A New Nakagami-Exponential Distribution: Theoretical and Statistical Properties

IBRAHIM ABDULLAHI[1], WIKANDA PHAPHAN[2,3*]
[1]Department of Mathematics and Statistics, Faculty of Science,
Yobe State University, Damaturu, 500501,
NIGERIA
[2]Department of Applied Statistics, Faculty of Applied Science,
King Mongkut's University of Technology North Bangkok, Bangkok, 10800,
THAILAND
[3]Research Group in Statistical Learning and Inference,
King Mongkut's University of Technology North Bangkok, Bangkok, 10800,
THAILAND

**Corresponding author*

Abstract: This article proposes the new Nakagami-generated family of distribution (NNak-G) and the new Nakagami-Exponential (NNak-E) distribution for fitting skewed and heavy-tailed data. By integrating the flexibility of the Nakagami distribution with the useful properties of the exponential distribution, the NNak-E distribution provides enhanced adaptability for fitting with right-skewed data structures. The theoretical properties of the NNak-E distribution are developed, including its distribution function, density function, moments, and hazard function. Parameter estimation is constructed using the Maximum Likelihood Estimation (MLE) method, and a comprehensive simulation study is performed to evaluate the performance of the MLE estimates. Furthermore, the NNak-E distribution is applied to real-world datasets, where it demonstrates superior goodness-of-fit compared to existing distributions. The results confirm that NNak-E is a choice distribution for practitioners and researchers dealing with asymmetric and heavy-tailed data distributions in diverse fields such as engineering, medical research, and risk assessment.

Key-Words: - Nakagami-Exponential Distribution, Probability Distributions, Survival Analysis, Reliability Engineering, Maximum Likelihood Estimation (MLE), Skewed Data Modeling, Heavy-Tailed Distributions.

Received: October 28, 2025. Revised: December 19, 2025. Accepted: January 13, 2026. Published: May 5, 2026.

1 Introduction

Probability distributions are fundamental tools for explaining the data, playing a critical role in diverse fields such as engineering, finance, medical research, and reliability analysis. Classical distributions often struggle to fit the skewed and heavy-tailed nature of real data, necessitating the development of new distributions that provide greater fitting and flexibility.

The Nakagami distribution has gained prominence in statistical modeling, especially in applications related to signal processing and wireless communications, due to its ability to model fading phenomena. Meanwhile, the exponential distribution serves as a fundamental model in reliability theory and survival analysis because of its simplicity and memoryless property. However, both distributions have inherent limitations when dealing with highly skewed or heavy-tailed data. Recent studies, such as [1], introduced the odd generalized Nakagami (Nak-G) family, expanding the flexibility of the Nakagami distribution to capture diverse data behaviors. The study, [2], introduced the three-parameter Nakagami–Rayleigh distribution as a powerful distribution for right-skewed distributions. The study, [3], proposed the mixture of Burr XII and Nakagami distributions based on properties from earlier Nakagami models. Moreover, [4], studied properties of the Nakagami distribution and its relations to gamma, Rayleigh, Weibull, chi-square, and exponential distributions, including moment derivations and confidence intervals for medians from independent Nakagami samples. Other related works on the Nakagami distribution include, [5], [6], [7], [8], [9].

To address these challenges, this article introduces the new Nakagami-Exponential (NNak-E)

distribution, which combines the strengths of the Nakagami and exponential distributions to enhance the modeling of asymmetric and heavy-tailed data. The NNak-E distribution enhances flexibility through the inclusion of additional parameters, enabling a more accurate representation of real-world datasets. To provide a comprehensive understanding of the behavior of the proposed distribution, its theoretical properties are developed, including the density function, distribution function, moments, and hazard function.

Parameter estimation plays a vital role in the development of any statistical model. In this article, the Maximum Likelihood Estimation (MLE) method is applied to estimate the NNak-E parameters. A comprehensive simulation study is carried out to assess the performance of the MLE approach. Furthermore, the practicality of the NNak-E distribution is illustrated using a real-world dataset, where it demonstrates a superior goodness of fit compared to existing models.

Therefore, this article presents an alternative distribution for skewed and heavy-tailed data, that efforts in statistical modeling and distribution theory. The findings suggest that the NNak-E distribution is a valuable tool for researchers and practitioners seeking improved methods for data analysis in various scientific and engineering disciplines.

This article is structured as follows: Section 2 presents the theoretical background of the Nakagami distribution. Section 3 discusses the formulation of the NNak-G family of distributions. Section 4 examines the properties of the NNak-E distribution, including its moments and quantile function. Section 5 presents the parameter estimation process using MLE. Section 6 details the simulation study, followed by Section 7, which applies the NNak-E model to real-world data. Section 8 discusses the results and their implications, and Section 9 concludes the study with final remarks and future research directions.

2 Nakagami Distribution

The Nakagami distribution, [10], a statistical model closely associated with the gamma distribution, has gained significant attention due to its applications in modeling wireless signal propagation, particularly in environments where signals traverse multiple paths. This distribution is defined by two key parameters: the shape parameter λ, which must satisfy $\lambda \geq 0.5$, and the scale parameter β, which is restricted to $\beta > 0$.

The distribution function (CDF) of the Nakagami distribution is mathematically expressed as:

$$F\left(x;\lambda,\beta\right)=\frac{\gamma\left(\lambda,\lambda x^{2}/\beta\right)}{\Gamma\left(\lambda\right)}, \tag{1}$$

where $\gamma\left(\lambda,\lambda x^{2}/\beta\right)$ represents the lower incomplete gamma function, and $\Gamma\left(\lambda\right)$ denotes the gamma function. Complementing this, the density function (PDF) is given by:

$$f\left(x;\lambda,\beta\right)=\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}x^{2\lambda-1}\exp\left(\frac{-\lambda}{\beta}x^{2}\right). \tag{2}$$

3 Developments of the NNak-G Family of Distributions

Various statistical functions of the NNak-G distribution, including the density function and distribution function, have been derived. This distribution is constructed using the G-family generation methodology proposed by [11], and is particularly useful for modeling datasets that exhibit NNak-G characteristics.

The derivation of the CDF for the NNak-G distribution relies on two fundamental functions: the cumulative distribution function, $GG(x;\eta)$, and the probability density function, $gg(x;\eta)$, where η represents the parameter vector of the baseline distribution. Following the G-family construction approach outlined by [11], the CDF of the NNak-G distribution is obtained by substituting x with $-\log GG(x;\eta)$ in equation (1):

$$F(x;\lambda,\beta,\eta)=\int_{0}^{-\log GG(x;\eta)}\left[\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}t^{2\lambda-1}\times\exp\left(\frac{-\lambda}{\beta}t^{2}\right)dt\right]. \tag{3}$$

Let

$$u=\frac{\lambda}{\beta}t^{2}\Rightarrow t=\left(\frac{u\beta}{\lambda}\right)^{1/2}, \tag{4}$$

and

$$\frac{\partial u}{\partial t}=\frac{2\lambda}{\beta}t\Rightarrow\partial t=\frac{\beta}{2\lambda t}\partial u, \tag{5}$$

substituting (4) and (5) into (3) yield

$$F(x;\lambda,\beta,\eta)=\frac{1}{\Gamma(\lambda)}\int_{0}^{-\log GG(x;\eta)}u^{\lambda-1}\mathrm{e}^{-u}\partial u, \tag{6}$$

when $t\to 0$, $u\to 0$; and when $t\to-\log GG(x;\eta)$, $u\to\frac{\lambda}{\beta}\left(-\log GG(x;\eta)\right)^{2}$;

$$F(x;\lambda,\beta,\eta)=\frac{1}{\Gamma(\lambda)}\int_{0}^{\frac{\lambda}{\beta}(-\log GG(x;\eta))^{2}}u^{\lambda-1}\mathrm{e}^{-u}\partial u. \tag{7}$$

Therefore,

$$F(x;\lambda,\beta,\eta)=\frac{1}{\Gamma(\lambda)}\gamma\left(\lambda,\frac{\lambda}{\beta}\left(-\log GG(x;\eta)\right)^2\right), \tag{8}$$

where $\gamma\left(\lambda,\frac{\lambda}{\beta}\left(-\log GG(x;\eta)\right)^2\right)$ is the lower incomplete gamma function.
Differentiate the CDF in equation (8) to obtain the PDF:

$$f(x;\lambda,\beta,\eta)=\frac{\partial}{\partial x}\left[\frac{1}{\Gamma(\lambda)} \times\gamma\left(\lambda,\frac{\lambda}{\beta}\left(-\log GG(x;\eta)\right)^2\right)\right]. \tag{9}$$

Thus,

$$f(x;\lambda,\beta,\eta)=\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}\left(-\log G(x;\eta)\right)^{2\lambda-1} \times\exp\left(-\frac{\lambda}{\beta}\left(-\log G(x;\eta)\right)^2\right)\frac{g(x;\eta)}{G(x;\eta)}. \tag{10}$$

4 New Nakagami-Exponential (NNak-E) Distribution

The exponential distribution serves as the baseline distribution with a parameter denoted as θ. Let

$$GG(x;\theta)=1-\mathrm{e}^{-\theta x}, \tag{11}$$

and

$$gg(x;\theta)=\theta\mathrm{e}^{-\theta x};\ \theta>0,\ x>0. \tag{12}$$

The PDF, CDF, survival function, and hazard function are provided as follows:

$$f(x;\lambda,\beta,\theta)=\frac{2\lambda^{\lambda}\theta\mathrm{e}^{-\theta x}}{\Gamma(\lambda)\beta^{\lambda}\left(1-\mathrm{e}^{-\theta x}\right)} \times\left(-\log\left(1-\mathrm{e}^{-\theta x}\right)\right)^{2\lambda-1} \times\exp\left(-\frac{\lambda}{\beta}\left(-\log\left(1-\mathrm{e}^{-\theta x}\right)\right)^2\right), \tag{13}$$

The shape of the PDF is shown in Figure 1, 2.

$$F(x;\lambda,\beta,\theta)=1-\frac{1}{\Gamma(\lambda)} \times\gamma\left(\lambda,\frac{\lambda}{\beta}\left(-\log\left(1-\mathrm{e}^{-\theta x}\right)\right)^2\right), \tag{14}$$

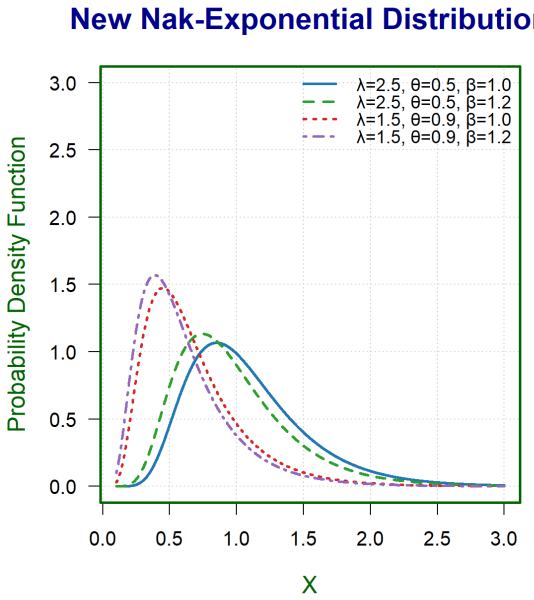

Fig. 1: (a) NNak-E PDF. Source: created by the authors.

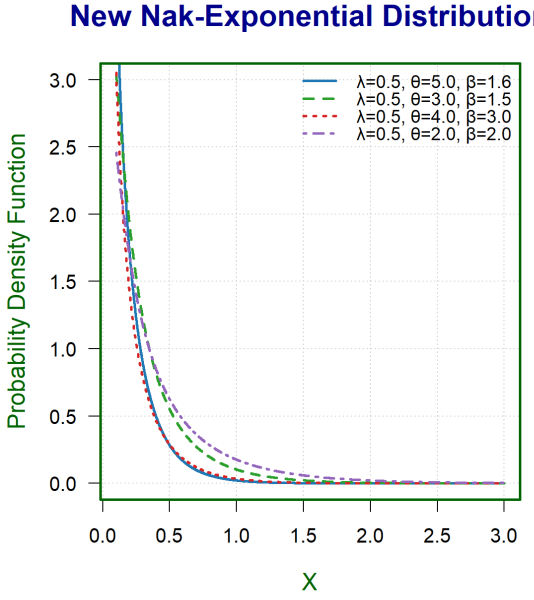

Fig. 2: (b) NNak-E PDF. Source: created by the authors.

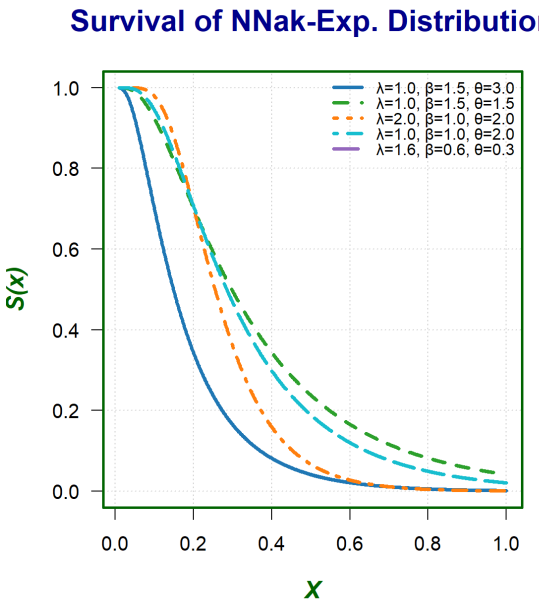

Fig. 3: NNak-E Survival Function. Source: created by the authors.

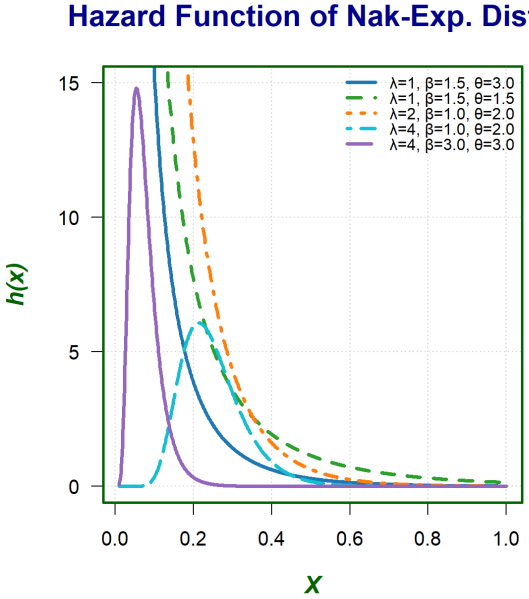

Fig. 4: NNak-E Hazard Function. Source: created by the authors.

$$S(x;\lambda,\beta,\theta) = 1 - \frac{1}{\Gamma(\lambda)} \times\gamma\left(\lambda, \frac{\lambda}{\beta}\left(-\log\left(1-\mathrm{e}^{-\theta x}\right)\right)^2\right). \tag{15}$$

$$h(x;\lambda,\beta,\theta) = \frac{\frac{2\lambda^\lambda\theta\,\mathrm{e}^{-\theta x}(-\log(1-\mathrm{e}^{-\theta x}))^{2\lambda-1}\Theta}{\Gamma(\lambda)\beta^\lambda(1-\mathrm{e}^{-\theta x})}}{1-\frac{1}{\Gamma(\lambda)}\gamma\left(\lambda,\frac{\lambda}{\beta}\left(-\log\left(1-\mathrm{e}^{-\theta x}\right)\right)^2\right)}, \tag{16}$$

where $\Theta = \exp\left(-\frac{\lambda}{\beta}\left(-\log\left(1-\mathrm{e}^{-\theta x}\right)\right)^2\right)$. The shape of the survival and hazard functions are shown in Figure 3 and 4, respectively.

4.1 Main Features

4.2 Quantile Function of the NNak-E Distribution

The quantile function derived from the cumulative distribution function (CDF) is given by:

$$Q = -\frac{1}{\theta}\log\left(1-\exp\left(-\sqrt{\frac{\beta}{\lambda}\cdot\gamma^{-1}\left(\lambda, u\cdot\Gamma(\lambda)\right)}\right)\right), \tag{17}$$

where $\gamma^{-1}(\lambda,\cdot)$ is the inverse lower incomplete gamma function.

4.3 Linear Expansion for NNak-E Distribution

A simplified expression for the probability density function of the NNak-E distribution.

Note that: $-\log(1-v) = \sum_{t=0}^{\infty}\frac{v^t}{t!}$ valid for $|v|<1$.

Hence, the pdf of NNak-E is simplified to

$$f(x) = \frac{2\lambda^\lambda\theta^{2\lambda}x^{2\lambda-2}\exp\left(-\theta x - \frac{\lambda}{\beta}(\mathrm{e}^{-\theta x})^2\right)}{\Gamma(\lambda)\beta^\lambda}. \tag{18}$$

4.3.1 Moment for NNak-E Distribution

The moment may be determined by using the PDF in equation (18) in the following manner.

The r-th moment of X is defined as:

$$E(X^r) = \int_0^\infty x^r f(x)\,dx. \tag{19}$$

$$E(X^r) = \frac{2\lambda^\lambda \theta^{2\lambda}}{\Gamma(\lambda)\beta^\lambda} \int_0^\infty x^{r+2\lambda-2} \times \exp\left(-\theta x - \frac{\lambda}{\beta}(\mathrm{e}^{-\theta x})^2\right) dx. \quad (20)$$

Let $u = \mathrm{e}^{-\theta x}$, so that:

$$x = -\frac{\log(u)}{\theta}, \quad dx = -\frac{1}{\theta u}\, du. \quad (21)$$

when

$$x \to 0 \implies u \to 1, \quad x \to \infty \implies u \to 0.$$

$$E(X^r) = \frac{2\lambda^\lambda \theta^{2\lambda}}{\Gamma(\lambda)\beta^\lambda} \int_0^1 \left(-\frac{\log(u)}{\theta}\right)^{r+2\lambda-2} \times \exp\left(-\frac{\lambda}{\beta}u^2\right)\left(-\frac{1}{\theta u}\right) du. \quad (22)$$

$$E(X^r) = \frac{2\lambda^\lambda \theta^{1-r}}{\Gamma(\lambda)\beta^\lambda} \int_0^1 \frac{(-\log(u))^{r+2\lambda-2}}{u} \times \exp\left(-\frac{\lambda}{\beta}u^2\right) du. \quad (23)$$

The term $\exp\left(-\frac{\lambda}{\beta}u^2\right)$ can be expanded into a Taylor series for small values of $\frac{\lambda}{\beta}$:

$$\exp\left(-\frac{\lambda}{\beta}u^2\right) \approx 1 - \frac{\lambda}{\beta}u^2 + \frac{1}{2}\left(\frac{\lambda}{\beta}\right)^2 u^4 + \cdots \quad (24)$$

Substitute equation (4.14) into (4.13)

$$E(X^r) = \frac{2\lambda^\lambda \theta^{1-r}}{\Gamma(\lambda)\beta^\lambda} \int_0^1 \frac{(-\log(u))^{r+2\lambda-2}}{u} \left(1 - \frac{\lambda}{\beta}u^2 + \frac{1}{2}\left(\frac{\lambda}{\beta}\right)^2 u^4\right) du. \quad (25)$$

$$E(X^r) = \frac{2\lambda^\lambda \theta^{1-r}}{\Gamma(\lambda)\beta^\lambda}\left[A_1 - \frac{\lambda}{\beta}A_2 + \frac{1}{2}\left(\frac{\lambda}{\beta}\right)^2 A_3\right], \quad (26)$$

where:

$$A_1 = \int_0^1 \frac{(-\log(u))^{r+2\lambda-2}}{u}\, du,$$
$$A_2 = \int_0^1 \frac{(-\log(u))^{r+2\lambda-2}}{u} u^2\, du,$$
$$A_3 = \int_0^1 \frac{(-\log(u))^{r+2\lambda-2}}{u} u^4\, du.$$

For A_1, let $v = -\log(u)$, so that $u = e^{-v}$ and $du = -e^{-v}dv$. This gives:

$$A_1 = \int_0^\infty v^{r+2\lambda-2}\, dv = \Gamma(r+2\lambda-1) = \frac{2\lambda^\lambda \theta^{1-r}}{\Gamma(\lambda)\beta^\lambda}\left[\Gamma(r+2\lambda-1) - \frac{\lambda}{\beta}\frac{\Gamma(r+2\lambda-1)}{2^{r+2\lambda-1}} + \frac{1}{2}\left(\frac{\lambda}{\beta}\right)^2 \frac{\Gamma(r+2\lambda-1)}{4^{r+2\lambda-1}}\right]. \quad (27)$$

The definition of the r^{th} ordinary moment of the NNak-E distribution is provided.

$$E(X^r) = \frac{2\lambda^\lambda \theta^{1-r}\Gamma(r+2\lambda-1)}{\Gamma(\lambda)\beta^\lambda}\left[1 - \frac{\lambda}{\beta \cdot 2^{r+2\lambda-1}} + \frac{1}{2}\left(\frac{\lambda}{\beta \cdot 4^{r+2\lambda-1}}\right)\right]. \quad (28)$$

The mean of the NNak-E distribution takes the form:

$$E(X) = \frac{2\lambda^\lambda \Gamma(2\lambda)}{\Gamma(\lambda)\beta^\lambda}\left[1 - \frac{\lambda}{\beta \cdot 2^{2\lambda}} + \frac{1}{2}\left(\frac{\lambda}{\beta \cdot 4^{2\lambda}}\right)\right]. \quad (29)$$

The expected value of the random variable X^2, the second moment of the NNak-E distribution may be derived by substituting the value of $r = 2$ into equation (4.18).

$$E(X^2) = \frac{2\lambda^\lambda \Gamma(1+2\lambda)}{\Gamma(\lambda)\beta^\lambda \theta} \times \left[1 - \frac{\lambda}{\beta \cdot 2^{1+2\lambda}} + \frac{1}{2}\left(\frac{\lambda}{\beta \cdot 4^{1+2\lambda}}\right)\right]. \quad (30)$$

The variance of the NNak-E distribution is obtained, respectively, as follows:

$$\mathrm{Var}(X) = \frac{2\lambda^\lambda \Gamma(1+2\lambda)}{\Gamma(\lambda)\beta^\lambda \theta} \times \left[1 - \frac{\lambda}{\beta \cdot 2^{1+2\lambda}} + \frac{1}{2}\left(\frac{\lambda}{\beta \cdot 4^{1+2\lambda}}\right)\right] - \left(\frac{2\lambda^\lambda \Gamma(2\lambda)}{\Gamma(\lambda)\beta^\lambda}\left[1 - \frac{\lambda}{\beta \cdot 2^{2\lambda}} + \frac{1}{2}\left(\frac{\lambda}{\beta \cdot 4^{2\lambda}}\right)\right]\right)^2. \quad (31)$$

5 Maximum Likelihood Estimation (MLE) Method for Parameter Estimation

In this section, we apply the MLE method to estimate the parameters of the PDF in equation

(4.3). For a set of observations $x_1, x_2, \ldots, x_n$, the likelihood function is defined as the product of the PDF evaluated at each observation:

$$L(\lambda, \beta, \theta) = \prod_{i=1}^{n} f(x_i; \lambda, \beta, \theta),$$

where $f(x_i; \lambda, \beta, \theta)$ is the PDF of the distribution.

To simplify the maximization process, we take the natural logarithm of the likelihood function:

$$\log L(\lambda, \beta, \theta) = \sum_{i=1}^{n} \log \left[f(x_i; \lambda, \beta, \theta) \right].$$

Substituting the expression of $f(x_i; \lambda, \beta, \theta)$, we have

$$\log L(\lambda, \beta, \theta) = \sum_{i=1}^{n} \left[\log \left(\frac{2\lambda^{\lambda} \theta \mathrm{e}^{-\theta x_i}}{\Gamma(\lambda) \beta^{\lambda} \left(1 - \mathrm{e}^{-\theta x_i}\right)} \right) + \left(-\log \left(1 - \mathrm{e}^{-\theta x_i}\right) \right)^{2\lambda - 1} \exp \left(-\frac{\lambda}{\beta} \left(-\log \left(1 - \mathrm{e}^{-\theta x_i}\right) \right)^2 \right) \right]. \tag{32}$$

To estimate the parameters λ, β, and θ, we maximize the log-likelihood function. This is achieved by taking the partial derivatives of $\log L(\lambda, \beta, \theta)$ with respect to each parameter and setting them equal to zero:

$$\begin{aligned} \frac{\partial}{\partial \lambda} \log L(\lambda, \beta, \theta) &= 0, \\ \frac{\partial}{\partial \beta} \log L(\lambda, \beta, \theta) &= 0, \\ \frac{\partial}{\partial \theta} \log L(\lambda, \beta, \theta) &= 0. \end{aligned} \tag{33}$$

The solutions to these equations provide the Maximum Likelihood Estimates (MLEs) for the parameters λ, β, and θ. Given the complexity of the log-likelihood function, numerical optimization techniques, such as the Newton-Raphson method or gradient-based methods, are typically employed to solve these equations. The results of these estimations will serve as the basis for evaluating the fit of the model to the observed data.

6 Simulation Study

A comprehensive simulation study was conducted to examine the performance of the Maximum Likelihood Estimation (MLE) method for the New Nakagami-Exponential distribution. The study employed the acceptance-rejection method to generate random samples, with simulations repeated 1,000 times for sample sizes of $n = 10, 50, 100$. The true parameter values were set as $\lambda = 1.5$, $\beta = 2$, and $\theta = 3$. The results, as presented in Figure 5, Figure 6, Figure 7, Figure 8 , Figure 9, Figure 10, Figure 11, Figure 12, and Figure 13, indicate that the estimated means for all parameters approached the true values as the sample size increased. Furthermore, both the variance and Mean Squared Error (MSE) of the estimates decreased consistently with larger sample sizes.

The findings underscore the robustness and reliability of the MLE method for parameter estimation in the NNak-E distribution, particularly for larger datasets. Figure 5, Figure 6, Figure 7, Figure 8 , Figure 9, Figure 10, Figure 11, Figure 12, and Figure 13, provide a detailed visualization of accuracy in parameter estimation as sample size increases. These results affirm the suitability of the MLE method for estimating parameters in practical applications involving this distribution.

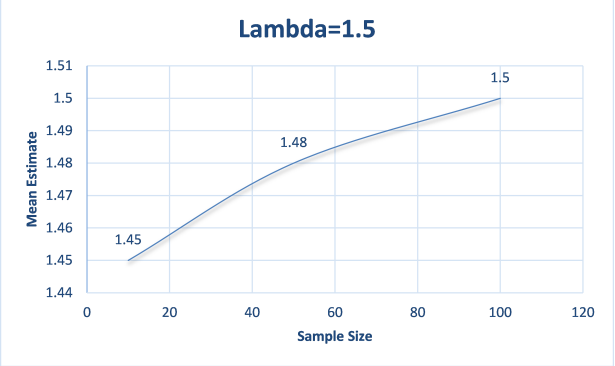

Fig. 5: Mean Estimate Lambda. Source: created by the authors.

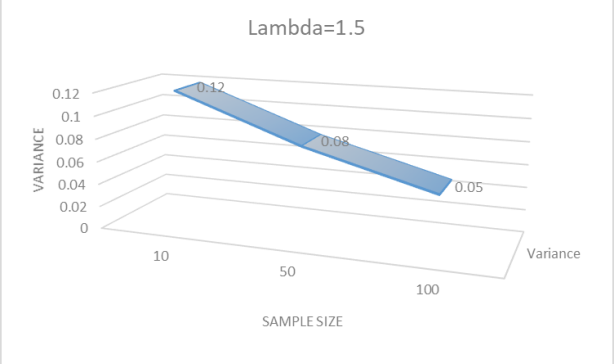

Fig. 6: Variance of Lambda. Source: created by the authors.

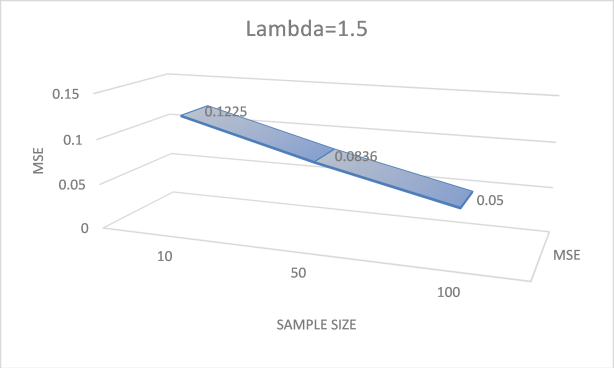

Fig. 7: MSE Lambda. Source: created by the authors.

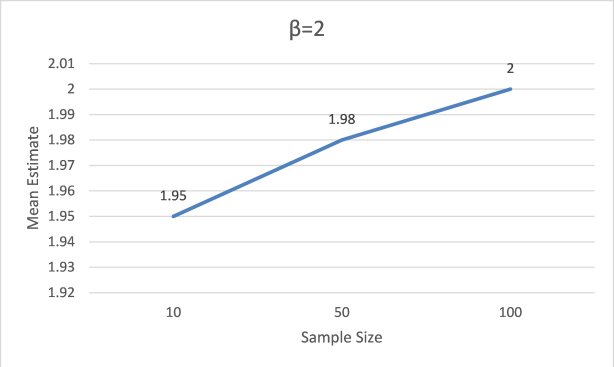

Fig. 8: Mean Estimate Bete. Source: created by the authors.

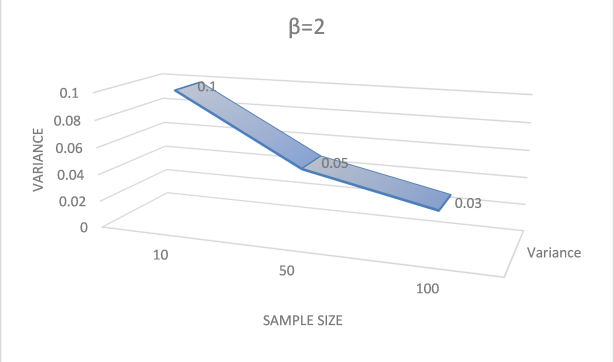

Fig. 9: Variance of Beta. Source: created by the authors.

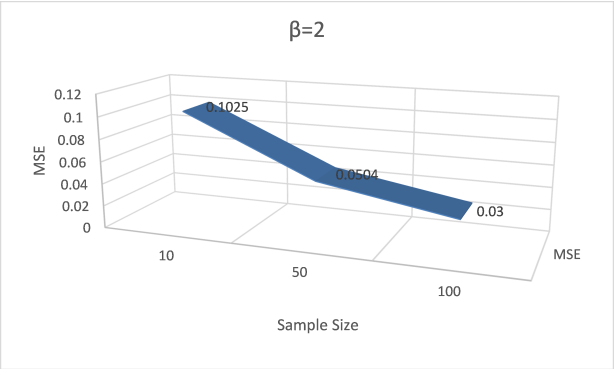

Fig. 10: MSE of Beta. Source: created by the authors.

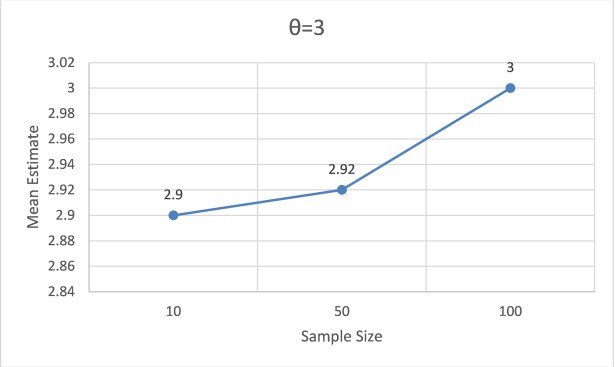

Fig. 11: Mean Estimate Theta. Source: created by the authors.

7 Application

In this section, we demonstrate that the new Nakagami-Exponential distribution provides a superior modeling framework when applied to a real dataset. For illustration, we utilize the survival data originally presented by [12], which includes 16 observations of survival times (in weeks) for AG-negative leukemia patients. The AG-negative condition was characterized by the absence of Auer rods and significant granulation in leukemic cells within the bone marrow at diagnosis. The observed survival times (in weeks) for the AG-negative patients are as follows: 56, 65, 17, 17, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, and 43. This dataset serves as a basis for comparing the performance of the proposed Nakagami-Exponential distribution with existing survival models, demonstrating its ability to capture the variability and skewness inherent in survival times.

The survival times for AG-negative leukemia patients (16 observations) were analyzed, and their summary statistics are presented in Table 1. And fitting the NNak-E distribution of survival times for AG-negative leukemia patients' data (Dataset 1) is shown in Figure 14. The dataset exhibits

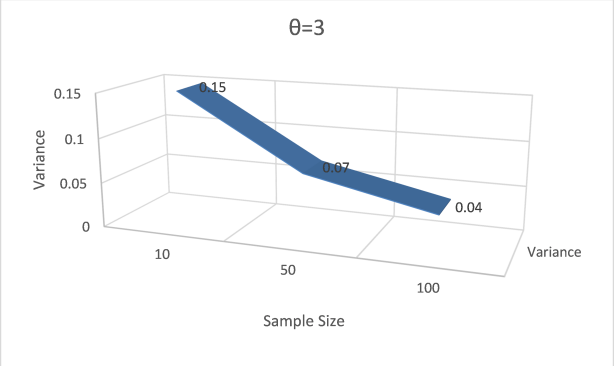

Fig. 12: Variance of Theta. Source: created by the authors.

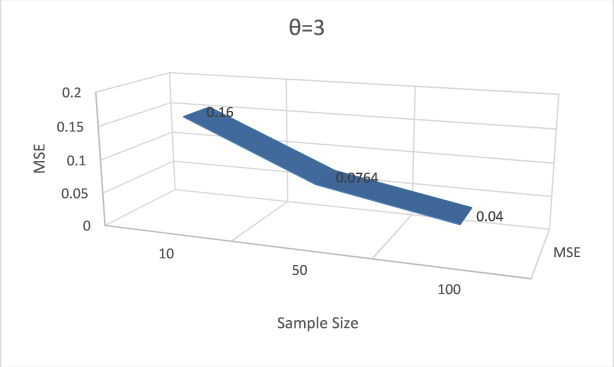

Fig. 13: MSE of Theta. Source: created by the authors.

Table 1. Summary Statistics for Survival Times. Source: created by the authors.

Statistic	**Value**
Count	16
Average (Mean)	18.5625
Standard Deviation	20.0963
Coefficient of Variation	108.263%
Minimum	2.0
Maximum	65.0
Range	63.0
Standardized Skewness	2.18331
Standardized Kurtosis	0.656219

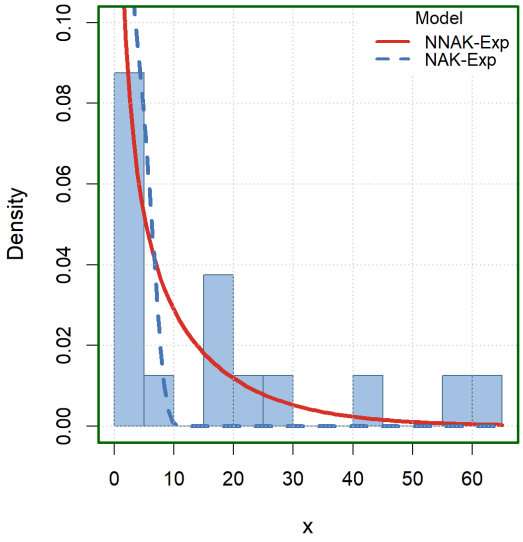

Fig. 14: NNak-E Distribution Dataset 1. Source: created by the authors.

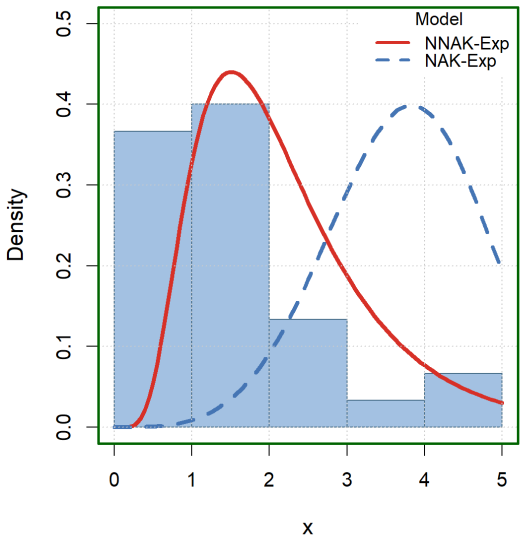

Fig. 15: NNak-E Distribution Dataset 2. Source: created by the authors.

a mean survival time of 18.5625 weeks, with a standard deviation of 20.0963 weeks, reflecting high variability. The coefficient of variation (108.263%) indicates that the data is highly dispersed relative to the mean. The range of survival times is 63 weeks, with minimum and maximum values of 2 and 65 weeks, respectively. Of particular importance are the measures of shape. The standardized skewness value of 2.18331 falls outside the acceptable range of -2 to +2, indicating significant positive skewness. This suggests that the data does not follow a normal distribution. On the other hand, the standardized kurtosis value of 0.656219 lies within the range expected for a normal distribution, suggesting no significant departure in kurtosis. These results suggest that methods relying on the assumption of normality may not be suitable for this dataset. The significant positive skewness in the data justifies the application of the Nakagami-Exponential distribution, which is capable of modeling skewed data effectively.

The second dataset provided by [13], represents the time intervals between successive failures of a repairable component. These intervals, measured in hours, highlight the component's reliability and serve as a benchmark for analyzing failure patterns. The dataset is as follows:

1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74,
4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

The dataset representing the time intervals between successive failures of a repairable component was analyzed, and the summary statistics of the failures data (Dataset 2) are presented in Table 2, fitting the NNak-E distribution for the failures data is shown in Figure 15. The dataset exhibits a mean time between failures of 1.54267, with a standard deviation of 1.12769, indicating moderate variability. The coefficient of variation (73.0998%) suggests that the data is dispersed relative to the mean. The range of the data is 4.62, with a minimum and maximum of 0.11 and 4.73, respectively.

Table 2. Summary Statistics for Time Between Failures. Source: created by the authors.

Statistic	Value
Count	30
Average (Mean)	1.54267
Standard Deviation	1.12769
Coefficient of Variation	73.0998%
Minimum	0.11
Maximum	4.73
Range	4.62
Standardized Skewness	3.05149
Standardized Kurtosis	2.01118

The measures of shape, specifically the standardized skewness (3.05149) and standardized kurtosis (2.01118), indicate significant departures from normality. Both values exceed the expected range of -2 to +2 for data from a normal distribution. This implies that methods assuming normality may not be suitable for analyzing this dataset.

These results highlight the need for robust statistical methods to handle non-normal data, such as the Nakagami-Exponential distribution, which is capable of accommodating skewed and heavy-tailed datasets effectively.

This dataset provides valuable insights into the repairable component's reliability performance. Its analysis is crucial for understanding failure mechanisms, which can assist in improving component design and maintenance strategies. Furthermore, it allows for evaluating the suitability of various probability distributions, including the newly proposed Nakagami-Exponential distribution, in modeling reliability data.

8 Results and Discussion

This section presents the Maximum Likelihood Estimators (MLEs) and goodness-of-fit statistics for the Nakagami-Exponential (Nak-E) and New Nakagami-Exponential (NNak-E) models applied to two datasets. The results, summarized in Tables 3 and 4 (Appendix), provide insights into the effectiveness of these models in capturing data patterns.

Table 3 (Appendix) presents the estimation results for Dataset 1. The negative log-likelihood ($-\ell$) values indicate that the NNak-E model $-\ell = 61.8764$ fits the data better than the Nak-E model $-\ell = 64.8401$. Similarly, the goodness-of-fit statistics, including the W-statistic (0.1145 vs. 0.1535) and Anderson-Darling statistic (0.7482 vs. 0.9726), demonstrate a superior fit for NNak-E

Additionally, the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) values for NNak-E (AIC = 129.7527, BIC = 132.0705) are lower than those for Nak-E (AIC = 135.6803, BIC = 137.9980), further confirming that NNak-E provides a more optimal trade-off between model complexity and fit.

Table 4 (Appendix) presents the estimation results for Dataset 2. Similar to Dataset 1, the NNak-E model outperforms the Nak-E model, with a lower negative log-likelihood ($-\ell = 40.1447$ vs. 43.3456), indicating a better likelihood-based fit. The W-statistic (0.0300 vs. 0.1158) and Anderson-Darling statistic (0.2387 vs. 0.7915)

further confirm the superior goodness-of-fit of NNak-E

Additionally, the AIC (86.2894 vs. 92.6913) and BIC (90.4929 vs. 96.8949) values are significantly lower for NNak-E, reinforcing its efficiency in balancing goodness-of-fit and model complexity.

9 Conclusion

This article proposed the New Nakagami-Exponential (NNak-E) distribution as an alternative model designed to enhance the flexibility of existing distributions in capturing skewed and heavy-tailed data. By integrating the adaptability of the Nakagami distribution with the reliability properties of the exponential distribution, the NNak-E model provides a powerful tool for statistical modeling in various fields. Theoretical properties, including the probability density function, cumulative distribution function, moments, and hazard function, were rigorously derived to establish a solid mathematical foundation. Parameter estimation was developed using the Maximum Likelihood Estimation (MLE) method. The application to survival and reliability datasets demonstrated that NNak-E provides a superior goodness of fit compared to existing models. These findings suggest that NNak-E is a flexible model for real-world applications in survival analysis, reliability engineering, and related fields.

For future work, the proposed distribution should be extended to the form of a regression model or a spline regression model, [14], [15], [16]. In addition, Bayesian estimation approaches, such as those in [17], [18], [19], [20], should also be considered.

Acknowledgments:
This research was funded by Thailand Science Research and Innovation Fund (NSRF), and King Mongkut's University of Technology North Bangkok, with Contract no. KMUTNB-FF-68-B-35.

References:

[1] Abdullahi, I., Job, O.: A new family of odd generalized Nakagami (Nak-G) distributions. *Turkish Journal of Science* 5(2), 85–101 (2020)

[2] Abdullahi, I., Simmachan, T., Phaphan, W.: On the novel three-parameter Nakagami–Rayleigh distribution and its applications. *Lobachevskii Journal of Mathematics* 45(9), 4001–4017 (2024)

[3] Sharma, H., Kumar, P.: Mixture of burr xii and nakagami distributions: Properties and applications. *Reliability: Theory & Applications* 18(4), 821–841 (2023)

[4] Huang, L.-F.: The nakagami and its related distributions. *WSEAS Transactions on Mathematics* 15, 477–485 (2016)

[5] Bashir, S., Waseem, N., Rasul, M.: A three parameter power nakagami distribution: Properties and application on the tax revenue data. *International Journal of Analysis and Applications* 21, 97 (2023)

[6] Pandey, R., Ali, D.: Bayesian analysis of nakagami distribution under generalized type-ii progressive hybrid censoring scheme. *Journal of Scientific Research of the Banaras Hindu University* 65(5), 252–264 (2021). doi:10.37398/JSR.2021.650531

[7] Kumar, K., Garg, R., Krishna, H.: Nakagami distribution as a reliability model under progressive censoring. *International Journal of System Assurance Engineering and Management* 8(1), 109–122 (2017). doi:10.1007/s13198-016-0494-3

[8] Hadziahmetovic, N., Milisic, M., Ahic Dokic, M., Hadzialic, M.: Estimation of nakagami distribution parameters based on signal samples corrupted with multiplicative and additive disturbances. In: ELMAR 2007, pp. 235–238 (2007). doi:10.1109/ELMAR.2007.4418838

[9] Badarneh, O.S., Almehmadi, F., da Costa, D.B.: The nakagami–lomax distribution: A hyper composite fading model. *IEEE Transactions on Vehicular Technology* 74(10), 16601–16606 (2025). doi:10.1109/TVT.2025.3573445

[10] Nakagami, M.: The m-distribution: A general formula of intensity distribution of rapid fading. Statistical Methods in Radio Wave Propagation, 3–36 (1960)

[11] Alzaatreh, A., Lee, C., Famoye, F.: A new method for generating families of continuous distributions. *Metron* 71(1), 63–79 (2013)

[12] Feigl, P., Zelen, M.: Estimation of exponential survival probabilities with concomitant information. Biometrics, 826–838 (1965)

[13] Murthy, D.N.P., Xie, M., Jiang, R.: Weibull Models. John Wiley & Sons, Hoboken, NJ (2004)

[14] Pailoung, C., Kamoljitprapa, P., Sookkhee, S.: Tuning spline smoothing parameters in gwas using replication-based approach. *Thailand Statistician* 23(3), 643–656 (2025). Contributed paper

[15] Pailoung, C., Kamoljitprapa, P., Sookkhee, S.: Optimization of smoothing parameters in splines in gwas using a replication strategy. In: Proceedings of the 2024 Research, Invention, and Innovation Congress: Innovative Electricals and Electronics (RI2C), pp. 188–192. IEEE, Bangkok, Thailand (2024). doi:10.1109/RI2C64012.2024.10784419

[16] Sookkhee, S., Kirdwichai, P., Baksh, F.: The optimal parameters of spline regression for snp-set analysis in genome-wide association study. *Science & Technology Asia* 26(1), 39–52 (2021)

[17] Duangsaphon, M., Sokampang, S., Na Bangchang, K.: Bayesian estimation for median discrete Weibull regression model. *AIMS Mathematics* 9(1), 270–288 (2024)

[18] Srisuradetchai, P., Niyomdecha, A.: Bayesian inference for the gamma zero–truncated poisson distribution with an application to real data. *Symmetry* 16(4), 417 (2024)

[19] Duangsaphon, M., Sokampang, S., Na Bangchang, K.: Bayesian estimation for median discrete Weibull regression model. *AIMS Mathematics* 9(1), 270–288 (2024). doi:10.3934/math.2024016

[20] Rerngchaiyaphum, P., Duangsaphon, M.: Bayesian estimation for zero-truncated bivariate poisson regression model. *Engineering Science Journal* 9(3), 1247–1265 (2025). doi:10.28991/ESJ-2025-09-03-07

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors jointly conducted the research and manuscript drafting. Ghostwriting did not occur.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This research was funded by Thailand Science Research and Innovation Fund (NSRF), and King Mongkut's University of Technology North Bangkok with Contract no. KMUTNB-FF-68-B-35.

Conflicts of Interest

The authors declare no conflict of interest.

Creative Commons Attribution License 4.0 (Attribution 4.0 International , CC BY 4.0)

This article is published under the terms of the Creative Commons Attribution License 4.0
https://creativecommons.org/licenses/by/4.0/deed.en_US

APPENDIX

The tables referenced in Sections 8 are presented here.

Table 3. Maximum Likelihood Estimators (MLEs) and Goodness-of-Fit Statistics for Dataset 1. Source: created by the authors.

Models	MLE	$-\ell$	W	A	AIC	BIC
Nak-E	$\lambda = 0.2212$ $\beta = 4.1802$ $\theta = 0.0285$	64.8401	0.1535	0.9726	135.6803	137.9980
NNak-E	$\lambda = 0.7356$ $\beta = 3.5814$ $\theta = 0.0228$	61.8764	0.1145	0.7482	129.7527	132.0705

Table 4. Maximum Likelihood Estimators (MLEs) and Goodness-of-Fit Statistics for Dataset 2. Source: created by the authors.

Models	MLE	$-\ell$	W	A	AIC	BIC
Nak-E	$\lambda = 0.3999$ $\beta = 0.9307$ $\theta = 0.2767$	43.3456	0.1158	0.7915	92.6913	96.8949
NNak-E	$\lambda = 2.4135$ $\beta = 5.6846$ $\theta = 0.0948$	40.1447	0.0300	0.2387	86.2894	90.4929