New Conditions for Stability of Neutral Systems With Multiple Delays

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Abstract: - This paper aims to make some contributions to the stability problems associated with linear neutral systems comprising multiple time delays in the states together with involving multiple neutral delays in time derivatives of states of system. We first develop an appropriate Lyapunov functional, and then derive new global asymptotic stability conditions for such a model of systems with the help of this functional. The proposed stability results basically depend on the constraint conditions that are imposed on the components of constant system matrices without involving delay parameters. It is also shown that the obtained stability conditions improve some formerly published corresponding stability conditions for the same model of neutral systems. This work also studies a comparative numerical example to indicate advantage of the results of this paper over the past results and demonstrate the applicability of the derived stability criteria of the algebraic forms.

Key-Words: - Neutral Systems, Time Delays, Neutral Delays, Algebraic Stability Criteria, Delay-Independent Criteria, Lyapunov–Krasovskii Method, Global Asymptotic Stability, Lyapunov Functionals, M-Matrices.

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1 Introduction

Recently, many research articles have intensively dealt with the analysis of the various dynamical properties of neutral-type systems, that possess delay parameters both in states and in time derivatives of states. The functional differential equations in the forms of neutral systems are sometimes employed to model various types of physical processes such as flexible systems, VLSI (Very Large Scale Integration) systems, partial element equivalent circuits and population ecology [1], [2], [3], [4], [5]. It should be pointed out here that stability is one of the major issues in analysis of dynamical properties of neutral-type systems. It is a fact that stability analysis in neutral systems is mainly carried out by exploiting appropriate Lyapunov functional candidates. The derivation of stability conditions using standard Lyapunov theorems completely depends on the mathematical model of the system. In particular, when establishing the stability of these systems, the way of introduction of the delay components into dynamical system equations plays a critical role in derivation of stability conditions associated with the system parameters of considered neutral systems.

In the past literature, a considerable number of research articles have essentially focused on different stability analyses of neutral system possessing a single time delay argument and a single neutral delay argument [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]. In these papers, different Lyapunov functionals of the quadratic forms have been combined with some useful mathematical techniques to derive various sets of sufficient conditions guaranteeing global asymptotic stability of such model of neutral systems. Stability of such neutral systems, that basically involve n time delay components and n neutral delay components, where n denotes the number of states possessed by the neutral system, has also been analyzed [20], [21], [22], [23], [24], [25]. In these papers, the quadratic-type Lyapunov functionals have been utilized to determine various sets of sufficient conditions guaranteeing global asymptotic stability of this model of neutral systems. We should mention that quadratic-type Lyapunov functionals usually involve a main term which is a linear combination of the squares of the system states.

In case of a neutral system possessing n^2 time delays and n^2 neutral delays, the mathematical statement of neutral systems turns into a very complicated form, and stability analysis of neutral systems becomes a very complicated and difficult task to achieve. Such classes of neutral systems require exploiting more advanced mathematical methods and techniques to be able to achieve a suitable stability analysis of equilibrium points. In the past literature, two papers have addressed stability issue of neutral systems including n^2 time delay terms in states and n^2 neutral delay terms in time derivatives of states. In [26], a quadratic-type Lyapunov functional has been utilized to present sufficient criteria regarding global stability of this model of neutral systems. In [27], a Lyapunov functional that involves a main term which is a linear combination of the absolute values of system states is utilized. This Lyapunov functional candidate has been used for stability analysis of such neutral systems only in [27]. In our paper, we propose a novel Lyapunov functional, whose main term is also a linear combination of the absolute values of system states, and obtain new conditions for global asymptotic stability of neutral systems admitting n^2 time delay terms and n^2 neutral delay terms. We show that our stability conditions generalize the stability conditions presented in [27].

Notations: This paper considers the following notations for vectors and matrices: $P = diag(p_i > 0)$ will represent a positive diagonal matrix. For a real matrix $A = (a_{ij})_{n\times n}$, $|A|$ is the matrix that has form $|A| = (|a_{ij}|)_{n\times n}$. For a real vector $x = (x_1, x_2, ..., x_n)^T$, the following norm is defined

$$||x||_1 = \sum_{i=1}^{n} |x_i|$$

2 Stability Analysis of Neutral Systems with Multiple Delay Terms

The current section aims to propose some stability conditions for neutral systems that mainly involve multiple delay terms. We consider the neutral system:

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}x_j(t) + \sum_{j=1}^{n} b_{ij}x_j(t-\tau_{ij}) + \sum_{j=1}^{n} c_{ij}\dot{x}_j(t-\zeta_{ij}), \quad i = 1,\dots,n. \tag{1}$$

where $x_i(t)$ denotes the i-th state variable of the system, a_{ij}, b_{ij} and c_{ij} are some real constants, τ_{ij} and ζ_{ij} are the delay components. Define $\iota = \max\{\tau,\zeta\}$ with $\tau = \max\{\tau_{ij}\}$ and $\zeta = \max\{\zeta_{ij}\}$, $\forall i, j$. System (1) possesses the initial conditions $x_i(t) = \theta_i(t)$ and $\dot{x}_i(t) = \rho_i(t)$ in $C([-\iota, 0], \mathbb{R})$. $C([-\iota, 0], \mathbb{R})$ is defined to consist of continuous real functions which are from $[-\iota, 0]$ to $\mathbb{R}$.

The system matrices associated with (1) are usually defined by $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $C = (c_{ij})_{n\times n}$. It may be of interest to indicate that the parameters τ_{ij} and ζ_{ij} are respectively referred to as time delays and neutral delays of the system.

In this paper, the stability conditions to be proposed for system (1) will make use of the properties of M-matrices. In this context, we first give some useful properties associated with nonsingular M-matrices.

Lemma 1 ([28]). Consider the real matrix $S = (s_{ij})_{n\times n}$ with the elements $s_{ii} > 0$ and $s_{ij} \le 0$ for $i \ne j$. S is said to be a nonsingular M-matrix if S possesses all positive real parted eigenvalues.

Lemma 2 ([28]). If a real matrix S is a nonsingular M-matrix, then there exist constants $p_i > 0$ such that

$$p_i s_{ii} - \sum_{\substack{j=1\\ j\ne i}}^{n} p_j|s_{ji}| > 0, \quad i = 1,\dots,n.$$

The key goal of our work is to obtain new conditions that assure global asymptotic stability of system expressed in (1). A set of criteria proving stability of system (1) is derived as given below:

Theorem 1. Define the matrix $S = (s_{ij})_{n\times n}$ such that $s_{ii} = -a_{ii} - |b_{ii}| > 0$ and $s_{ij} = -|a_{ij}| - |b_{ij}|$ for $i \ne j, (i, j = 1,\dots,n)$. Then, system (1) is globally asymptotically stable if there exist constants $p_i > 0$ such that the following criteria are satisfied:

$$\begin{aligned}\alpha_i =& p_i s_{ii} - \sum_{\substack{j=1\\ j\ne i}}^{n} p_j|s_{ji}| \\ =& p_i(-a_{ii} - |b_{ii}|) - \sum_{\substack{j=1\\ j\ne i}}^{n} p_j(|a_{ji}| + |b_{ji}|) > 0\end{aligned}$$

and

$$\beta_i = p_i - \sum_{j=1}^{n} p_j|c_{ji}| > 0, \quad \forall i.$$

Proof. A Lyapunov functional that involves two major terms will be constructed:

$$V(t) = V_1(t) + V_2(t) \tag{2}$$

We will define $V_1(t)$ and $V_2(t)$ in (2) as follows

$$V_1(t) = \sum_{i=1}^{n}\Big(p_i sgn(x_i(t))x_i(t) - p_i sgn(\dot{x}_i(t))x_i(t) + k sgn(x_i(t))x_i(t)\Big) \quad (3)$$

and

$$V_2(t) = \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\int_{t-\tau_{ij}}^{t} p_i|b_{ij}||x_j(\varrho)|d\varrho + \int_{t-\zeta_{ij}}^{t} p_i|c_{ij}||\dot{x}_j(\varphi)|d\varphi + \epsilon\int_{t-\tau_{ij}}^{t}|x_j(\varrho)|d\varrho + \varepsilon\int_{t-\zeta_{ij}}^{t}|\dot{x}_j(\varphi)|d\varphi\Big) \quad (4)$$

where k, ϵ and ε are some sufficiently small positive numbers, whose exact numerical values will be determined later.

Since $V_2(t)$, given by (4), is a nonnegative functional, it can be observed that the Lyapunov functional $V(t)$, defined in (2), satisfies the condition

$$\begin{aligned} V(t) &\geq \sum_{i=1}^{n}\Big(p_i|x_i(t)| - p_i sgn(\dot{x}_i(t))x_i(t) + k|x_i(t)|\Big) \\ &\geq \sum_{i=1}^{n}\Big(p_i|x_i(t)| - p_i|x_i(t)| + k|x_i(t)|\Big) \\ &= k\,||x(t)||_1 \end{aligned} \quad (5)$$

Hence, it is apparent to conclude from (5) that $V(t)$ is positive definite, that simply assures that $V(t) > 0$ whenever $x(t) \neq 0$.

The Dini derivative of $V_1(t)$, expressed in (3), is determined to possess the form:

$$\begin{aligned} \dot{V}_1(t) &= \sum_{i=1}^{n}\Big(p_i sgn(x_i(t))\dot{x}_i(t) - p_i sgn(\dot{x}_i(t))\dot{x}_i(t) + k sgn(x_i(t))\dot{x}_i(t)\Big) \\ &= \sum_{i=1}^{n}\Big(p_i sgn(x_i(t))\dot{x}_i(t) - p_i|\dot{x}_i(t)| + k sgn(x_i(t))\dot{x}_i(t)\Big) \\ &\leq \sum_{i=1}^{n}\Big(p_i sgn(x_i(t))\dot{x}_i(t) - p_i|\dot{x}_i(t)| + k|\dot{x}_i(t)|\Big) \end{aligned} \quad (6)$$

The following equality is noted

$$\begin{aligned} &\sum_{i=1}^{n} p_i sgn(x_i(t))\dot{x}_i(t) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} p_i sgn(x_i(t))\Big(a_{ij}x_j(t) + b_{ij}x_j(t-\tau_{ij}) + c_{ij}\dot{x}_j(t-\zeta_{ij})\Big) \\ &= \sum_{i=1}^{n} p_i a_{ii}|x_i(t)| + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} p_i a_{ij} sgn(x_i(t))x_j(t) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} p_i b_{ij} sgn(x_i(t))x_j(t-\tau_{ij}) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} p_i sgn(x_i(t))c_{ij}\dot{x}_j(t-\zeta_{ij}) \end{aligned} \quad (7)$$

From (7), one can derive the inequality

$$\begin{aligned} &\sum_{i=1}^{n} p_i sgn(x_i(t))\dot{x}_i(t) \\ &\leq \sum_{i=1}^{n} p_i a_{ii}|x_i(t)| + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} p_i|a_{ij}||x_j(t)| \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} p_i|b_{ij}||x_j(t-\tau_{ij})| \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} p_i|c_{ij}||\dot{x}_j(t-\zeta_{ij})| \end{aligned} \quad (8)$$

Inserting (8) into (6) directly leads to

$$\begin{aligned} \dot{V}_1(t) &\leq \sum_{i=1}^{n} p_i a_{ii}|x_i(t)| + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} p_i|a_{ij}||x_j(t)| \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} p_i|b_{ij}||x_j(t-\tau_{ij})| - \sum_{i=1}^{n} p_i|\dot{x}_i(t)| \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} p_i|c_{ij}||\dot{x}_j(t-\zeta_{ij})| + k\sum_{i=1}^{n}|\dot{x}_i(t)| \end{aligned} \quad (9)$$

The time derivative of $V_2(t)$, defined by (4), is determined to have the form

$$\dot{V}_2(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} p_i\Big(|b_{ij}||x_j(t)| - |b_{ij}||x_j(t-\tau_{ij})| + |c_{ij}||\dot{x}_j(t)| - |c_{ij}||\dot{x}_j(t-\zeta_{ij})|\Big) + \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\epsilon|x_j(t)| - \epsilon|x_j(t-\tau_{ij})| + \varepsilon|\dot{x}_j(t)| - \varepsilon|\dot{x}_j(t-\zeta_{ij})|\Big) \quad (10)$$

Combining (10) with (9) yields

$$\begin{aligned}
\dot{V}(t) =\ & \dot{V}_1(t) + \dot{V}_2(t) \\
\le & \sum_{i=1}^{n}\Big(p_i a_{ii}|x_i(t)| + \sum_{\substack{j=1\\ j\ne i}}^{n} p_i|a_{ij}||x_j(t)| + \sum_{j=1}^{n} p_i|b_{ij}||x_j(t)| - p_i|\dot{x}_i(t)| + \sum_{j=1}^{n} p_i|c_{ij}||\dot{x}_j(t)| + k|\dot{x}_i(t)|\Big) \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\epsilon|x_j(t)| - \epsilon|x_j(t-\tau_{ij})| + \varepsilon|\dot{x}_j(t)| - \varepsilon|\dot{x}_j(t-\zeta_{ij})|\Big) \\
\le & \sum_{i=1}^{n}\Big(p_i a_{ii}|x_i(t)| + \sum_{\substack{j=1\\ j\ne i}}^{n} p_i|a_{ij}||x_j(t)| - p_i|\dot{x}_i(t)| + \sum_{j=1}^{n} p_i|b_{ij}||x_j(t)| + \sum_{j=1}^{n} p_i|c_{ij}||\dot{x}_j(t)| + k|\dot{x}_i(t)|\Big) \\
& + \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\epsilon|x_j(t)| + \varepsilon|\dot{x}_j(t)|\Big) \\
= & \sum_{i=1}^{n}\Big(p_i(a_{ii} + |b_{ii}|)|x_i(t)| + k|\dot{x}_i(t)| - p_i|\dot{x}_i(t)| + \sum_{\substack{j=1\\ j\ne i}}^{n} p_j(|a_{ji}| + |b_{ji}|)|x_i(t)| + \sum_{j=1}^{n} p_j|c_{ji}||\dot{x}_i(t)|\Big) \\
& + \epsilon n\sum_{i=1}^{n}|x_i(t)| + \varepsilon n\sum_{i=1}^{n}|\dot{x}_i(t)| \\
= & -\sum_{i=1}^{n}\Big(p_i(-a_{ii} - |b_{ii}|) - \sum_{\substack{j=1\\ j\ne i}}^{n} p_j(|a_{ji}| + |b_{ji}|)\Big)|x_i(t)| \\
& + \epsilon n\sum_{i=1}^{n}|x_i(t)| - \sum_{i=1}^{n}\Big(p_i - \sum_{j=1}^{n} p_j|c_{ji}|\Big)|\dot{x}_i(t)| + (k + \varepsilon n)\sum_{i=1}^{n}|\dot{x}_i(t)| \\
= & -\sum_{i=1}^{n}\Big(p_i s_{ii} - \sum_{\substack{j=1\\ j\ne i}}^{n} p_j|s_{ji}|\Big)|x_i(t)| + \epsilon n\sum_{i=1}^{n}|x_i(t)| \\
& - \sum_{i=1}^{n}\Big(p_i - \sum_{j=1}^{n} p_j|c_{ji}|\Big)|\dot{x}_i(t)| + (k + \varepsilon n)\sum_{i=1}^{n}|\dot{x}_i(t)| \\
= & \sum_{i=1}^{n}\Big((-\alpha_i + \epsilon n)|x_i(t)| + (-\beta_i + k + \varepsilon n)|\dot{x}_i(t)|\Big) \\
\le & \sum_{i=1}^{n}\Big((-\alpha_m + \epsilon n)|x_i(t)| + (-\beta_m + k + \varepsilon n)|\dot{x}_i(t)|\Big) \\
= & -(\alpha_m - \epsilon n)\|x(t)\|_1 - (\beta_m - k - \varepsilon n)\|\dot{x}(t)\|_1 \quad (11)
\end{aligned}$$

where $\alpha_m = \min_{1\le i\le n}\{\alpha_i\} > 0$ and $\beta_m = \min_{1\le i\le n}\{\beta_i\} > 0$. We first examine the time derivative of $V(t)$ for either $x(t) \ne 0$ or $\dot{x}(t) \ne 0$. In any of these cases, if we select $\epsilon n < \alpha_m$ and $k + \varepsilon n < \beta_m$, then (11) will directly imply that $\dot{V}(t) < 0$. When both $x(t) = 0$ and $\dot{x}(t) = 0$, (6) will simply indicate the result of $\dot{V}_1(t) = 0$. Therefore, in the light of (10), for the time derivative of $V(t)$, we obtain

$$\begin{aligned}
\dot{V}(t) = & -\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(p_i|b_{ij}||x_j(t-\tau_{ij})| + p_i|c_{ij}||\dot{x}_j(t-\zeta_{ij})| + \epsilon|x_j(t-\tau_{ij})| + \varepsilon|\dot{x}_j(t-\zeta_{ij})|\Big) \\
\le & -\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\epsilon|x_j(t-\tau_{ij})| + \varepsilon|\dot{x}_j(t-\zeta_{ij})|\Big) \quad (12)
\end{aligned}$$

We can obviously conclude from inequality (12) that $\dot{V}(t)$ is negative definite if $x_j(t-\tau_{ij}) \ne 0$ or $\dot{x}_j(t-\zeta_{ij}) \ne 0$ for some pairs of i and j. Besides, if $x(t) = \dot{x}(t) = 0$, $x_j(t-\tau_{ij}) = 0$ and $\dot{x}_j(t-\zeta_{ij}) = 0$ for all i, j, then one may directly see from the equations (6) and (10) that $\dot{V}(t) = 0$. Thus, we can immediately draw the conclusion from the analysis of these two cases that $\dot{V}(t) = 0$ solely at the equilibrium point associated with system (1). This indicates that $\dot{V}(t) < 0$ except for the origin. Thus, system (1) has been proved to possess an asymptotically stable equilibrium point. In the next step, it is needed to ensure that $V(t)$ justifies the requirement of being radially unbounded, namely, $V(t) \to \infty$ as $||x(t)|| \to \infty$. To this end, it is only needed to show that $V_1(t)$, defined by (3), is radially unbounded since $V_2(t)$ is a nonnegative functional. From (5), we know that $V(t) \geq k\|x(t)\|_1$. Since k is a positive constant, one can immediately confirm that $V(t) \to \infty$ as $||x(t)||_1 \to \infty$. Hence, global asymptotic stability of neutral system of the form given by (1) has been proved.

Remarks: According to Lemmas 1 and 2, in order that the stability conditions given in Theorem 1 be satisfied, it is necessary that both the matrix S defined in Theorem 1 and the matrix $I - |C|$ be nonsingular M-matrices. On the other hand, these necessary conditions may not be sufficient for stability of system (1). Therefore, we now suggest a systematic procedure for justification of stability criteria proposed in Theorem 1. Since S is justified to be a nonsingular M-matrix, according to Lemma 2, it is always possible to find the appropriate constant elements $p_i > 0, (i = 1, \dots, n)$, such that the first set of stability conditions in Theorem 1 is satisfied, that is to say, $\alpha_i > 0$ holds for all i. Then, for these determined fixed constants $p_i > 0, (i = 1, \dots, n)$, select the elements of the matrix C in a way such that the second set of stability conditions in Theorem 1 is satisfied, that is to say, $\beta_i > 0$ holds for all i. Alternatively, since $I - |C|$ is a nonsingular M-matrix, according to Lemma 2, the appropriate elements $p_i > 0, (i = 1, \dots, n)$, can be found such that the second set of stability conditions of Theorem 1 is satisfied, that is to say, $\beta_i > 0$ holds for all i. Then, for these determined fixed constants $p_i > 0, (i = 1, \dots, n)$, select the components of constant system matrices A and B such that the first set of conditions obtained in Theorem 1 is satisfied, i.e., $\alpha_i > 0$ holds $\forall i$.

We now compare the results derived in Theorem 1 with the stability results reported in [27]. The author of [27] has obtained the following results for stability of neutral system stated in (1):

Theorem 2 ([27]). Define the matrix $S = (s_{ij})_{n\times n}$ such that $s_{ii} = -a_{ii} - |b_{ii}| > 0$ and $s_{ij} = -|a_{ij}| - |b_{ij}|$ for $i \neq j, (i, j = 1, \dots, n)$. The neutral system stated in (1) is asymptotically stable if the following criteria hold:

$$\begin{aligned}\hat{\alpha}_i &= s_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n} |s_{ji}| \\ &= (-a_{ii} - |b_{ii}|) - \sum_{\substack{j=1\\ j\neq i}}^{n} (|a_{ji}| + |b_{ji}|) > 0\end{aligned}$$

and

$$\hat{\beta}_i = 1 - \sum_{j=1}^{n} |c_{ji}| > 0, \quad i = 1, \dots, n.$$

In the statements of the conditions of Theorem 1, if we select $p_i = 1, (i = 1, \dots, n)$, then Theorem 1 directly yields the conditions of Theorem 2. It is also important to point out that Theorem 1 assures global asymptotic stability of the system given with (1). Since Theorem 2 simply refers to the proof of asymptotic stability of neutral system (1), this comparison indicates another advantage of the criteria of Theorem 1 over the conditions of Theorem 2.

3 A Comparative Numerical Example and Simulation Results

This section analyses a comparative numerical example to point out the applicability and reduced conservatism of the presented stability criteria in Theorem 1 in comparison with the existing conditions given in Theorem 2 [27]. To this end, consider the two-dimensional neutral system of the form (1) that possesses the system matrices given below:

$$A = \begin{bmatrix} -2 & 0.5 \\ 0.6 & -6 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0.5 \\ -0.6 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 0.2 & 0.1 \\ 0.9 & 0.1 \end{bmatrix}.$$

The time delay and neutral delay parameters are chosen as $\tau_{11} = 1$, $\tau_{12} = 1.2$, $\tau_{21} = 0.8$, $\tau_{22} = 1.5$ and $\zeta_{11} = 0.5$, $\zeta_{12} = 0.6$, $\zeta_{21} = 0.4$, $\zeta_{22} = 0.7$ where $\tau = \max\{\tau_{ij}\} = 1.5$ and $\zeta = \max\{\zeta_{ij}\} = 0.7$, for $i, j = 1, 2$ and thus $\iota = \max\{\tau, \zeta\} = 1.5$.

For the matrices A and B, given above, the elements of the matrix S in Theorems 1 and 2 are computed as follows:

$$s_{11} = -(-2) - |1| = 1, \qquad s_{12} = -|0.5| - |0.5| = -1,$$
$$s_{21} = -|0.6| - |-0.6| = -1.2, \; s_{22} = -(-6) - |1| = 5.$$

We first examine the conditions of Theorem 2 for the parameters described in this example. According to Theorem 2, the values $\hat{\alpha}_1$ and $\hat{\beta}_1$ are obtained to satisfy

$$\hat{\alpha}_1 = s_{11} - |s_{21}| = 1 - 1.2 = -0.2 < 0$$

and

$$\hat{\beta}_1 = 1 - |c_{11}| - |c_{21}| = 1 - 0.2 - 0.9 = -0.1 < 0.$$

Since the required conditions of Theorem 2 are violated, Theorem 2 cannot be used to imply stability of system (1) for this example.

We now proceed with applying the criteria developed in Theorem 1 to the case of this example. Choosing the positive constants as $p_1 = 2$ and $p_2 = 1$, we obtain the following conditions

$$\alpha_1 = p_1 s_{11} - p_2|s_{21}| = 0.8 > 0,$$

$$\alpha_2 = p_2 s_{22} - p_1|s_{12}| = 3 > 0,$$

$$\beta_1 = p_1 - p_1|c_{11}| - p_2|c_{21}| = 0.7 > 0,$$

and

$$\beta_2 = p_2 - p_1|c_{12}| - p_2|c_{22}| = 0.7 > 0.$$

Thus, the conditions obtained in Theorem 1 are directly established for the system parameters of this example.

To illustrate the theoretical results, time-domain simulations are conducted for different choices of delay parameters and initial functions. First, let $x(t) = \theta(t) = (2, -1.5)^T$, $t \in [-1.5, 0]$. The time response of the states of this neutral system is shown in Figure 1.

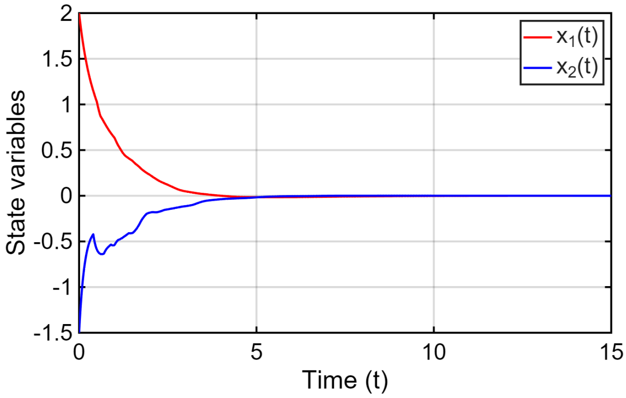

Fig. 1: Time response of the system states.
Source: Created by the authors.

For a different choice of the initial function, $x(t) = \theta(t) = (-1.2, 1.8)^T$, $t \in [-1.5, 0]$, the time response of the states is illustrated in Figure 2.

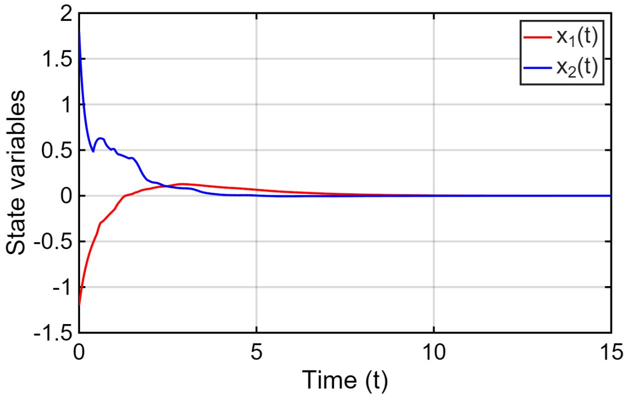

Fig. 2: Time response of the system states.
Source: Created by the authors.

It is also of interest to examine the system behavior under another set of delay parameters for which the maximum delay is attained by a neutral delay term. To this end, the delay parameters are chosen as $\tau_{11} = 2.2$, $\tau_{12} = 1.6$, $\tau_{21} = 2.8$, $\tau_{22} = 2.9$, $\zeta_{11} = 3.6$, $\zeta_{12} = 4.7$, $\zeta_{21} = 2.2$ and $\zeta_{22} = 2.8$, where $\tau = \max\{\tau_{ij}\} = 2.9$ and $\zeta = \max\{\zeta_{ij}\} = 4.7$, and thus $\iota = \max\{\tau, \zeta\} = 4.7$. For the same initial function used in Figure 2 and these delay parameters, $x(t) = \theta(t) = (-1.2, 1.8)^T$, $t \in [-4.7, 0]$, the time response of the system states is presented in Figure 3.

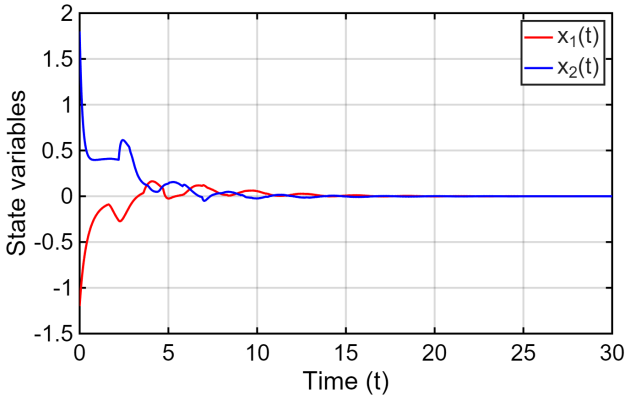

Fig. 3: Time response of the system states.
Source: Created by the authors.

For yet another choice of the initial function, $x(t) = \theta(t) = (-0.5, 0.4)^T, t \in [-4.7, 0]$, the time response of the system states is given in Figure 4.

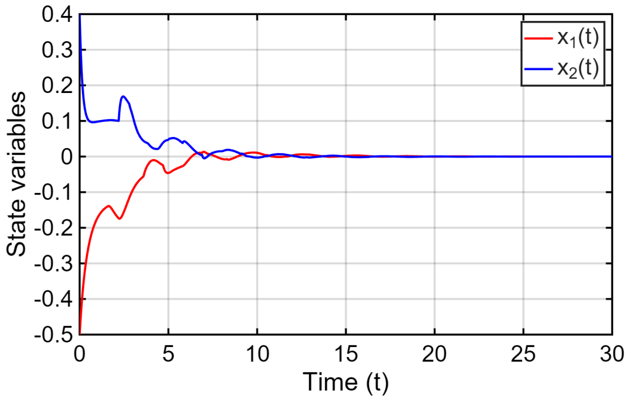

Fig. 4: Time response of the system states.
Source: Created by the authors.

4 Conclusions

This research paper has ultimately paid attention to stability problems for linear neutral systems having a set of multiple time delay parameters in state variables and a set of multiple neutral delay parameters in the time derivatives of the states of this system. We first define a novel Lyapunov functional, whose main term is a linear combination of the absolute values of system states, and then derive new alternative sufficient stability conditions. These conditions proved to be easily verifiable and are independent of delay components. We have also shown that our stability conditions improve some previously published corresponding stability conditions for the same model of neutral systems. A comparative numerical example has also been given to determine advantage of the results of this paper over the previously published results and show the applicability of the stability conditions. Since obtaining stability criteria regarding a linear neutral system that includes multiple time and neutral delay elements is a difficult task to accomplish, the stability criteria we have presented in this paper can be assessed as useful contributions to stability issues for linear neutral systems.

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Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.