{1}{2}\right)}{\Gamma(\lambda)} \left(\frac{\lambda}{\beta}\right)^{\frac{1}{2}} + \frac{(1-\theta)}{\delta}\sqrt{\frac{\pi}{2}}. \quad (24)$$

3.11 Moments generating function

We can calculate the moments generating function of the mNR distribution, ($M_T(x)$) using the following formula:

$$M_T(x) = E(e^{xT}) = \sum_{r=0}^{\infty} \frac{x^r \mu_r'}{r!}, \quad (25)$$

By substituting μ_r' presented in (16) into (25), we obtain

$$\begin{aligned} M_T(x) = &\sum_{r=0}^{\infty} \frac{x^r}{r!} \Bigg[\theta \frac{\Gamma\left(\lambda + \frac{r}{2}\right)}{\Gamma(\lambda)} \left(\frac{\beta}{\lambda}\right)^{\frac{r}{2}} \\ &+ (1-\theta)\delta^r 2^{\frac{r}{2}} \Gamma\left(\frac{r}{2}+1\right)\Bigg]. \end{aligned} \quad (26)$$

3.12 Characteristic function

The characteristic function of the mNR distribution $\phi_T(x)$, is a function that completely specifies the distribution and can be written as:

$$\begin{aligned} \phi_T(x) = &\sum_{r=0}^{\infty} \frac{(ix)^r}{r!} \Bigg[\theta \frac{\Gamma\left(\lambda + \frac{r}{2}\right)}{\Gamma(\lambda)} \left(\frac{\beta}{\lambda}\right)^{\frac{r}{2}} \\ &+ (1-\theta)\delta^r 2^{\frac{r}{2}} \Gamma\left(\frac{r}{2}+1\right)\Bigg]. \end{aligned} \quad (27)$$

3.13 Cumulant generating function

The cumulant generating function of the mNR distribution $K_T(x)$ can be written as:

$$\begin{aligned} K_T(x) = &\log \sum_{r=0}^{\infty} \frac{x^r}{r!} \Bigg[\theta \frac{\Gamma\left(\lambda + \frac{r}{2}\right)}{\Gamma(\lambda)} \left(\frac{\beta}{\lambda}\right)^{\frac{r}{2}} \\ &+ (1-\theta)\delta^r 2^{\frac{r}{2}} \Gamma\left(\frac{r}{2}+1\right)\Bigg]. \end{aligned} \quad (28)$$

3.14 Incomplete moments

The incomplete moments $\Lambda_r(x)$ is calculated using the following equation:

$$\Lambda_r(x) = \int_0^x t^r f(t) dt. \quad (29)$$

By substituting equation (6) in (26), we can write $\Lambda_r(x)$ as follows:

$$\begin{aligned} \Lambda_r(x) = \int_0^x \Bigg[&\frac{2\theta\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda} t^{2\lambda+r-1} \exp\left(\frac{-\lambda}{\beta}t^2\right) \\ &+ (1-\theta)\frac{t^{r+1}}{\delta^2} \exp\left(-\frac{t^2}{2\delta^2}\right)\Bigg] dt. \end{aligned} \quad (30)$$

By simplifying (30), the incomplete moments can be written as

$$\begin{aligned} \Lambda_r(x) = &\frac{\theta}{\Gamma(\lambda)} \left(\frac{\beta}{\lambda}\right)^{\frac{r}{2}} \gamma\left(\lambda + \frac{r}{2}, \frac{\lambda}{\beta}x^2\right) \\ &+ (1-\theta)\delta^r 2^{\frac{r}{2}} \gamma\left(\frac{r+2}{2}, \frac{x^2}{2\delta^2}\right). \end{aligned} \quad (31)$$

3.15 Mean residual life (MRL)

The mean residual life (MRL) is a fundamental metric in reliability analysis. It represents the conditional expectation of an item's future lifetime, given that it has operated without failure until time x. The MRL is given by:

$$m_1(x) = E[(T-x)|T > x]. \quad (32)$$

For the mNR distribution, the MRL function can be calculated using the following formula:

$$m_1(x) = \frac{\theta \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)} \left(\frac{\beta}{\lambda}\right)^{\frac{1}{2}} + (1-\theta)\delta\sqrt{\frac{\pi}{2}} - \zeta_x}{1 - \frac{\theta}{\Gamma(\lambda)}\gamma\left(\lambda, \frac{\lambda}{\beta}t^2\right) - (1-\theta)(1-\mathcal{E})}. \quad (33)$$

where

$$\begin{aligned} \zeta_x = &\frac{\theta}{\Gamma(\lambda)} \left(\frac{\beta}{\lambda}\right)^{\frac{1}{2}} \gamma\left(\lambda + \frac{1}{2}, \frac{\lambda}{\beta}x^2\right) \\ &- (1-\theta)\delta 2^{\frac{1}{2}} \gamma\left(\frac{3}{2}, \frac{x^2}{2\delta^2}\right), \\ \mathcal{E} = &\exp\left(-\frac{t^2}{2\delta^2}\right). \end{aligned}$$

3.16 Mean deviations

In this subsection, we present the mean deviation (MD) of the mNR distribution. We consider two measures: the mean deviation about the mean ($MD(\mu)$) and about the median ($MD(M)$), respectively. The $MD(\mu)$ can be calculated using the following formula:

$$MD(\mu) = 2\mu G(\mu) - 2\int_0^\mu t f(t) dt. \quad (34)$$

Therefore, we obtain the mean as follows:

$$\begin{aligned} MD(\mu) =& 2\left[\theta\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)}\left(\frac{\beta}{\lambda}\right)^{\frac{1}{2}}+(1-\theta)\delta\sqrt{\frac{\pi}{2}}\right] \\ &\times\left[\frac{\theta}{\Gamma(\lambda)}\gamma\left(\lambda,\frac{\lambda}{\beta}\mu^2\right)\right. \\ &\left.+(1-\theta)\left(1-\exp\left(-\frac{\mu^2}{2\delta^2}\right)\right)\right] \\ &-2\left[\frac{\theta}{\Gamma(\lambda)}\left(\frac{\beta}{\lambda}\right)^{\frac{1}{2}}\gamma\left(\lambda+\frac{1}{2},\frac{\lambda}{\beta}\mu^2\right)\right. \\ &\left.+(1-\theta)\delta 2^{\frac{1}{2}}\gamma\left(\frac{3}{2},\frac{\mu^2}{2\delta^2}\right)\right]. \end{aligned} \tag{35}$$

Similarly, the MD about median $MD(M)$ can be calculated using the following formula:

$$MD(\mu) = \mu - 2\int_0^M tf(t)dt. \tag{36}$$

Therefore, we obtain the median as follows:

$$\begin{aligned} MD(M) =& \left[\theta\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)}\left(\frac{\beta}{\lambda}\right)^{\frac{1}{2}}\right. \\ &\left.+(1-\theta)\delta\sqrt{\frac{\pi}{2}}\right] \\ &-2\left[\frac{\theta}{\Gamma(\lambda)}\left(\frac{\beta}{\lambda}\right)^{\frac{1}{2}}\gamma\left(\lambda+\frac{1}{2},\frac{\lambda}{\beta}M^2\right)\right. \\ &\left.+(1-\theta)\delta 2^{\frac{1}{2}}\gamma\left(\frac{3}{2},\frac{M^2}{2\delta^2}\right)\right]. \end{aligned} \tag{37}$$

3.17 Bonferroni Curve

For the mNR distribution, the Bonferroni curve $B(q)$ of a random variable T is given by

$$B(q) = \frac{1}{q\mu}\int_0^r tf(t)dt. \tag{38}$$

Simplifying the above equation yields:

$$B(q) = \frac{\left[\frac{\theta}{\Gamma(\lambda)}\left(\frac{\beta}{\lambda}\right)^{\frac{1}{2}}\mathcal{G}+(1-\theta)\delta 2^{\frac{1}{2}}\gamma\left(\frac{3}{2},\frac{r^2}{2\delta^2}\right)\right]}{q\left[\theta\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)}\left(\frac{\beta}{\lambda}\right)^{\frac{1}{2}}+(1-\theta)\delta\sqrt{\frac{\pi}{2}}\right]}, \tag{39}$$

where $\mathcal{G} = \gamma\left(\lambda+\frac{1}{2},\frac{\lambda}{\beta}r^2\right)$.

4 MLE for the mNR distribution

In this section, we apply the method of the MLE, which is a standard and most widely used approach, to estimate the parameters of the mNR distribution. The PDF of the mNR distribution presented in equation (6) includes four parameters: λ, β, δ, and θ. The corresponding log-likelihood function is presented as follows:

$$\begin{aligned} \ln\mathcal{L}(\boldsymbol{t},\Theta) =& \sum_{i=1}^{n}\ln\left\{\frac{2\theta\lambda^\lambda}{\Gamma(\lambda)\beta^\lambda}t_i^{2\lambda-1}\exp\left(\frac{-\lambda}{\beta}t_i^2\right)\right. \\ &\left.+(1-\theta)\frac{t_i}{\delta^2}\exp\left(-\frac{t_i^2}{2\delta^2}\right)\right\}, \end{aligned} \tag{40}$$

where $\boldsymbol{t} = (t_1, t_2, t_3, ..., t_n)$ and $\Theta = (\lambda, \beta, \delta, \theta)$. Let $\bar{l} = \ln L(\mathbf{t}, \Theta)$, and

$$\mathcal{E}_i = \exp\left(-\frac{t_i^2}{2\delta^2}\right), \quad \mathcal{F}_i = \exp\left(-\frac{\lambda t_i^2}{\beta}\right). \tag{41}$$

By differentiating the log-likelihood function, shown in (40), with respect to each parameter (λ, β, δ, θ) and setting the resulting derivatives to zero, we obtain the system of equations as follows:

$$
\begin{aligned}
\frac{\partial \bar{l}}{\partial \theta} =& \sum_{i=1}^{n} \frac{1}{\frac{2\theta\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}} t_i^{2\lambda-1}\mathcal{F}_i + (1-\theta)\frac{t_i}{\delta^2}\mathcal{E}_i} \\
& \times \left(\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}} t_i^{2\lambda-1}\mathcal{F}_i - \frac{t_i}{\delta^2}\mathcal{E}_i \right) = 0, \\
\frac{\partial \bar{l}}{\partial \lambda} =& \sum_{i=1}^{n} \frac{1}{\frac{2\theta\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}} t_i^{2\lambda-1}\mathcal{F}_i + (1-\theta)\frac{t_i}{\delta^2}\mathcal{E}_i} \\
& \times \left(- \frac{2\theta \left(\frac{\lambda}{\beta}\right)^{\lambda} t_i^{2\lambda-1}\mathcal{F}_i \Psi(\lambda)}{\Gamma(\lambda)} \right. \\
& + \frac{2\theta \left(\frac{\lambda}{\beta}\right)^{\lambda} t_i^{2\lambda-1} \left(\ln\left(\frac{\lambda}{\beta}\right) + 1\right)\mathcal{F}_i}{\Gamma(\lambda)} \\
& + \frac{4\theta \left(\frac{\lambda}{\beta}\right)^{\lambda} \ln(t_i) t_i^{2\lambda-1}\mathcal{F}_i}{\Gamma(\lambda)} \qquad (42) \\
& \left. - \frac{2\theta \left(\frac{\lambda}{\beta}\right)^{\lambda} t_i^{2\lambda+1}\mathcal{F}_i}{\Gamma(\lambda)\beta} \right) = 0, \\
\frac{\partial \bar{l}}{\partial \beta} =& \sum_{i=1}^{n} \frac{1}{\frac{2\theta\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}} t_i^{2\lambda-1}\mathcal{F}_i + (1-\theta)\frac{t_i}{\delta^2}\mathcal{E}_i} \\
& \times \left(\frac{2\,\theta\,\lambda^{\lambda+1}\mathcal{F}_i t_i^{2\,\lambda-1}}{\Gamma\left(\lambda\right)\beta^{\lambda+1}} \left(\frac{t_i^2}{\beta} - 1 \right) \right) = 0, \\
\frac{\partial \bar{l}}{\partial \delta} =& \sum_{i=1}^{n} \frac{1}{\frac{2\theta\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}} t_i^{2\lambda-1}\mathcal{F}_i + (1-\theta)\frac{t_i}{\delta^2}\mathcal{E}_i} \\
& \times \left((1-\theta)\frac{t_i^3}{\delta^5}\mathcal{E}_i - 2(1-\theta)\frac{t_i}{\delta^3}\mathcal{E}_i \right) = 0,
\end{aligned}
$$

where $\mathcal{E}_i$ and $\mathcal{F}_i$ are defined as (41). $\Psi(\lambda)$ is the digamma function, denoted by

$$\Psi(\lambda) = \frac{d}{d\lambda}\ln(\Gamma(\lambda)) = \frac{\frac{d}{d\lambda}\Gamma(\lambda)}{\Gamma(\lambda)}.$$

However, solving the system equations (42) is challenging due to its complexity. This system is difficult to solve because it involves both gamma and logarithmic function terms. Therefore, a closed-form solution for the MLE does not exist. For this reason, it is necessary to use numerical methods such as the Newton-Raphson algorithm, the Fisher scoring algorithm, the expectation-maximization algorithm, the Quasi-Newton methods, or other iterative techniques, [35], [36].

5 Numerical simulations

In this section, we present a numerical study using R version 4.4.1, [37].

5.1 Graphical representations

In this subsection, we visualize the shape of the significant functions of the new distribution presented in Section 3. These graphical shapes illustrate the variation in the four distribution parameters: λ, β, δ, and θ. The two-dimensional (2D) graphical simulations are presented under different parameter values. Additionally, contour plots are simulated to explore the dynamical behavior of these parameters over time. The baseline parameters used for the plot are: $\lambda = 2.5, \theta = 0.5, \delta = 0.8$ and $\beta = 1$. The PDF, CDF, survival function, and hazard function of the mNR distribution are plotted in Figure 1 (Appendix), Figure 2 (Appendix), Figure 3 (Appendix), and Figure 4 (Appendix), respectively.

5.2 Parameters simulation

In this subsection, numerical simulations of the parameter estimation are performed to demonstrate the accuracy of the parameter estimation of the new distribution using the MLE presented in Section 4. Synthetic data were generated using Monte Carlo simulation, [38]. The parameters estimation will be implemented via the *maxLik* package using the Broyden–Fletcher–Goldfarb–Shanno (BFGS) optimization algorithm, [39], [40].

Six different parameter sets, shown in Table 1, are simulated to systematically explore the distribution's behavior. The analysis begins with baseline parameter Set 1: $\{\theta = 0.5, \lambda = 2.0, \beta = 2.0, \delta = 4.0\}$. From this baseline, the impact of the mixing weight θ is examined by increasing its value to 0.8. This gives Set 2: $\{\theta = 0.8, \lambda = 2.0, \beta = 2.0, \delta = 4.0\}$. Similarly, decreasing θ to 0.2 defines Set 3: $\{\theta = 0.2, \lambda = 2.0, \beta = 2.0, \delta = 4.0\}$. To study the changes in component characteristics, λ is adjusted to 1.0 in Set 4: $\{\theta = 0.5, \lambda = 1.0, \beta = 2.0, \delta = 4.0\}$. Next, we investigate the effect of δ by setting δ to 2.0; Set 5: $\{\theta = 0.5, \lambda = 2.0, \beta = 2.0, \delta = 2.0\}$. Finally, Set 6 is used to study the impact of β by adjusting β to 4.0; Set 6: $\{\theta = 0.5, \lambda = 2.0, \beta = 4.0, \delta = 4.0\}$. The sample sizes considered are $n = 25, 50, 100, 500, 1000$. A total of 1,000 replications were performed for each scenario to evaluate the influence of sample size on the estimated parameters: $\hat{\lambda}$, $\hat{\beta}$, $\hat{\delta}$ and $\hat{\theta}$.

The parameter simulation results are summarized in Table 2 (Appendix). The

Table 1. Parameter sets for simulation study

Set	θ	λ	β	δ
1	0.5	2.0	2	4
2	0.8	2.0	2	4
3	0.2	2.0	2	4
4	0.5	1.0	2	4
5	0.5	2.0	2	2
6	0.5	2.0	4	4

table compares the fixed parameters against the estimated values using three performance metrics: Mean Squared Error (MSE), Variance (Var), and Bias. From Table 2 (Appendix), it is observed that lower sample sizes (n) result in higher MSE and variance. When n increases, both MSE and variance decrease, indicating that the sample size significantly impacts the estimation's performance. Additionally, the bias is presented to further validate the quality of the estimator. Additionally, plots for the Bias, MSE, and variance are presented in Figure 5 (Appendix), Figure 6 (Appendix), and Figure 7 (Appendix), respectively.

6 Conclusions and future work

In this study, we propose a new mixture Nakagami-Rayleigh distribution that differs from previous research. The mathematical properties of the proposed distributions are also presented and their validity is verified. Additionally, the MLE for estimating the parameters of the new distribution is provided. Furthermore, we conducted a simulation study using varying sample sizes to demonstrate the performance of the MLE method. Since the closed-form expressions of the estimated parameters cannot be obtained, this study employs numerical methods for parameter estimation using the *maxLik* package in R. The results from the simulation show that when the number of samples increases, the MLE estimation method has better performance according to the sample size. In addition, the significant functions of the mNR distribution, as PDF, CDF, survival function, and hazard function, are plotted as 2D and contour plots.

A limitation of this study is that it does not include an application with real-world data. Therefore, a primary direction for future research is to apply this distribution to various datasets, such as those in the engineering, medical and agricultural fields. Furthermore, future research could employ other estimation methods such as Bayesian methods, [41], method of moments, [42], and robust estimation methods, [43].

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Conflicts of Interest

The authors have no conflicts of interest to declare.

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APPENDIX

Figures and table mentioned in section 5 are presented in this section.

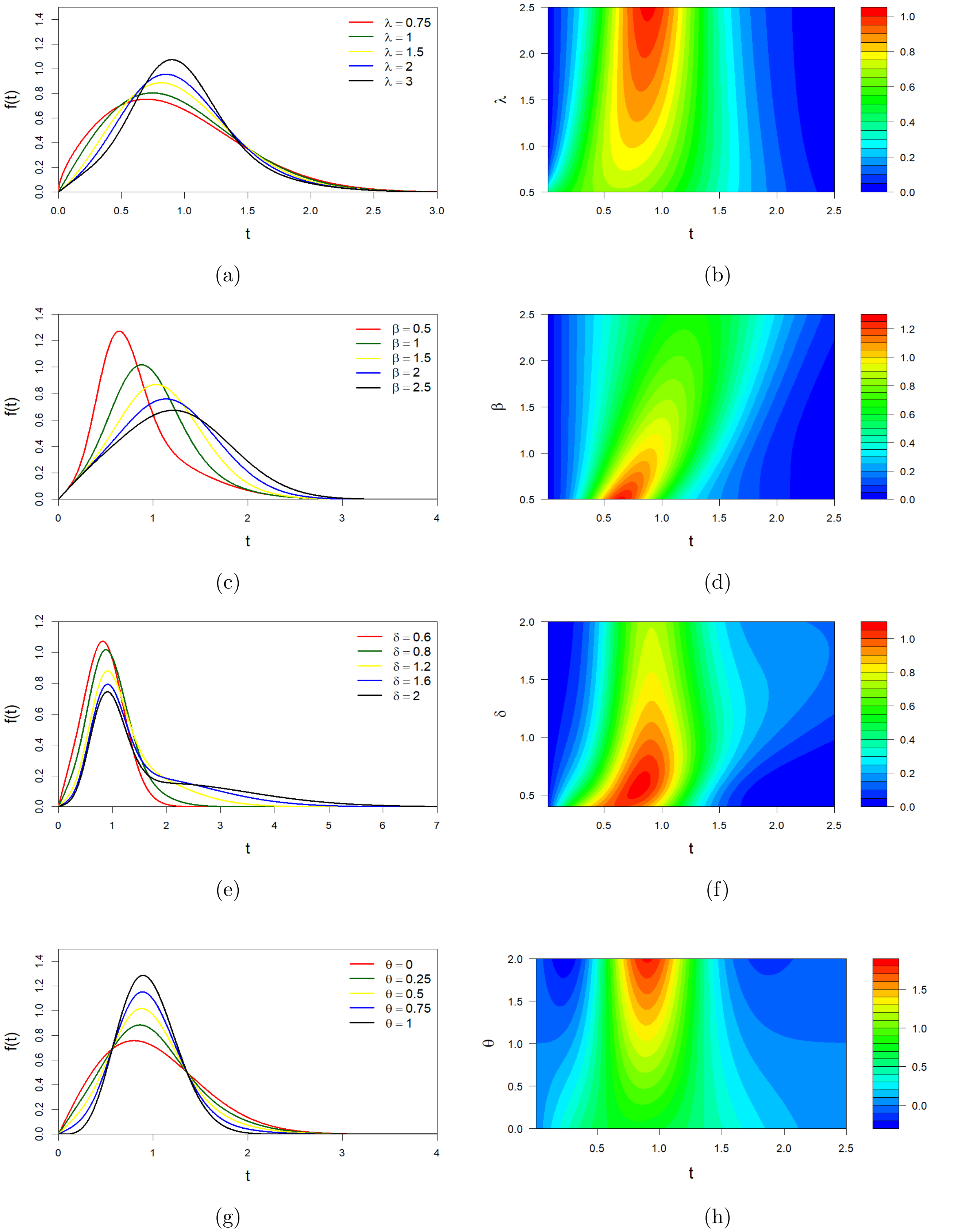

Fig. 1: Visualization of the CDF of the mNR distribution presented in (6). Left panels (a, c, e, g) show line plots for selected parameter values. Right panels (b, d, f, h) show the contour plots.

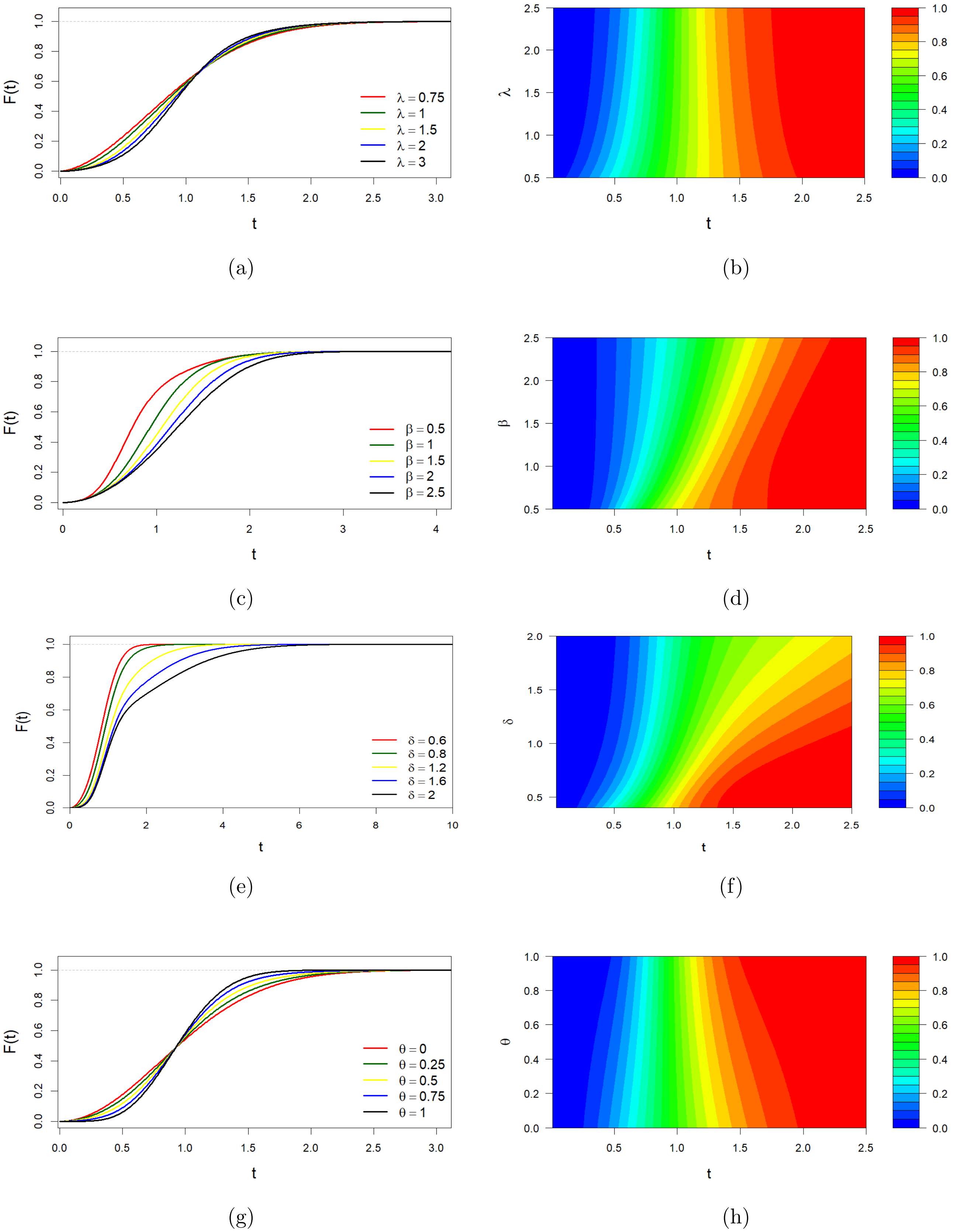

Fig. 2: Visualization of the CDF of the mNR distribution presented in (9). Left panels (a, c, e, g) show line plots for selected parameter values. Right panels (b, d, f, h) show the contour plots.

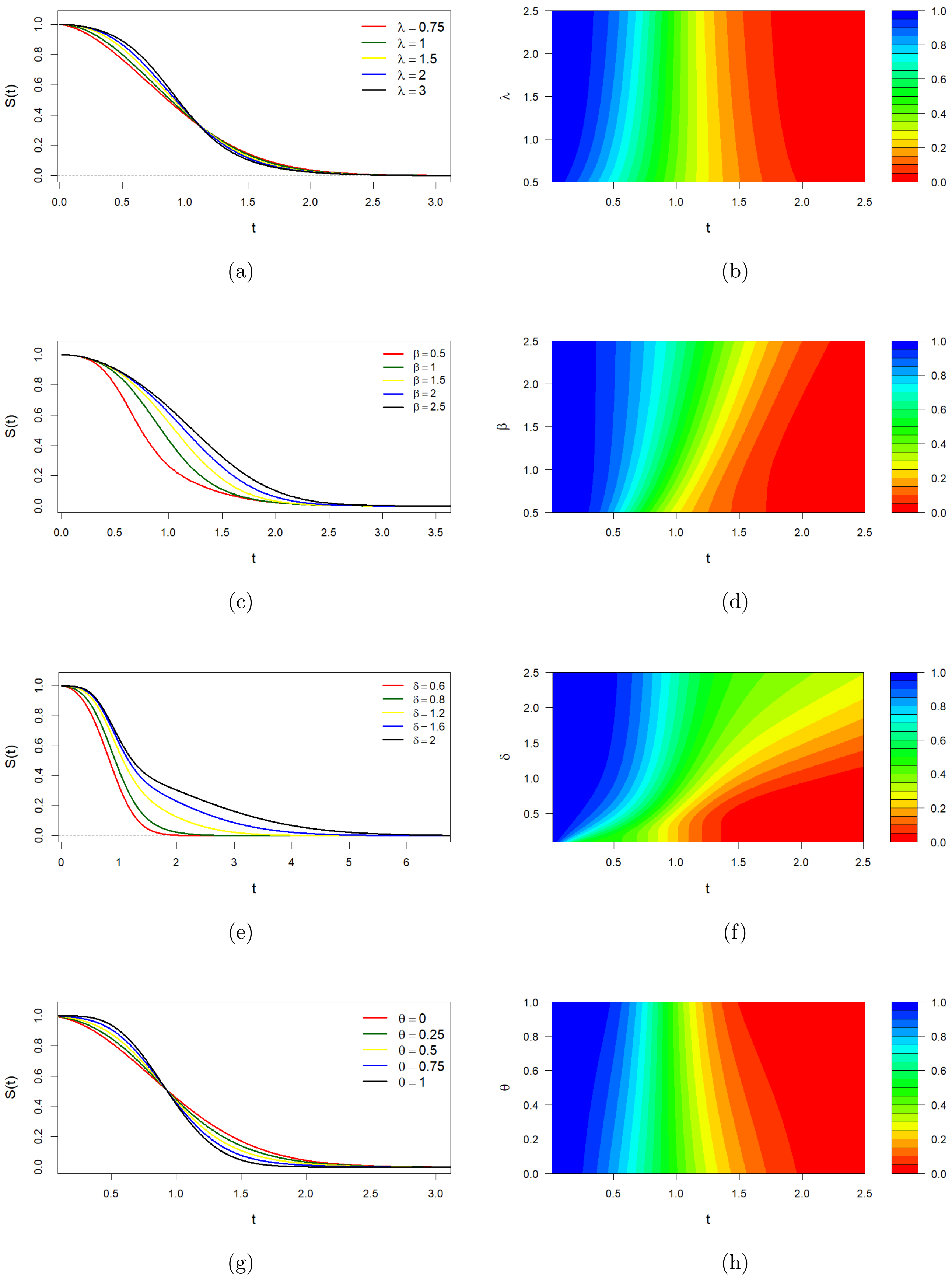

Fig. 3: Visualization of the survival function for the mNR distribution presented in (11). Left panels (a, c, e, g) show line plots for selected parameter values. Right panels (b, d, f, h) show the contour plots.

DOI: 10.37394/23202.2026.25.28

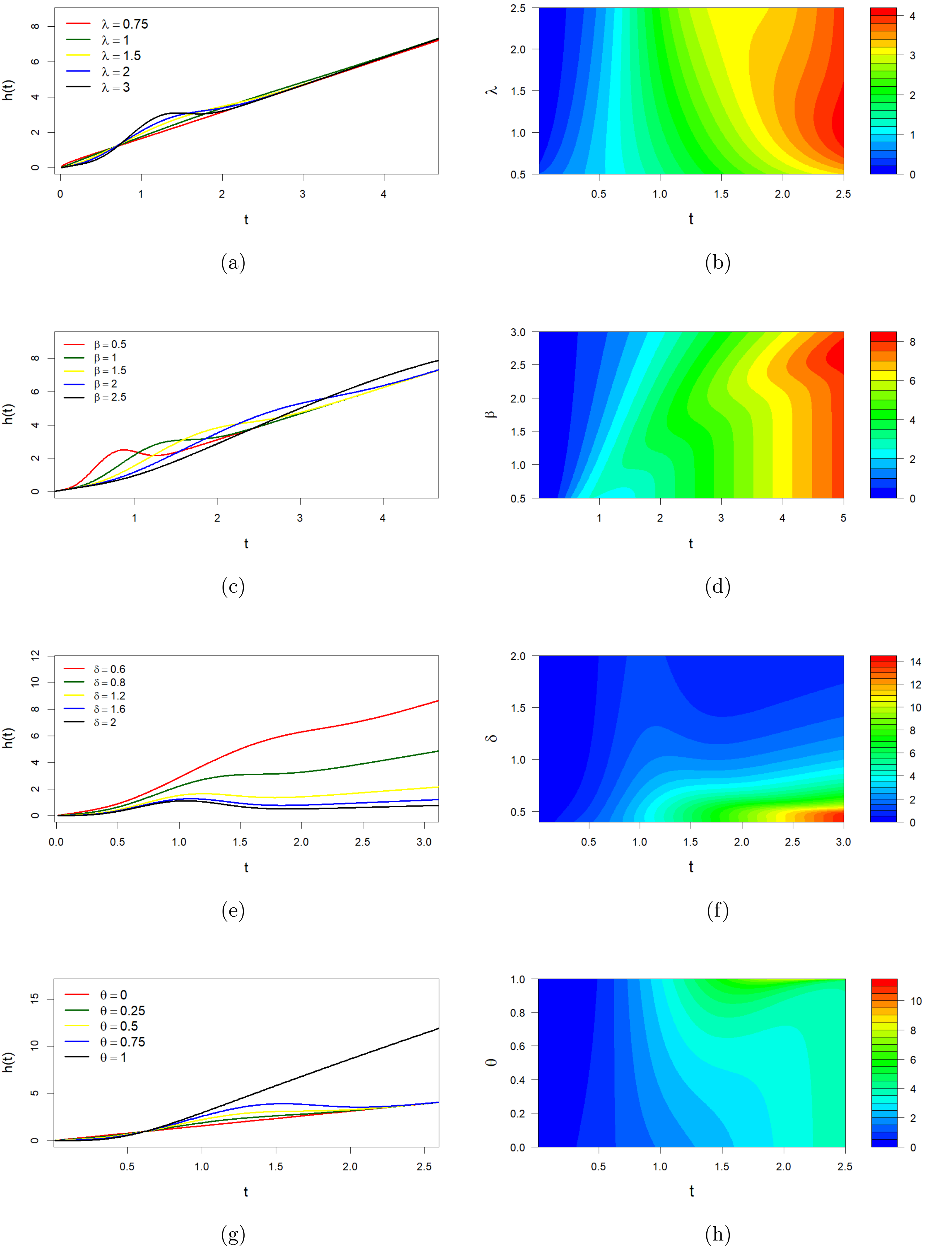

Fig. 4: Visualization of the hazard function for the mNR distribution presented in (13). Left panels (a, c, e, g) show line plots for selected parameter values. Right panels (b, d, f, h) show the contour plots.

Table 2. Simulation Results

n	Parameters				Estimated Parameters				Bias				MSE				Variance			
	θ	λ	β	δ	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\delta}$	Bias($\hat{\theta}$)	Bias($\hat{\lambda}$)	Bias($\hat{\beta}$)	Bias($\hat{\delta}$)	MSE($\hat{\theta}$)	MSE($\hat{\lambda}$)	MSE($\hat{\beta}$)	MSE($\hat{\delta}$)	Var($\hat{\theta}$)	Var($\hat{\lambda}$)	Var($\hat{\beta}$)	Var($\hat{\delta}$)
25	0.5	2	2	4	0.498681	4.016259	2.006856	3.993806	-0.001319	2.016259	0.006856	-0.006194	0.017175	90.157974	0.495067	0.443453	0.017190	86.178851	0.495515	0.443858
	0.8	2	2	4	0.784662	2.607722	1.965614	3.888866	-0.015338	0.607722	-0.034386	-0.111134	0.011166	2.828222	0.158114	1.192718	0.010942	2.461358	0.157088	1.181549
	0.2	2	2	4	0.222138	13.391576	2.151134	4.050681	0.022138	11.391576	0.151134	0.050681	0.015319	1215.610000	2.531666	0.235378	0.014844	1086.929000	2.511336	0.233042
	0.5	1	2	4	0.496852	2.263114	2.018436	3.965211	-0.003149	1.263114	0.018436	-0.034789	0.022225	52.448865	1.429312	0.545519	0.022238	50.904311	1.430403	0.544853
	0.5	2	2	2	0.510172	8.812698	2.009374	2.031594	0.010172	6.812698	0.009373	0.031594	0.052612	629.092827	0.528665	0.180847	0.052561	583.263242	0.529107	0.180028
	0.5	2	4	4	0.490466	6.267608	3.964138	4.023291	-0.009534	4.267608	-0.035862	0.023291	0.029771	302.121591	2.174933	0.564584	0.029710	284.193304	2.175822	0.564606
50	0.5	2	2	4	0.501384	2.453323	2.034410	3.996048	0.001384	0.453323	0.034410	-0.003952	0.008490	2.241476	0.198097	0.211873	0.008497	2.038012	0.197110	0.212069
	0.8	2	2	4	0.795820	2.210873	1.991961	3.977339	-0.004180	0.210873	-0.008039	-0.022661	0.004760	0.437130	0.074274	0.612162	0.004748	0.393055	0.074284	0.612260
	0.2	2	2	4	0.216275	8.810716	2.177812	4.043126	0.016274	6.810716	0.177812	0.043126	0.009918	784.367300	1.592000	0.138770	0.009663	738.720100	1.561945	0.137047
	0.5	1	2	4	0.503889	1.192697	2.079098	4.017503	0.003889	0.192697	0.079098	0.017503	0.012854	0.276052	0.945343	0.299128	0.012852	0.239159	0.940027	0.299121
	0.5	2	2	2	0.488965	6.411332	2.013806	2.034330	-0.011035	4.411332	0.013806	0.034330	0.040312	333.412346	0.409625	0.127763	0.040231	314.266759	0.409844	0.126711
	0.5	2	4	4	0.495428	3.010650	4.051626	4.006077	-0.004572	1.010650	0.051626	0.006077	0.015543	15.832922	1.189100	0.293772	0.015538	14.826336	1.187622	0.294029
100	0.5	2	2	4	0.502926	2.189981	2.022606	4.014711	0.002926	0.189981	0.022606	0.014711	0.004472	0.425593	0.095695	0.115146	0.004467	0.389890	0.095280	0.115044
	0.8	2	2	4	0.797447	2.093073	2.003154	4.010650	-0.002553	0.093073	0.003154	0.010649	0.002491	0.147164	0.035737	0.260629	0.002487	0.138640	0.035763	0.260777
	0.2	2	2	4	0.208612	3.806254	2.150761	4.020819	0.008612	1.806254	0.150761	0.020819	0.005689	45.941075	1.133880	0.076927	0.005620	42.721243	1.112263	0.076570
	0.5	1	2	4	0.503859	1.085657	2.025447	4.006181	0.003859	0.085657	0.025447	0.006181	0.005675	0.080031	0.334631	0.132229	0.005666	0.072766	0.334318	0.132323
	0.5	2	2	2	0.493655	3.903515	2.015945	2.026510	-0.006345	1.903515	0.015945	0.026510	0.022914	289.928129	0.226097	0.068414	0.022897	286.591350	0.226069	0.067779
	0.5	2	4	4	0.500547	2.324956	4.043499	4.019770	0.000547	0.324956	0.043499	0.019770	0.007727	1.256011	0.572270	0.149136	0.007735	1.151567	0.570949	0.148894
500	0.5	2	2	4	0.501374	2.027836	2.002985	4.000087	0.001374	0.027836	0.002985	0.000087	0.000850	0.049032	0.015667	0.020678	0.000849	0.048306	0.015674	0.020699
	0.8	2	2	4	0.801041	2.008547	2.001928	4.007831	0.001041	0.008547	0.001928	0.007831	0.000463	0.022832	0.006688	0.053390	0.000462	0.022782	0.006691	0.053382
	0.2	2	2	4	0.201122	2.175640	2.017443	4.003029	0.001122	0.175640	0.017443	0.003029	0.000840	0.347803	0.086754	0.013297	0.000840	0.317270	0.086536	0.013301
	0.5	1	2	4	0.500128	1.014543	2.009731	4.000833	0.000128	0.014543	0.009731	0.000833	0.001008	0.010087	0.047557	0.020462	0.001009	0.009885	0.047510	0.020482
	0.5	2	2	2	0.502209	2.081701	2.002548	2.006246	0.002209	0.081702	0.002548	0.006246	0.004569	0.144727	0.034343	0.010768	0.004569	0.138190	0.034371	0.010740
	0.5	2	4	4	0.501522	2.042887	4.005432	4.002578	0.001522	0.042887	0.005432	0.002578	0.001497	0.070698	0.091683	0.026188	0.001496	0.068927	0.091745	0.026208
1000	0.5	2	2	4	0.499024	2.015715	1.998391	3.998743	-0.000976	0.015715	-0.001609	-0.001257	0.000396	0.021900	0.007756	0.009640	0.000395	0.021675	0.007762	0.009648
	0.8	2	2	4	0.799122	2.001775	1.998358	4.004580	-0.000878	0.001775	-0.001642	0.004580	0.000254	0.011185	0.003483	0.025126	0.000253	0.011193	0.003483	0.025130
	0.2	2	2	4	0.200786	2.074218	2.007621	3.998760	0.000786	0.074217	0.007621	-0.001240	0.000478	0.124794	0.042285	0.006184	0.000478	0.119405	0.042270	0.006189
	0.5	1	2	4	0.500894	1.005566	2.004274	3.999754	0.000894	0.005566	0.004274	-0.000246	0.000517	0.004707	0.024388	0.011121	0.000517	0.004680	0.024395	0.011132
	0.5	2	2	2	0.498226	2.038903	2.004735	2.000523	-0.001774	0.038903	0.004735	0.000523	0.002085	0.052289	0.016447	0.005039	0.002084	0.050827	0.016441	0.005044
	0.5	2	4	4	0.498717	2.019388	4.003779	3.998894	-0.001283	0.019388	0.003779	-0.001106	0.000708	0.031001	0.044744	0.012777	0.000707	0.030656	0.044775	0.012788

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Ibrahim Abdullahi, Chalermrat Nontapa

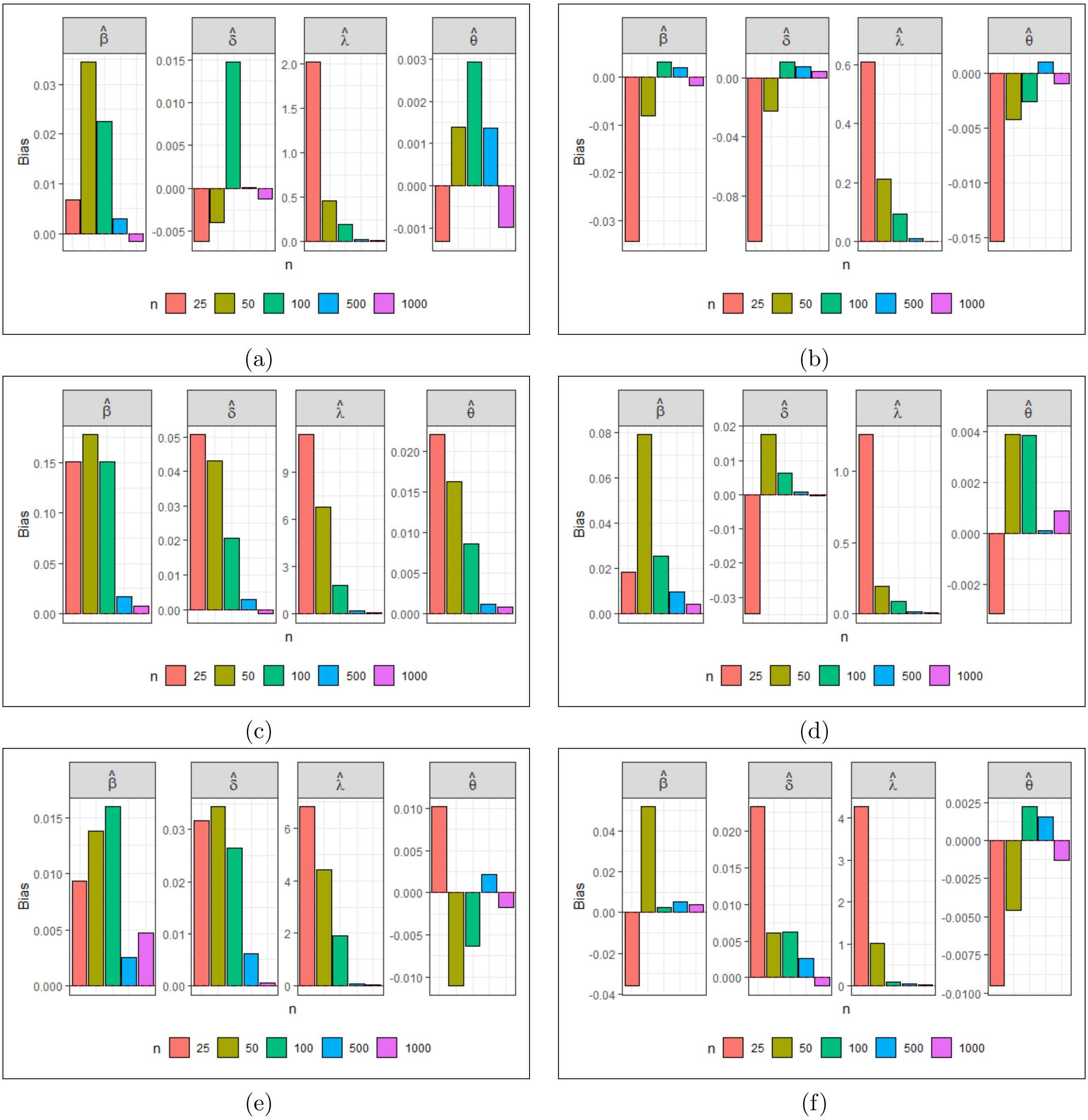
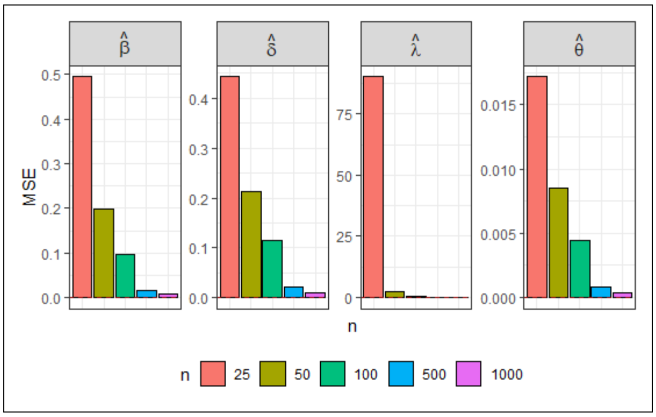
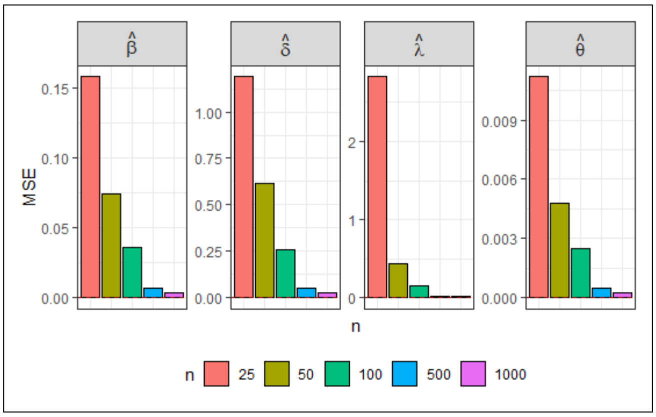


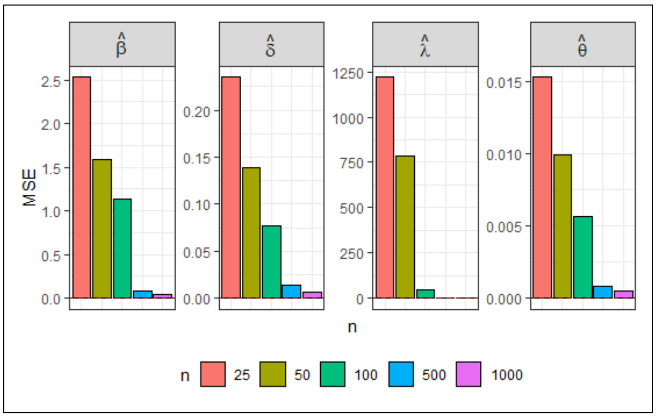
Fig. 5: Bias of the estimated parameters $\hat{\lambda}$, $\hat{\beta}$, $\hat{\delta}$ and $\hat{\theta}$ for the six different parameter sets: (a) Set 1, (b) Set 2, (c) Set 3, (d) Set 4, (e) Set 5, (f) Set 6. Within each set, the bias is shown for various sample sizes (n).



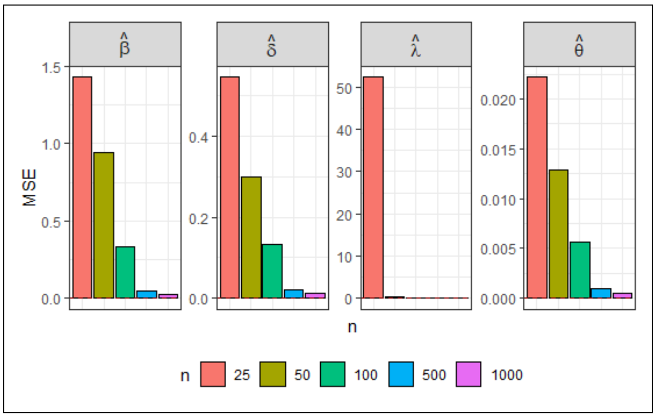
(a)



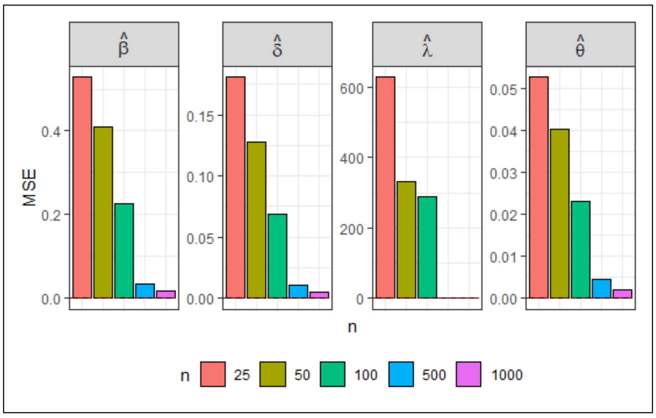
(b)



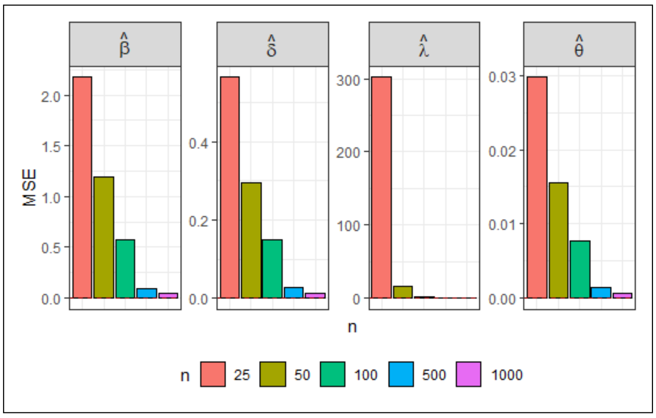
(c)



(d)

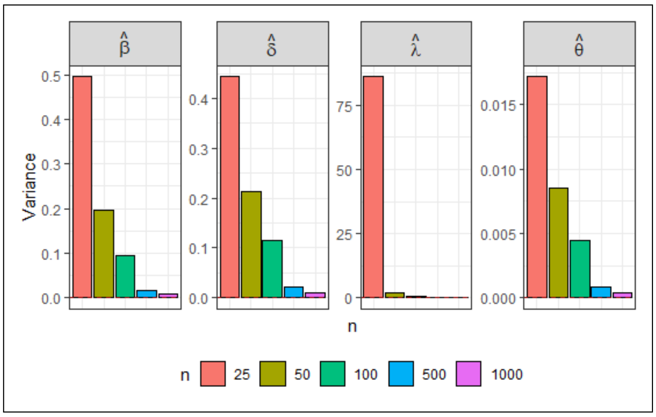


(e)

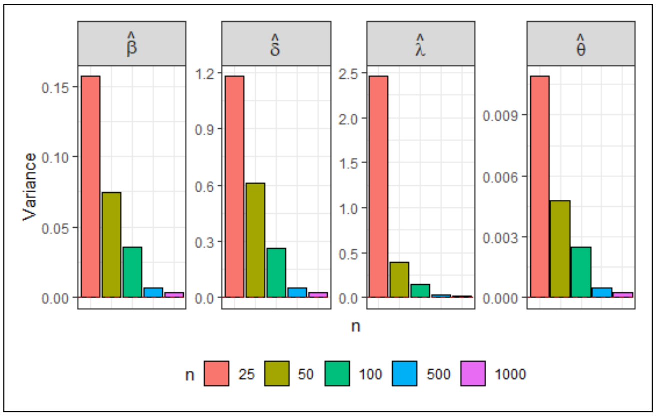


(f)

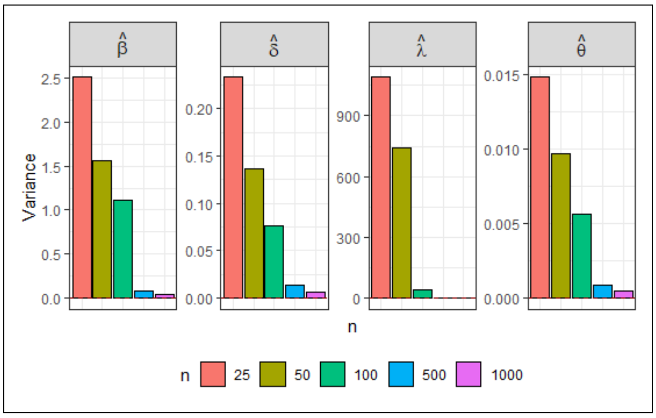
Fig. 6: MSE of the estimated parameters $\hat{\lambda}$, $\hat{\beta}$, $\hat{\delta}$ and $\hat{\theta}$ for the six different parameter sets: (a) Set 1, (b) Set 2, (c) Set 3, (d) Set 4, (e) Set 5, (f) Set 6. Within each set, the bias is shown for various sample sizes (n).



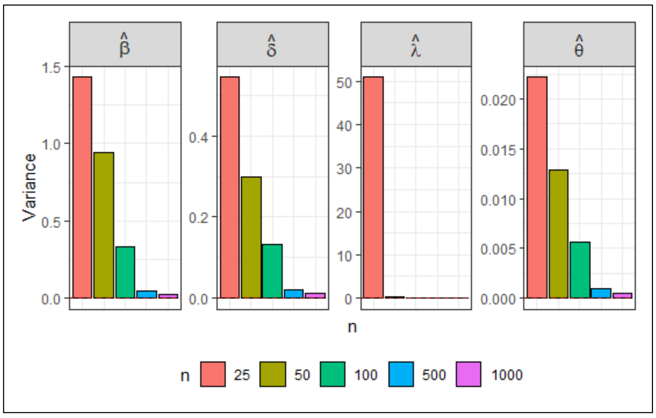
(a)



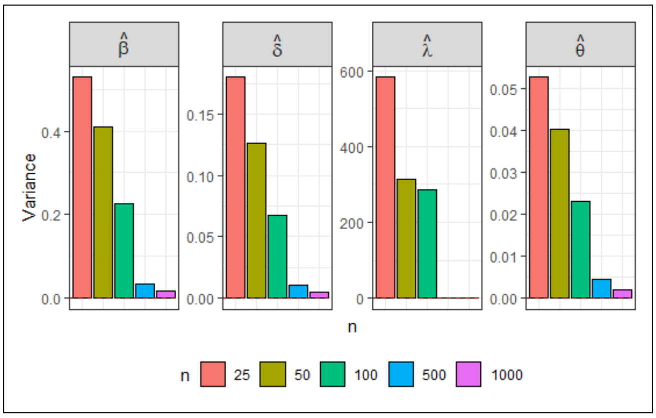
(b)



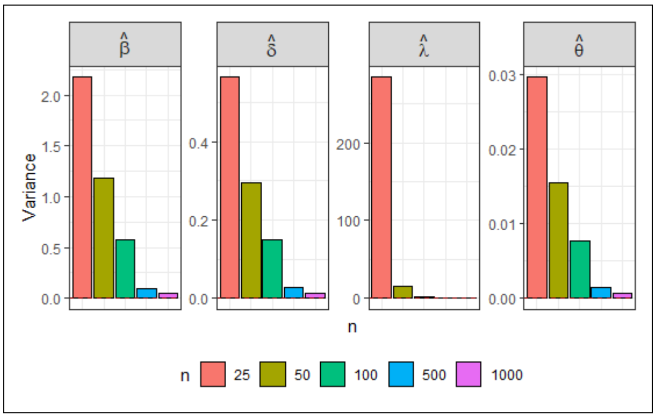
(c)



(d)



(e)



(f)

Fig. 7: Variance of the estimated parameters $\hat{\lambda}$, $\hat{\beta}$, $\hat{\delta}$ and $\hat{\theta}$ for the six different parameter sets: (a) Set 1, (b) Set 2, (c) Set 3, (d) Set 4, (e) Set 5, (f) Set 6. Within each set, the bias is shown for various sample sizes (n).