A Novel Numerical Integration Equation Approach using the Gauss-Legendre Quadrature to Approximate the Average Run Length of Time-Series Model Running on a CUSUM Control Chart

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Abstract: - Herein, we present the comprehensive validation of an approximation of the average run length (ARL) for monitoring small-to-moderate changes in the process means of long-memory fractionally integrated moving average with exogenous variables (LFIMAX) processes with different types of long memory on a cumulative sum (CUSUM) control chart based on the numerical integration equation (NIE) method. The ARL approximation using the NIE method was obtained by resolving a system of linear equations and performing integration through the partitioning and summing of the area under the curve of a function produced from the Gauss-Legendre quadrature rule. The performances of the proposed NIE and an established analytical method were compared for mean shifts of varying sizes for LFIMAX processes on a CUSUM control chart. The numerical results indicate that the proposed ARL method performed comparably with the analytical method regarding percentage accuracy Acc(%)). Moreover, a small percentage relative deviation (DEV%) is indicated, i.e., a change in magnitude of less than 0.25 can be detected rapidly in all situations. A numerical example using real-world scenario data is also provided to illustrate the practicability of the proposed method.

Key-Words: - Approximated ARL, numerical integral equation, analytical method, long-memory process, LFIMAX(d, q, r) process, exponential white noise.

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1 Introduction

Statistical process control (SPC) tools are crucial for overseeing various industrial processes. First proposed by Walter A. Shewhart in the 1920s, SPC enhances productivity and quality by monitoring process variation, possibly due to natural or assignable causes such as tool wear. Assignable causes hinder optimal performance, making the process out of control. Control charts are essential in SPC for signaling when processes deviate, enabling prompt action when necessary, [1].

Memory-type control charts, such as cumulative sum (CUSUM) [2] and exponentially weighted moving averages (EWMA) [3], can effectively detect small-to-moderate shifts in process parameters. Using these is crucial for industries to rapidly identify process parameter shifts that could result in significant financial losses. Unlike the Shewhart control chart, which only considers the current monitoring data, past monitoring data are also considered in memory-type control charts.

When evaluating control chart performance, observations are typically assumed to be independent and identically distributed (i.i.d.) with a normal distribution. However, in practice, process data (such as autocorrelated data) often take the form of non-normal distributions, [4], [5], [6].

Time-series analysis examines the behavior of autocorrelation functions, particularly those that decrease hyperbolically. This pattern appears in various data types, such as wind speed, air temperature, air quality, economic indicators, and hydrological events. These processes exhibit long-memory properties, where past values continuously influence future ones. Researchers often use the autoregressive fractionally integrated moving-average (ARFIMA) model to analyze long-memory processes. Notably, utilizing fractional differencing parameter d helps to capture the time-series dynamics, [7], [8], [9] and [10]. In addition, external factors can significantly correlate with the primary time series; including them can enhance the analysis context and significantly improve performance and prediction accuracy, [11]. The study addresses the long-memory fractionally integrated moving-average model with an exogenous variable

(LFIMAX). Previous studies have used control charts to detect shifts in the mean of long-memory processes with normally distributed white noise, [12], [13], [14]. However, cases where the white noise is exponentially distributed have also been studied, [15], [16], which is especially important for various real-life scenarios.

Control chart performance for a particular process can be evaluated in terms of the average run length (ARL), the median run length (MRL), or the standard deviation of the run length (SDRL), with the ARL being the most often used.

For detecting deviations in the process parameter, the in-control ARL (ARL_0) must be sufficiently large to minimize false alarms, and the out-of-control ARL (ARL_1) must be as small as possible. Methods for calculating the ARL include Monte Carlo simulation, integral equations (IEs), and the Markov chain approach. The latter has been applied to analyze the ARL of various processes on CUSUM control charts while assuming that the observations are independently and identically distributed (i.i.d.), [17], [18]. Moreover, this method has been enhanced by applying Richardson extrapolation for other distributions, such as Chi-squared.

The integral equation (IE) approach, which can be numerical (NIE) or analytical, has been used to compute the ARL, [19]. Thus, in 1999, [20] employed Fredholm IEs of the second kind in NIE calculations, while others [21] approximated the IE approach using the Gauss-Legendre quadrature. A piecewise collocation method is recommended over the traditional Gauss-Legendre quadrature for the NIE method, [22].

The NIE method approximates the ARL using IE and the midpoint rule, which has served as the foundation for validating the analytical method for deriving the ARL, [23]. Most ARL derivation methods have primarily been concentrated on the process mean, [24], [25]. The present study aims to approximate the ARL using the NIE method that provides sufficient sensitivity for detecting small-to-moderate shifts in the LFIMAX process on a CUSUM control chart. To achieve this, the integral equation is formulated based on the characteristics of the CUSUM control chart applied to the LFIMAX process. Subsequently, the NIE method, employing the Gauss-Legendre quadrature for numerical integration, is implemented to solve this integral equation and obtain ARL approximations. The performance of the NIE-approximated ARL will be evaluated by comparing it with analytical ARL under various shift scenarios in the LFIMAX process.

2 Time-series Models

As mentioned previously, we have limited our study to approximate the ARL via the NIE method for a specific set of models. The model selection is sufficiently comprehensive to ensure common possible features of real data.

Table 1 details the parameters for the $\text{LFIMAX}(d,q,r)$ process, a long-memory fractionally integrated moving-average model with exogenous variables of order d, q and r, [11]. The table will outline the different parameters used to define and estimate the LFIMAX model. These parameters will include the long-memory parameter (d), the moving average parameters (q), and the parameters related to the exogenous variables (r). A clear understanding of these parameters is crucial for adequately specifying, estimating, and interpreting the results obtained from the $\text{LFIMAX}(d,q,r)$ process. Further sections will elaborate on the estimation techniques and the interpretation of these parameters within the context of time series analysis. Fractional differencing parameter (d) for $\text{LFIMAX}(d,q,r)$ can be $d \in (-0.5,\ 0.5)$, [8]; the model is categorized as exhibiting long-memory characteristics when $d \in (0, 0.5)$, giving rise to the $\text{LFIMAX}(d,q,r)$ model defined as

$$(1-B)^d Y_t = \theta_0 + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \ldots - \theta_q \varepsilon_{t-q} + \omega_1 X_{1t} + \omega_2 X_{2t} + \ldots + \omega_r X_{rt}, \quad (1)$$

where θ_0 is a constant process, $\{\theta_i\}_{i=1}^{q}$ are moving-average (MA) coefficient parameters, q is the MA order, $\{X_{rt}\}_{i=1}^{r}$ are exogenous variables of Y_t, $\{\omega_j\}_{i=1}^{r}$ are exogenous coefficient parameters, r is the exogenous order, and ε_t is assumed to comprise i.i.d. observations that are exponentially distributed ($\varepsilon_t \sim Exp(\lambda)$). Meanwhile, $(1-B)^d$ can be extended via a binomial series expansion as follows:

$$(1-B)^d := \sum_{i=0}^{\infty} \binom{d}{i} (-B)^i = 1 - dB + \frac{1}{2!} B^2 d(d-1) - \ldots, \quad (2)$$

The general $\text{LFIMAX}(d,q,r)$ model with exponential white noise can be expressed as

$$Y_t = \theta_0 + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \ldots - \theta_q \varepsilon_{t-q} + \omega_1 X_{1t} + \omega_2 X_{2t} + \ldots + \omega_r X_{rt} + (dY_{t-1} - \frac{1}{2!} d(d-1) Y_{t-2} + \ldots), \quad (3)$$

where $Y_{t-1}, Y_{t-2}, \ldots$, are initial values and $X_{1t}, X_{2t}, \ldots, X_{rt}$ are equal to 1.

Table 1. The parameters and their values for the $\text{LFIMAX}(d,q,r)$ model on a CUSUM control chart

Coefficient parameters of LFIMAX model				
θ_0	d	θ_1	θ_2	ω_1
-1	1/5	0.1		0.2
		0.1	0.2	0.2
	1/4	0.1		0.2
		0.1	0.2	0.2
	1/3	0.1		0.2
		0.1	0.2	0.2

3 Designing of the One-Sided CUSUM Chart for the LFIMAX Process with Exponential White Noise

This is based on the IE methodology introduced by [19]. Statistic C_t for the one-sided upper CUSUM chart can be computed as follows [2]:

$$C_t = \max\{0,\, C_{t-1} + Y_t - \kappa\},\ t = 1,2,\dots, \tag{4}$$

where $Y_t, t = 1,2,\dots$ is the sequence for the general form of a $\text{LFIMAX}(d,q,r)$ model with exponential white noise (*i.e.*, $\varepsilon_t \sim Exp(\lambda)$), C_0 is the starting value when $C_0 = \psi$, ψ is the initial value ($\psi \in [0,b]$) and κ is the reference value for $\kappa > 0$. As the value of κ increases, CUSUM control charts exhibit a decrease in sensitivity to small process parameter shifts while demonstrating an increased sensitivity to larger shifts.

We have assumed that the observations of the white noise are exponentially distributed with known in-control mean λ_0. When a process mean shift occurs, the value of the mean (λ) changes accordingly:

$$\lambda_1 = (1+\delta)\lambda_0,$$

where $\delta = 0.$ for the in-control process.

The performance of a control chart, both in-control and out-of-control, is typically assessed by using the ARL. ARL denotes the average number of samples required before a control chart signals that a process is out-of-control: the higher the ARL, the less sensitive the control chart is at detecting process shifts, and vice versa.

The one-sided CUSUM chart triggers an out-of-control signal whenever $C_t > b$, where b is the decision interval or upper control limit (UCL) ($b > 0$), indicating an upward shift in the process mean. The computation of b and selection of κ can be adjusted to achieve the target in-control ARL performance.

The stopping time (τ_b) used for the alarm signal for the CUSUM control chart characterized in Equation (4)) is defined as:

$$\tau_b = \inf\{\, t > 0;\, C_t > b \,\}, \tag{5}$$

If statistic C_t falls within the range of $A < C_t < B,$ the process is assumed to be in-control at time t. When assuming that $A = 0$ and $B = b$ are the lower and upper limits, respectively, the process is in the in-control state ($C_0 = \psi$) and

$$0 < C_{t-1} + \theta_0 + \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} - \dots - \theta_q\varepsilon_{t-q} + \omega_1 X_{1t} + \omega_2 X_{2t} + \dots + \omega_r X_{rt} + (dY_{t-1} - \frac{1}{2!}d(d-1)Y_{t-2} + \dots) - \kappa < b.$$

4 The Numerical Solution for the ARL IE

Let $\mathrm{L}(\psi)$ represent the ARL for a $\text{LFIMAX}(d,q,r)$ process with initial value ψ on a CUSUM control chart. The function $\mathrm{L}(\psi)$ is defined as follows:

$$\mathrm{L}(\psi) = \mathrm{E}_\infty(\tau_b) < \infty, \tag{6}$$

where $\mathrm{E}_\infty(.)$ denotes the expectation of τ_b.

Using a solution to the IE, we can derive a Fredholm IE of the second kind for the ARL accordingly [19]:

$$\mathrm{L}(\psi) = 1 + \int_0^b \mathrm{L}(u) f\begin{pmatrix} u + \kappa - \psi - \theta_0 + \theta_1\varepsilon_{t-1} + \dots + \theta_q\varepsilon_{t-q} - \omega_1 X_{1t} - \dots \\ -\omega_r X_{rt} - (dY_{t-1} - \frac{1}{2!}d(d-1)Y_{t-2} + \dots) \end{pmatrix} du + F\begin{pmatrix} \kappa - \psi - \theta_0 + \theta_1\varepsilon_{t-1} + \dots + \theta_q\varepsilon_{t-q} - \omega_1 X_{1t} - \dots \\ -\omega_r X_{rt} - (dY_{t-1} - \frac{1}{2!}d(d-1)Y_{t-2} + \dots) \end{pmatrix} \mathrm{L}(0), \tag{7}$$

where $\psi \in [0,b], \varepsilon_t$ are continuously distributed i.i.d random variables for exponential distribution $\varepsilon_t \sim Exp(\lambda),$ where $f(\psi) = \lambda\exp\{-\lambda\psi\}$ and $F(\psi) = 1 - \exp\{-\lambda\psi\}.$

4.1 Analytical Method

Here, we provide the methodology for the analytical IE method described in [26]. The analytical approach obtained from resolving IEs entails segmenting the ARL into the in-control state (ARL_0 or $\text{ARL}_{\text{Ana-in-control}}$) and out-of-control state (ARL_1 or $\text{ARL}_{\text{Ana-out-of}}$) as follows:

$$\text{ARL}_0 = \exp\{\lambda_0 b\}(1 + \exp\begin{Bmatrix} \lambda_0(\kappa - \theta_0 + \theta_1\varepsilon_{t-1} + \dots + \theta_q\varepsilon_{t-q} - \omega_1 X_{1t} - \dots \\ -\omega_r X_{rt} - (dY_{t-1} - \frac{1}{2!}d(d-1)Y_{t-2} + \dots)) \end{Bmatrix} - \lambda_0 b) - \exp\{\lambda_0\psi\}. \tag{8}$$

and

$$\mathrm{ARL}_1 = \exp\{\lambda_1 b\}\left(1+\exp\begin{Bmatrix}\lambda_1(\kappa-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1 X_{1t}-\ldots \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots))\end{Bmatrix}-\lambda_1 b\right)$$
$$-\exp\{\lambda_1\psi\}. \quad (9)$$

4.2 The NIE Method

Here, we propose a numerical technique for solving the IE, [5]. We can closely approximate the integral using the quadrature rule, which calculates the finite sum of rectangle areas with a base of b/m and heights based on f values at the midpoints of intervals of length b/m starting from zero. Once the quadrature rule is established, interval $[0,b]$ is divided into parts $0 \le \{a_j, j=1,2,...,m\} \le b$ accompanied by a set of constant weights $\{w_j,\ j=1,2,...,m\}$.

The ARL approximation is accomplished as follows:

$$\int_0^b W(u)f(u)du \approx \sum_{j=1}^m w_j f(a_j),$$

where $a_j = \dfrac{b(2j-1)}{2m}$ and $w_j = \dfrac{b}{m}$.

Let $\mathrm{L}_{NIE}(\psi)$ represent the approximate ARL method obtained using the NIE approach by applying the Gauss-Legendre rule. This is accomplished by solving a system of algebraic linear equations:

$$\mathrm{L}_{NIE}(a_i) = 1+\mathrm{L}_{NIE}(a_1)F\begin{pmatrix}\kappa-a_i-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1X_{1t}-\ldots \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots)\end{pmatrix}$$
$$+\sum_{j=1}^m w_j \mathrm{L}_{NIE}(a_j)f\begin{pmatrix}a_j+\kappa-a_i-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1X_{1t}-\ldots \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots)\end{pmatrix} \quad (10)$$

Thus,

$$\mathrm{L}_{NIE}(a_1) = 1+\mathrm{L}_{NIE}(a_1)\Big[F\begin{pmatrix}\kappa-a_1-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1X_{1t}-\ldots \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots)\end{pmatrix}$$
$$+w_1 f\begin{pmatrix}\kappa-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1X_{1t}-\ldots \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots)\end{pmatrix}\Big]$$
$$+\sum_{j=2}^m w_j \mathrm{L}_{NIE}(a_j)f\begin{pmatrix}a_j+\kappa-a_1-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1X_{1t}-\ldots \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots)\end{pmatrix}$$

$$\mathrm{L}_{NIE}(a_2) = 1+\mathrm{L}_{NIE}(a_1)\Big[F\begin{pmatrix}\kappa-a_2-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1X_{1t}-\ldots \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots)\end{pmatrix}$$
$$+w_1 f\begin{pmatrix}a_1+\kappa-a_2-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1X_{1t}-\ldots \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots)\end{pmatrix}\Big]$$
$$+\sum_{j=2}^m w_j \mathrm{L}_{NIE}(a_j)f\begin{pmatrix}a_j+\kappa-a_2-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1X_{1t}-\ldots \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots)\end{pmatrix}$$

$$\vdots \qquad\qquad \vdots$$

$$\mathrm{L}_{NIE}(a_m) = 1+\mathrm{L}_{NIE}(a_1)\Big[F\begin{pmatrix}\kappa-a_m-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1X_{1t}-\ldots \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots)\end{pmatrix}$$
$$+w_1 f\begin{pmatrix}a_1+\kappa-a_m-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1X_{1t}-\ldots \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots)\end{pmatrix}\Big]$$
$$+\sum_{j=2}^m w_j \mathrm{L}_{NIE}(a_j)f\begin{pmatrix}a_j+\kappa-a_2-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1X_{1t}-\ldots \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots)\end{pmatrix}$$

Alternatively, this can be represented in matrix form as:

$$\mathbf{L}_{m\times1} = \mathbf{1}_{m\times1}+\mathbf{H}_{m\times m}\mathbf{L}_{m\times1} \quad \text{or} \quad (\mathbf{I}_m-\mathbf{H}_{m\times m})\mathbf{L}_{m\times1}=\mathbf{1}_{m\times1}, \quad (11)$$

where $\mathbf{I}_m = diag(1,1,\ldots,1)$ is a unit matrix of order m. Matrix $\mathbf{H}_{m\times m}$ contains entries defined by

$$\mathbf{H}_{m\times m} = \begin{bmatrix} h_{11} & h_{12} & \ldots & h_{1m} \\ h_{21} & h_{22} & \ldots & h_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ h_{m1} & h_{m2} & \ldots & h_{mm}\end{bmatrix}.$$

Thus, $h_{ij};\ i,j=1,2,....,m$ can be expanded as:

$$h_{ij} = F\begin{pmatrix}\kappa-a_i-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1X_{1t}-\ldots \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots)\end{pmatrix}$$
$$+w_j f\begin{pmatrix}a_j+\kappa-a_i-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1X_{1t}-\ldots \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots)\end{pmatrix}.$$

Next, we define column vectors $\mathbf{L}_{m\times1}$ and $\mathbf{1}_{m\times1}$ as:

$$\mathbf{L}_{m\times1} = \begin{bmatrix}\mathrm{L}_p(a_1) \\ \mathrm{L}_p(a_2) \\ \vdots \\ \mathrm{L}_p(a_m)\end{bmatrix} \quad \text{and} \quad \mathbf{1}_{m\times1} = \begin{bmatrix}1 \\ 1 \\ \vdots \\ 1\end{bmatrix}.$$

If $(\mathbf{I}_m-\mathbf{H}_{m\times m})^{-1}$, exists, then the solution of the matrix equation becomes:

$$\mathbf{L}_{m\times1} = (\mathbf{I}_m-\mathbf{H}_{m\times m})^{-1}\mathbf{1}_{m\times1}. \quad (12)$$

We solve a system of equations to approximate the values for $\mathrm{L}_p(a_1), \mathrm{L}_p(a_2), ..., \mathrm{L}_p(a_m)$.

Thus, the approximate ARL derived from the NIE method for a LFIMAX(d,q,r) process on a CUSUM chart becomes:

$$\mathrm{ARL}_{\mathrm{NIE}} = \mathrm{L}_{NIE}(\varphi) = 1+\mathrm{L}_{NIE}(a_1)F\begin{pmatrix}\kappa-\psi-\theta_0+\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\omega_1X_{1t} \\ -\ldots-\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+\ldots)\end{pmatrix}$$

$$+\sum_{j=1}^{m} w_j \mathrm{L}_{NIE}(a_j)\, f\begin{pmatrix} a_j+\kappa-\psi-\theta_0+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\omega_1 X_{1t}-... \\ -\omega_r X_{rt}-(dY_{t-1}-\frac{1}{2!}d(d-1)Y_{t-2}+...) \end{pmatrix}, \quad (13)$$

where $w_j=\frac{b}{m}$ and $a_j=\frac{b(2j-1)}{2m}; j=1,2,\ldots,m.$

5 Numerical Study

We compared the performances of the approximated and analytical ARL methods via a numerical simulation study. The NIE method was calculated using Equation (13) with $m=800$ nodes in the Gauss-Legendre quadrature rule. The CUSUM control chart was optimized using parameters $\kappa\in\{1.5, 2.0, 2.5\}$ to minimize the ARL and compute the upper control limit (b), thereby specifying ARL_0 $\in\{100,370,500\}$ when $\lambda_0=1$. We assigned LFIMAX(d, 1, 1), LFIMAX(d, 2, 1) models with $d=1/5, 1/4, 1/3; \theta_1=0.1, \theta_2=0.2; \omega_1=0.3$.

We concentrated on scenarios with increasing shifts in the process mean. The out-of-control process mean was $\lambda_1=(1+\delta)\lambda_0$, where shift size δ was set as {0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2}.

The following performance metrics were used in the evaluation. The percentage accuracy (Acc(%)):

$$\mathrm{Acc}(\%)=100-\left|\frac{\mathrm{ARL}_{\mathrm{Ana}}-\mathrm{ARL}_{\mathrm{NIE}}}{\mathrm{ARL}_{\mathrm{Ana}}}\right|\times 100\% \quad (14)$$

where $\mathrm{ARL}_{\mathrm{NIE}}$ and $\mathrm{ARL}_{\mathrm{Ana}}$ represent the ARL results using the NIE and analytical methods, respectively.

The percent relative deviation (Dev(%), [27]):

$$\mathrm{Dev}(\%)=\left|\frac{\mathrm{ARL}_{\mathrm{NIE\ out\text{-}of}}-\mathrm{ARL}_{\mathrm{NIE\ in\text{-}control}}}{\mathrm{ARL}_{\mathrm{NIE\ in\text{-}control}}}\right|\times 100\%, \quad (15)$$

where $\mathrm{ARL}_{\mathrm{NIE\ in\text{-}control}}$ and $\mathrm{ARL}_{\mathrm{NIE\ out\text{-}of}}$ denote the ARL results using the NIE method for in-control and out-of-control states, respectively.

Table 2, Table 3, Table 4, Table 5, Table 6 and Table 7 in Appendix provide the ARL_1 results for the NIE and analytical methods. Both effectively detected upward shifts in the mean across all scenarios tested. The ARL_1 results for NIE closely align with those for the analytical method, with the The percentage accuracy (Acc(%)) exceeding 95%, thereby indicating excellent agreement between the two methods. We determined that δ should have a maximum value of 2 for rapid signalling with ARL_1 approaching 3.

Furthermore, extensive computation results show that the in-control parameters for the CUSUM chart (κ, b) affected the performance of the CUSUM chart better in detecting the mean shifts for the LFIMAX processes. For $\kappa=1.5,\ 2.0,$ or 2.5, the upper control limit (b) is calculated using Equation (13) for the target ARL_0 values of 100, 370, or 500. The results reveal that as κ increased, b decreased for the upper-sided CUSUM chart with both models. Moreover, the ARL decreased as the shift size (δ) increased for $\kappa=1.5$, 2.0, and 2.5. Notably, $\kappa=1.5$ outperformed $\kappa=2.0$ and 2.5.

Both methods could rapidly detect small-to-moderate shifts ($\delta\le 0.25$). For instance, the ARL_1 results of the NIE method for ($\kappa=1.5$, b = 2.82038) and ARL_0 = 100 provided percent relative deviation (Dev(%)) values in a minimum 64.16% reduction and a maximum 96.14% reduction (Table 2, Appendix). That is, changes of magnitudes in shift size smaller than 0.25 can be rapidly detected for all scenarios.

Altogether, the results indicate that the proposed method could efficiently detect small-to-moderate changes in the mean of LFIMAX(d, q, r) processes on a CUSUM control chart.

6 Application of the Proposed Method to a Real-World Scenario

We illustrate the application of the proposed method using real data from [28]. In this example, the relationship between monthly data on the crude oil WTI futures price (response) and PTT stock price (exogenous variable) from December 2017 to March 2021 can be well described by using a LFIMAX model, as determined by using the Eviews statistical software package. Based on a retrospective analysis of a set of 40 observations as historical data, the p-values for the model parameters were all less than 0.05 (Table 8), indicating that they were all statistically significant. For the resulting LFIMAX(0.40335, 1, 1) model and using parameter values of $\theta_0=29.61795$, $\theta_1=0.47462$, and X_{1t} = 0.26983, the relationship between the response and exogenous variable was elucidated as follows:

$$\begin{aligned} Y_t &= 29.61795+\varepsilon_t-0.47462\varepsilon_{t-1}+0.26983X_{1t} \\ &+0.40334Y_{t-1}+0.1203Y_{t-2}+0.06404Y_{t-3}+... \end{aligned} \quad (16)$$

After that, the distribution of the residuals (white noise) (3.56019, 8.09521, 3.47684, 3.77699, 1.88500, 2.88198, 1.30553, 0.92787, 1.58690, 0.63425, 3.46934, 1.46899, 3.84035, 0.42863) from the LFIMAX(0.40334, 1, 1) model was analyzed using the SPSS software package (Table 9). The

Kolmogorov-Smirnov test p-values were more than 0.05, thus indicating no statistical significance. The white noise was exponentially distributed, with a mean of 2.6670, $\varepsilon_t \sim Exp(2.6670)$

Table 8. Parameter estimates for the LFIMAX(d, q, r) model based on real data

Parameters	Coefficient	Std. Error	t-Statistic	Prob.
C	29.61795	5.669256	5.22431	0.0000
PTT	0.26983	0.068729	3.92607	0.0004
d	0.40334	0.142053	2.83940	0.0074
MA(1)	0.47462	0.176445	2.68990	0.0108

**A significance level of 0.05.*

Table 9. White noise distribution test results for the LFIMAX(0.40334, 1, 1) process

Testing exponential white noise.	
Exponential Parameter ($\lambda = \lambda_0$)	2.6670
Kolmogorov-Smirnov	0.6465
Asymptotic Significance (2-Sided)	0.7974[ns]

[ns] non-significance level of 0.05.

By using $\kappa = 1.5$ and an in-control ARL of 100, 370, or 500, the fitted FIMAX (0.40334, 1, 1) model provided upper control limit (b) values of 3.193552, 3.434366, and 3.592500, respectively, as calculated by using Equation (17), the results of which are reported in Table 10.

$$\mathrm{ARL}_{\mathrm{NIE}} = 1 + \mathrm{L}_{NIE}(a_1)F\begin{pmatrix}\kappa - \psi - 29.61795 + 0.47462\varepsilon_{t-1} \\ -0.26983X_{1t} - 0.40334Y_{t-1} - 0.1203Y_{t-2} - \ldots\end{pmatrix}$$
$$+\sum_{j=1}^{m} w_j \mathrm{L}_{NIE}(a_j) f\begin{pmatrix}\kappa - \psi - 29.61795 + 0.47462\varepsilon_{t-1} \\ -0.26983X_{1t} - 0.40334Y_{t-1} - 0.1203Y_{t-2} - \ldots\end{pmatrix} \quad (17)$$

where $w_j = \dfrac{b}{m}$ and $a_j = \dfrac{b(2j-1)}{2m}; j = 1, 2, \ldots, 800.$

The results in Table 10 demonstrate that the ARL_1 results derived using both NIE and analytical methods decreased with increasing shift size, which is consistent with the numerical results in Appendix in Table 2, Table 3, Table 4, Table 5, Table 6 and Table 7. Furthermore, the ARL_1 results for both methods resulted in an Acc(%) above 95%, thereby indicating their excellent agreement. Interestingly, the proposed method enabled the rapid detection of small shifts $(\delta \le 0.25)$, [29], with $Dev(\%)$ minimized across all predetermined ARL_0 values. Thus, the NIE method shows strong potential for use in practical applications.

Table 10. The ARL_1 results were derived using the NIE and analytical methods for the LFIMAX(0.40334, 1, 1) model on a CUSUM control chart with κ = 1.5

		Parameter of CUSUM chart (k = 1.5)		
	ARL_0	100	370	500
δ	b	3.193552	3.434366	3.592500
0.25	NIE	27.368	81.004	96.320
	Analytical	27.392	81.011	96.382
	Acc(%)	99.91	99.99	99.94
	Dev(%)	72.63	78.11	80.74
0.50	NIE	12.459	30.513	33.704
	Analytical	12.473	30.52	33.725
	Acc(%)	99.89	99.98	99.94
	Dev(%)	87.54	91.75	93.26
0.75	NIE	7.557	15.895	16.942
	Analytical	7.581	15.913	16.967
	Acc(%)	99.68	99.89	99.85
	Dev(%)	92.44	95.70	96.61
1.00	NIE	5.415	10.104	10.623
	Analytical	5.436	10.136	10.657
	Acc(%)	99.61	99.68	99.68
	Dev(%)	94.59	97.27	97.88
1.25	NIE	4.264	7.315	7.633
	Analytical	4.297	7.324	7.671
	Acc(%)	99.23	99.88	99.50
	Dev(%)	95.74	98.02	98.47
1.50	NIE	3.588	5.708	6.004
	Analytical	3.61	5.743	6.016
	Acc(%)	99.39	99.39	99.80
	Dev(%)	96.41	98.46	98.80
1.75	NIE	3.148	4.751	4.974
	Analytical	3.155	4.757	4.991
	Acc(%)	99.78	99.87	99.66
	Dev(%)	96.85	98.72	99.01
2.00	NIE	2.83	4.09	4.301
	Analytical	2.833	4.095	4.302
	Acc(%)	99.89	99.88	99.98
	Dev(%)	97.17	98.89	99.14

7 Conclusion

We provided an approximate ARL method using the Gauss-Legendre quadrature for LFIMAX(d, q, r) processes with exponentially distributed white noise on a CUSUM chart. The novel NIE and established analytical methods showed excellent agreement, with Acc% of over 95%. Both methods exhibited a rapid decline in out-of-control ARL results and effectively minimized Dev% for detecting small shifts. The NIE method was also shown to be effective in real-world scenarios.

The value of design parameter κ for the CUSUM chart should be 1.5 to calculate the b value for effectively detecting upward shifts in the mean of LFIMAX(d, q, r) processes. This study approximates the ARL of the CUSUM control chart using the NIE method, although Monte Carlo simulation, Markov chains, or Martingale approaches are also viable. We suggest including

SDRL and MRL metrics alongside ARL for a more thorough performance assessment in future studies. Moreover, we will apply our approach to $\text{LFIMAX}(d, q, r)$ processes with exponential white noise on other control charts such as EWMA, GEMA [30], or DEWMA [31]. Further research will focus on optimizing the parameters of the NIE method, such as the step size, to enhance the accuracy and efficiency of ARL approximation for this specific process and control chart setting.

Finally, this research could be extended to creating commercial packages for ARL evaluation to analyze and control production processes and other related aspects.

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Conflict of Interest
The authors declare no conflict of interest.

APPENDIX

Table 2. The ARL_1 results for the NIE and analytical methods for the LFIMAX(1/5, 1,1) process on a CUSUM control chart

κ	b	ARL_0	Methods	δ							
				0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
1.5	2.82038	100	NIE	35.837	18.359	11.531	8.211	6.347	5.188	4.413	3.865
			Analytical	35.874	18.373	11.537	8.215	6.349	5.189	4.414	3.866
			Acc(%)	99.90	99.92	99.95	99.95	99.97	99.98	99.98	99.97
			Dev(%)	64.16	81.64	88.47	91.79	93.65	94.81	95.59	96.14
	4.409933	370	NIE	90.232	36.611	19.935	12.975	9.452	7.415	6.12	5.236
			Analytical	90.359	36.644	19.947	12.981	9.455	7.417	6.121	5.237
			Acc(%)	99.86	99.91	99.94	99.95	99.97	99.97	99.98	99.98
			Dev(%)	75.61	90.11	94.61	96.49	97.45	98.00	98.35	98.58
	4.798902	500	NIE	109.534	41.676	21.891	13.974	10.073	7.854	6.459	5.513
			Analytical	109.689	41.711	21.903	13.979	10.075	7.856	6.459	5.513
			Acc(%)	99.86	99.92	99.95	99.96	99.98	99.97	100.00	100.00
			Dev(%)	78.09	91.66	95.62	97.21	97.99	98.43	98.71	98.90
2	2.15211	100	NIE	37.488	19.535	12.309	8.737	6.714	5.453	4.61	4.012
			Analytical	37.522	19.549	12.315	8.741	6.717	5.454	4.61	4.013
			Acc(%)	99.91	99.93	99.95	99.95	99.96	99.98	100.00	99.98
			Dev(%)	62.51	80.47	87.69	91.26	93.29	94.55	95.39	95.99
	3.56708	370	NIE	101.271	43.046	23.62	15.21	10.887	8.38	6.794	5.722
			Analytical	101.422	43.094	23.64	15.221	10.893	8.384	6.797	5.723
			Acc(%)	99.85	99.89	99.92	99.93	99.94	99.95	99.96	99.98
			Dev(%)	72.63	88.37	93.62	95.89	97.06	97.74	98.16	98.45
	3.898257	500	NIE	126.668	51.191	27.153	17.083	12.028	9.149	7.351	6.147
			Analytical	126.872	51.253	27.178	17.096	12.035	9.153	7.353	6.149
			Acc(%)	99.84	99.88	99.91	99.92	99.94	99.96	99.97	99.97
			Dev(%)	74.67	89.76	94.57	96.58	97.59	98.17	98.53	98.77
2.5	1.58759	100	NIE	38.367	20.213	12.787	9.08	6.967	5.644	4.757	4.129
			Analytical	38.395	20.224	12.793	9.083	6.969	5.645	4.758	4.13
			Acc(%)	99.93	99.95	99.95	99.97	99.97	99.98	99.98	99.98
			Dev(%)	61.63	79.79	87.21	90.92	93.03	94.36	95.24	95.87
	2.94316	370	NIE	106.359	46.312	25.644	16.524	11.783	9.019	7.265	6.079
			Analytical	106.499	46.36	25.665	16.535	11.699	9.023	7.268	6.081
			Acc(%)	99.87	99.90	99.92	99.93	99.28	99.96	99.96	99.97
			Dev(%)	71.25	87.48	93.07	95.53	96.82	97.56	98.04	98.36
	3.258553	500	NIE	134.087	55.755	29.896	18.823	13.194	9.967	7.948	6.596
			Analytical	134.284	55.819	29.923	18.837	13.202	9.972	7.951	6.599
			Acc(%)	99.85	99.89	99.91	99.93	99.94	99.95	99.96	99.95
			Dev(%)	73.18	88.85	94.02	96.24	97.36	98.01	98.41	98.68

Table 3. The ARL_1 results for the NIE and analytical methods for the LFIMAX(1/5, 2,1) process on a CUSUM control chart

κ	b	ARL_0	Methods	δ							
				0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
1.5	2.531905	100	NIE	36.635	18.914	11.89	8.449	6.51	5.303	4.496	3.926
			Analytical	36.672	18.928	11.897	8.453	6.512	5.305	4.497	3.927
			Acc(%)	99.90	99.93	99.94	99.95	99.97	99.96	99.98	99.97
			Dev(%)	63.37	81.09	88.11	91.55	93.49	94.70	95.50	96.07
	4.02222	370	NIE	96.004	39.879	21.76	14.057	10.132	7.862	6.425	5.451
			Analytical	96.148	39.921	21.777	14.065	10.136	7.865	6.427	5.453
			Acc(%)	99.85	99.89	99.92	99.94	99.96	99.96	99.97	99.96
			Dev(%)	74.05	89.22	94.12	96.20	97.26	97.88	98.26	98.53
	4.375846	500	NIE	118.749	46.642	24.568	15.519	11.023	8.47	6.873	5.801
			Analytical	118.939	46.694	24.587	15.528	11.028	8.473	6.875	5.802
			Acc(%)	99.84	99.89	99.92	99.94	99.95	99.96	99.97	99.98
			Dev(%)	76.25	90.67	95.09	96.90	97.80	98.31	98.63	98.84
2	1.919192	100	NIE	37.904	19.851	12.528	8.892	6.827	5.538	4.674	4.063
			Analytical	37.936	19.864	12.535	8.896	6.83	5.539	4.675	4.064
			Acc(%)	99.92	99.93	99.94	99.96	99.96	99.98	99.98	99.98
			Dev(%)	62.10	80.15	87.47	91.11	93.17	94.46	95.33	95.94
	3.30444	370	NIE	103.685	44.567	24.548	15.805	11.288	8.663	7.000	5.877
			Analytical	103.834	44.616	24.569	15.816	11.294	8.667	7.003	5.879
			Acc(%)	99.86	99.89	99.91	99.93	99.95	99.95	99.96	99.97
			Dev(%)	71.98	87.95	93.37	95.73	96.95	97.66	98.11	98.41
	3.627536	500	NIE	130.205	53.326	28.416	17.874	12.552	9.512	7.613	6.343
			Analytical	130.409	53.389	28.443	17.887	12.559	9.517	7.616	6.345
			Acc(%)	99.84	99.88	99.91	99.93	99.94	99.95	99.96	99.97
			Dev(%)	73.96	89.33	94.32	96.43	97.49	98.10	98.48	98.73
2.5	1.37429	100	NIE	38.594	20.395	12.92	9.178	7.041	5.701	4.802	4.166
			Analytical	38.618	20.405	12.925	9.181	7.043	5.703	4.803	4.167
			Acc(%)	99.94	99.95	99.96	99.97	99.97	99.96	99.98	99.98
			Dev(%)	61.41	79.61	87.08	90.82	92.96	94.30	95.20	95.83
	2.716131	370	NIE	107.699	47.216	26.226	16.915	12.057	9.219	7.416	6.196
			Analytical	107.834	47.262	26.247	16.926	12.064	9.223	7.419	6.198
			Acc(%)	99.87	99.90	99.92	99.94	99.94	99.96	99.96	99.97
			Dev(%)	70.89	87.24	92.91	95.43	96.74	97.51	98.00	98.33
	3.028075	500	NIE	136.028	57.01	30.681	19.338	13.549	10.223	8.138	6.743
			Analytical	136.216	57.072	30.708	19.352	13.557	10.228	8.142	6.745
			Acc(%)	99.86	99.89	99.91	99.93	99.94	99.95	99.95	99.97
			Dev(%)	72.79	88.60	93.86	96.13	97.29	97.96	98.37	98.65

Table 4. The ARL_1 results for the NIE and analytical methods for the LFIMAX(1/4, 1,1) process on a CUSUM control chart

κ	b	ARL_0	Methods	δ							
				0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
1.5	2.933182	100	NIE	35.488	18.125	11.384	8.117	6.284	5.145	4.383	3.843
			Analytical	35.525	18.138	11.299	8.12	6.286	5.147	4.384	3.844
			Acc(%)	99.90	99.93	99.25	99.96	99.97	99.96	99.98	99.97
			Dev(%)	64.51	81.88	88.62	91.88	93.72	94.86	95.62	96.16
	4.58056	370	NIE	87.237	34.986	19.06	12.474	9.147	7.221	5.991	5.15
			Analytical	87.35	35.012	19.069	12.478	9.149	7.222	5.992	5.15
			Acc(%)	99.87	99.93	99.95	99.97	99.98	99.99	99.98	100.00
			Dev(%)	76.42	90.54	94.85	96.63	97.53	98.05	98.38	98.61
	4.994441	500	NIE	104.381	39.026	20.519	13.21	9.618	7.57	6.274	5.389
			Analytical	104.512	39.05	20.525	13.211	9.619	7.57	6.274	5.399
			Acc(%)	99.87	99.94	99.97	99.99	99.99	100.00	100.00	99.81
			Dev(%)	79.12	92.19	95.90	97.36	98.08	98.49	98.75	98.92
2	2.23736	100	NIE	37.316	19.407	12.221	8.676	6.67	5.42	4.583	3.993
			Analytical	37.351	19.421	12.228	8.68	6.672	5.422	4.585	3.994
			Acc(%)	99.91	99.93	99.94	99.95	99.97	99.96	99.96	99.97
			Dev(%)	62.68	80.59	87.78	91.32	93.33	94.58	95.42	96.01
	3.665772	370	NIE	100.251	42.417	23.243	14.973	10.729	8.27	6.716	5.662
			Analytical	100.402	42.465	23.263	14.983	10.734	8.274	6.717	5.664
			Acc(%)	99.85	99.89	99.91	99.93	99.95	99.95	99.99	99.96
			Dev(%)	72.91	88.54	93.72	95.95	97.10	97.76	98.18	98.47
	4.00074	500	NIE	125.159	50.302	26.637	16.765	11.82	9.006	7.249	6.073
			Analytical	125.363	50.362	26.661	16.777	11.827	9.01	7.252	6.074
			Acc(%)	99.84	99.88	99.91	99.93	99.94	99.96	99.96	99.98
			Dev(%)	74.97	89.94	94.67	96.65	97.64	98.20	98.55	98.79
2.5	1.663991	100	NIE	38.273	20.138	12.733	9.04	6.937	5.621	4.739	4.115
			Analytical	38.302	20.15	12.739	9.044	6.939	5.623	4.74	4.116
			Acc(%)	99.92	99.94	99.95	99.96	99.97	99.96	99.98	99.98
			Dev(%)	61.73	79.86	87.27	90.96	93.06	94.38	95.26	95.89
	3.025377	370	NIE	105.811	45.949	25.413	16.371	11.677	8.942	7.207	6.035
			Analytical	105.955	45.998	25.434	16.382	11.684	8.946	7.21	6.037
			Acc(%)	99.86	99.89	99.92	99.93	99.94	99.96	99.96	99.97
			Dev(%)	71.40	87.58	93.13	95.58	96.84	97.58	98.05	98.37
	3.342254	500	NIE	133.294	55.25	29.585	18.621	13.056	9.869	7.875	6.541
			Analytical	133.493	55.314	29.612	18.635	13.064	9.874	7.878	6.543
			Acc(%)	99.85	99.88	99.91	99.92	99.94	99.95	99.96	99.97
			Dev(%)	73.34	88.95	94.08	96.28	97.39	98.03	98.43	98.69

Table 5. The ARL_1 results for the NIE and analytical methods for the LFIMAX(1/4, 2,1) process on a CUSUM control chart

κ	b	ARL_0	Methods	δ							
				0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
1.5	2.62917	100	NIE	36.381	18.734	11.772	8.37	6.455	5.264	4.468	3.905
			Analytical	36.418	18.749	11.779	8.374	6.457	5.266	4.469	3.905
			Acc(%)	99.90	99.92	99.94	99.95	99.97	99.96	99.98	100.00
			Dev(%)	63.62	81.27	88.23	91.63	93.55	94.74	95.53	96.10
	4.147247	370	NIE	94.285	38.885	21.195	13.717	9.915	7.718	6.325	5.38
			Analytical	94.425	38.925	21.211	13.725	9.919	7.72	6.327	5.381
			Acc(%)	99.85	99.90	99.92	99.94	99.96	99.97	99.97	99.98
			Dev(%)	74.52	89.49	94.27	96.29	97.32	97.91	98.29	98.55
	4.51	500	NIE	116.079	45.169	23.759	15.045	10.727	8.275	6.74	5.707
			Analytical	116.261	45.217	23.777	15.053	10.731	8.277	6.742	5.708
			Acc(%)	99.84	99.89	99.92	99.95	99.96	99.98	99.97	99.98
			Dev(%)	76.78	90.97	95.25	96.99	97.85	98.35	98.65	98.86
2	1.99988	100	NIE	37.769	19.747	12.456	8.841	6.789	5.509	4.652	4.046
			Analytical	37.801	19.761	12.462	8.845	6.792	5.511	4.653	4.047
			Acc(%)	99.92	99.93	99.95	99.95	99.96	99.96	99.98	99.98
			Dev(%)	62.23	80.25	87.54	91.16	93.21	94.49	95.35	95.95
	3.394407	370	NIE	102.906	44.07	24.242	15.608	11.154	8.568	6.931	5.824
			Analytical	103.055	44.119	24.263	15.618	11.16	8.572	6.933	5.826
			Acc(%)	99.86	99.89	99.91	99.94	99.95	99.95	99.97	99.97
			Dev(%)	72.19	88.09	93.45	95.78	96.99	97.68	98.13	98.43
	3.719987	500	NIE	129.067	52.631	28.002	17.612	12.377	9.39	7.525	6.277
			Analytical	129.272	52.695	28.028	17.625	12.385	9.395	7.528	6.279
			Acc(%)	99.84	99.88	99.91	99.93	99.94	99.95	99.96	99.97
			Dev(%)	74.19	89.47	94.40	96.48	97.52	98.12	98.50	98.74
2.5	1.448865	100	NIE	38.52	20.335	12.877	9.146	7.016	5.682	4.787	4.154
			Analytical	38.546	20.346	12.882	9.149	7.018	5.684	4.788	4.155
			Acc(%)	99.93	99.95	99.96	99.97	99.97	99.96	99.98	99.98
			Dev(%)	61.48	79.67	87.12	90.85	92.98	94.32	95.21	95.85
	2.795125	370	NIE	107.259	46.917	26.033	16.784	11.965	9.151	7.365	6.157
			Analytical	107.397	46.965	26.054	16.795	11.972	9.156	7.368	6.159
			Acc(%)	99.87	99.90	99.92	99.93	99.94	99.95	99.96	99.97
			Dev(%)	71.01	87.32	92.96	95.46	96.77	97.53	98.01	98.34
	3.108172	500	NIE	135.392	56.595	30.42	19.166	13.429	10.137	8.074	6.693
			Analytical	135.585	56.659	30.447	19.179	13.438	10.142	8.077	6.695
			Acc(%)	99.86	99.89	99.91	99.93	99.93	99.95	99.96	99.97
			Dev(%)	72.92	88.68	93.92	96.17	97.31	97.97	98.39	98.66

Table 6. The ARL_1 results for the NIE and analytical methods for the LFIMAX(1/3, 1,1) process on a CUSUM control chart

κ	b	ARL_0	Methods	δ							
				0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
1.5	3.12304	100	NIE	34.854	17.71	11.13	7.958	6.181	5.077	4.337	3.812
			Analytical	34.89	17.723	11.136	7.961	6.183	5.078	4.338	3.813
			Acc(%)	99.90	99.93	99.95	99.96	99.97	99.98	99.98	99.97
			Dev(%)	65.15	82.29	88.87	92.04	93.82	94.92	95.66	96.19
	4.91739	370	NIE	80.283	31.37	17.183	11.433	8.535	6.844	5.753	4.996
			Analytical	80.359	31.38	17.184	11.433	8.535	6.844	5.753	4.996
			Acc(%)	99.91	99.97	99.99	100.00	100.00	100.00	100.00	100.00
			Dev(%)	78.30	91.52	95.36	96.91	97.69	98.15	98.45	98.65
	5.42048	500	NIE	90.474	32.228	17.142	11.399	8.581	6.946	5.885	5.142
			Analytical	90.521	32.218	17.132	11.393	8.577	6.943	5.883	5.141
			Acc(%)	99.95	99.97	99.94	99.95	99.95	99.96	99.97	99.98
			Dev(%)	81.91	93.55	96.57	97.72	98.28	98.61	98.82	98.97
2	2.37191	100	NIE	37.021	19.192	12.075	8.575	6.598	5.367	4.544	3.962
			Analytical	37.058	19.206	12.082	8.579	6.6	5.369	4.545	3.9623
			Acc(%)	99.90	99.93	99.94	99.95	99.97	99.96	99.98	99.99
			Dev(%)	62.98	80.81	87.93	91.43	93.40	94.63	95.46	96.04
	3.825187	370	NIE	98.465	41.335	22.603	14.574	10.466	8.089	6.585	5.567
			Analytical	98.614	41.38	22.622	14.583	10.471	8.093	6.587	5.567
			Acc(%)	99.85	99.89	99.92	99.94	99.95	99.95	99.97	100.00
			Dev(%)	73.39	88.83	93.89	96.06	97.17	97.81	98.22	98.50
	4.167395	500	NIE	122.491	48.755	25.752	16.227	11.472	8.77	7.082	5.951
			Analytical	122.691	48.813	25.774	16.237	11.478	8.774	7.085	5.953
			Acc(%)	99.84	99.88	99.91	99.94	99.95	99.95	99.96	99.97
			Dev(%)	75.50	90.25	94.85	96.75	97.71	98.25	98.58	98.81
2.5	1.782408	100	NIE	38.113	20.012	12.643	8.975	6.888	5.584	4.709	4.092
			Analytical	38.143	20.025	12.649	8.978	6.891	5.585	4.711	4.093
			Acc(%)	99.92	99.94	99.95	99.97	99.96	99.98	99.96	99.98
			Dev(%)	61.89	79.99	87.36	91.03	93.11	94.42	95.29	95.91
	3.15393	370	NIE	104.885	45.342	25.03	16.119	11.503	8.816	7.114	5.963
			Analytical	105.031	45.39	25.051	16.129	11.509	8.821	7.117	5.965
			Acc(%)	99.86	99.89	99.92	99.94	99.95	99.94	99.96	99.97
			Dev(%)	71.65	87.75	93.24	95.64	96.89	97.62	98.08	98.39
	3.473424	500	NIE	131.949	54.405	29.068	18.289	12.831	9.709	7.757	6.451
			Analytical	132.153	54.47	29.095	18.302	12.838	9.714	7.76	6.454
			Acc(%)	99.85	99.88	99.91	99.93	99.95	99.95	99.96	99.95
			Dev(%)	73.61	89.12	94.19	96.34	97.43	98.06	98.45	98.71

Table 7. The ARL_1 results for the NIE and analytical methods for the LFIMAX(1/3, 2,1) process on a CUSUM control chart

κ	b	ARL_0	Methods	δ							
				0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
1.5	2.786773	100	NIE	35.936	18.427	11.574	8.24	6.366	5.202	4.423	3.872
			Analytical	35.974	18.441	11.581	8.243	6.368	5.203	4.424	3.872
			Acc(%)	99.89	99.92	99.94	99.96	99.97	99.98	99.98	100.00
			Dev(%)	64.06	81.57	88.43	91.76	93.63	94.80	95.58	96.13
	4.361727	370	NIE	91.024	37.048	20.174	13.114	9.538	7.47	6.157	5.262
			Analytical	91.154	37.082	20.187	13.119	9.541	7.472	6.158	5.263
			Acc(%)	99.86	99.91	99.94	99.96	99.97	99.97	99.98	99.98
			Dev(%)	75.40	89.99	94.55	96.46	97.42	97.98	98.34	98.58
	4.744976	500	NIE	110.846	42.364	22.254	14.179	10.196	7.933	6.51	5.548
			Analytical	111.008	42.402	22.266	14.184	10.199	7.934	6.511	5.549
			Acc(%)	99.85	99.91	99.95	99.96	99.97	99.99	99.98	99.98
			Dev(%)	77.83	91.53	95.55	97.16	97.96	98.41	98.70	98.89
2	2.126025	100	NIE	37.538	19.573	12.335	8.755	6.727	5.463	4.616	4.018
			Analytical	37.572	19.587	12.342	8.759	6.73	5.464	4.617	4.019
			Acc(%)	99.91	99.93	99.94	99.95	99.96	99.98	99.98	99.98
			Dev(%)	62.46	80.43	87.67	91.25	93.27	94.54	95.38	95.98
	3.53719	370	NIE	101.568	43.23	23.731	15.281	10.934	8.413	6.818	5.739
			Analytical	101.718	43.278	23.751	15.291	10.94	8.417	6.82	5.741
			Acc(%)	99.85	99.89	99.92	99.93	99.95	99.95	99.97	99.97
			Dev(%)	72.55	88.32	93.59	95.87	97.04	97.73	98.16	98.45
	3.86731	500	NIE	127.104	51.451	27.305	17.177	12.09	9.191	7.381	6.17
			Analytical	127.309	51.513	27.331	17.19	12.097	9.196	7.384	6.172
			Acc(%)	99.84	99.88	99.90	99.92	99.94	99.95	99.96	99.97
			Dev(%)	74.58	89.71	94.54	96.56	97.58	98.16	98.52	98.77
2.5	1.56402	100	NIE	38.395	20.235	12.803	9.091	6.975	5.651	4.762	4.134
			Analytical	38.422	20.246	12.809	9.095	6.978	5.652	4.763	4.134
			Acc(%)	99.93	99.95	99.95	99.96	99.96	99.98	99.98	100.00
			Dev(%)	61.61	79.77	87.20	90.91	93.03	94.35	95.24	95.87
	2.9179	370	NIE	106.52	46.42	25.713	16.57	11.815	9.042	7.283	6.093
			Analytical	106.661	46.468	25.734	16.581	11.822	9.046	7.285	6.095
			Acc(%)	99.87	99.90	99.92	99.93	99.94	99.96	99.97	99.97
			Dev(%)	71.21	87.45	93.05	95.52	96.81	97.56	98.03	98.35
	3.232865	500	NIE	134.32	55.904	29.989	18.884	13.236	9.997	7.97	6.613
			Analytical	134.517	55.968	30.016	18.898	13.244	10.002	7.973	6.616
			Acc(%)	99.85	99.89	99.91	99.93	99.94	99.95	99.96	99.95
			Dev(%)	73.14	88.82	94.00	96.22	97.35	98.00	98.41	98.68