Exact Average Run Length Evaluation on a Two-Sided Extended EWMA Control Chart for the Moving Average Process

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Abstract: - The Extended Exponentially Weighted Moving Average (Extended EWMA) control chart is a type of control chart that is effective in promptly identifying minor deviations. The evaluation of control charts can be conducted using the average run length (ARL). Finding the explicit formulas for the ARL on a two-sided Extended EWMA control chart for the moving average process (MA(q)) has not been reported previously and the purpose of this research. The processes considered the MA(2) and MA(3) processes, all with exponential white noise. The accuracy of the analytical solution was assessed using the Extended EWMA control chart compared to the numerical integral equation (NIE) approach. The results show that there is minimal difference between the ARL by the explicit formula and the NIE. In addition, the performance of the Extended EWMA control chart is better than the EWMA control chart for all shift sizes, both the MA(2) and MA(3) processes. Finally, the analytical solution for the ARL is utilized using actual data on the monthly fuel price in Thailand, to showcase the effectiveness of the suggested approach.

Key-Words: - Average Run Length, Moving Average process, The Extended Exponentially Weighted Moving Average control chart, Explicit Formula, Numerical Integral Equation, The EWMA control chart, ARL, NIE method.

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1 Introduction

The Control chart is a statistical process control tool used in manufacturing. [1], introduced the control chart in 1931. The Shewhart control chart has performance in detecting shift sizes in the processes. The cumulative sum chart (CUSUM), [2], and EWMA control charts, [3], are efficient in detecting small changes. [4], presented the modified exponentially weighted moving average (Modified EWMA) control chart developed from the EWMA control chart. [5], developed a Modified EWMA control chart that can detect small changes. These are proficient at identifying little deviations. The Extended EWMA control chart, [6], is an effective performance management chart designed to detect subtle alterations in the monitored process.

The efficiency of control charts can be evaluated using the ARL, [7]. It is partitioned into two states. For instance, ARL_0 represents the anticipated number of observations required before a process that is in control is deemed to be out-of-control and should be substantial. On the other hand, ARL_1 denotes the expected number of observations gathered from an out-of-control process and it should be minimized. Prior studies have demonstrated that the ARL can be calculated using diverse methodologies. [8], in their study, presented explicit formulas and NIE for the ARL of the SARX(P,r)L model using a CUSUM. [9], developed the efficacy of the CUSUM for trend stationary seasonal autocorrelated data. [10], introduced a method to accurately calculate the run length on an EWMA control chart for a stationary MA(q) with exogenous variables. [11], developed a method to design the performance of an EWMA control chart for a seasonal moving average process with exogenous variables. [12], introduced the concept of the ARL for a multivariate EWMA control chart using Monte Carlo simulation. [13], devised the NIE technique to calculate the ARL on the Extended EWMA control chart for the AR(p) process.

There is a body of literature on evaluating the ARL with explicit formulas. [14], obtained the explicit formula for the ARL in the EWMA control chart for the trend exponential AR(1) process. [15], examined the ARL by applying the explicit formula to the EWMA control chart for a seasonal moving average model of order q with exponential white noise. In a recent study, [16], introduced a Modified EWMA control chart that derives the existing chart from its specific instances. [17], enhanced the Bayesian Modified EWMA control chart and its utilization in the mechanical and sports business.

However, the derivation of the explicit formula for the ARL on a two-sided Extended EWMA control chart for the MA(q) process has not been reported previously. The objective of this study is to determine the explicit formula for the ARL on a two-sided Extended EWMA control chart. This will be done for the MA(q) process, as well as the MA(2) and MA(3) processes. The explicit formula for the ARL was compared with the NIE technique for the purpose of benchmarking. In addition, the ability of explicit formulae to calculate the ARL on a two-sided Extended EWMA control chart was compared to the EWMA control chart using both simulated and real-world data on the monthly fuel price in Thailand.

2 Materials and Methods

The EWMA control chart, which is capable of detecting process variance. The EWMA control chart is illustrated below;

$$Z_t = (1-\lambda) Z_{t-1} + \lambda X_t, t = 1,2,3,... \quad (1)$$

where X_t is a process with mean, λ is exponential smoothing parameters with $0 < \lambda \le 1$ and Z_0 is the initial value of the EWMA statistics, $Z_0 = u$. The upper limit is UCL = *h*, respectively. The upper control limit (UCL) and the lower control limit (LCL) are

$$UCL/LCL = \mu_0 \pm L\sigma\sqrt{\frac{\lambda}{2-\lambda}} \quad (2)$$

where μ_0 is process mean of moving average process, σ is the process standard deviation parameter and *L* is a suitable control limit width.

It is derived from the EWMA control chart. This performance control chart is effective for detecting subtle deviations in the monitored process. The Extended EWMA statistic can be formulated in the following manner:

$$E_t = \lambda_1 X_t - \lambda_2 X_{t-1} + (1-\lambda_1+\lambda_2)E_{t-1}, t = 1,2,3,... \quad (3)$$

where X_t is a process with mean, λ_1 and λ_2 are exponential smoothing parameters with $0 \le \lambda_2 < \lambda_1 < 1$ and E_0 is the initial value of the Extended EWMA statistics, $E_0 = u$ and $\varepsilon_0 = v$. The UCL and the LCL are:

$$UCL/LCL = \mu_0 \pm Q\sigma\sqrt{\frac{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}}, \quad (4)$$

where μ_0 is process mean of moving average process, σ is the process standard deviation parameter and *Q* is a suitable control limit width.

3 The Explicit formulas of the ARL on a Two-Sided Extended EWMA Control Chart of MA(q) Process

3.1 The Exact Solution of the ARL on a Two-Sided Extended EWMA for MA(q) Process

The equation for the observations in the moving average process, or MA(q) process, where the noise follows an exponential distribution is defined as:

$$X_t = \eta + \varepsilon - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} - ... - \theta_q\varepsilon_{t-q} \text{ or}$$

$$X_t = \eta + \varepsilon_t - \sum_{i=1}^{q}\theta_i\varepsilon_{t-i} \quad (5)$$

where η is a suitable constant, θ_i is a moving average coefficient at $i = 1,2,3,...,q$ such that $|\theta_q| < 1$ and ε_t is white noise sequence of exponential $(\varepsilon_t \sim Exp(\alpha))$. The probability density function of ε_t is given by $f(x) = \frac{1}{\alpha}e^{-\frac{x}{\alpha}}$ where $x \ge 0$. The Extended EWMA statistics E_t can be written as:

$$E_t = \lambda_1(\eta + \varepsilon_t - \sum_{i=1}^{q}\theta_i\varepsilon_{t-i}) - \lambda_2\varepsilon_{t-1} + (1-\lambda_1+\lambda_2)E_{t-1}, t = 1,2,3,...$$

$$E_t = \lambda_1\eta + \lambda_1\varepsilon_t - \lambda_1\sum_{i=1}^{q}\theta_i\varepsilon_{t-i} - \lambda_2\varepsilon_{t-1} + (1-\lambda_1+\lambda_2)E_{t-1}$$

$$E_t = \lambda_1\eta + \lambda_1\varepsilon_t + (1-\lambda_1+\lambda_2)E_{t-1} - (\lambda_1\theta_i - \lambda_2)\varepsilon_{t-1} - \lambda_1\sum_{i=1}^{q}\theta_1\varepsilon_{t-1} - \lambda_1\varepsilon_1 \quad (6)$$

The upper limit and lower limit are LCL = *a* and UCL = *b*, respectively. For the Extended EWMA statistics E_t in an in-control:

$$a \le E_t \le b$$

$$a < \lambda_1\eta + \lambda_1\varepsilon_t + (1-\lambda_1+\lambda_2)E_{t-1} - (\lambda_1\theta_i - \lambda_2)\varepsilon_{t-1} - \lambda_1\sum_{i=1}^{q}\theta_1\varepsilon_{t-1} - \lambda_1\varepsilon_1 < b$$

$$a - \lambda_1\eta - \lambda_1\varepsilon_t - (1-\lambda_1+\lambda_2)E_{t-1} + (\lambda_1\theta_i - \lambda_2)\varepsilon_{t-1} + \lambda_1\sum_{i=1}^{q}\theta_1\varepsilon_{t-1} < \lambda_1\varepsilon_1 <$$

$$b - \lambda_1\eta - \lambda_1\varepsilon_t - (1-\lambda_1+\lambda_2)E_{t-1} + (\lambda_1\theta_i - \lambda_2)\varepsilon_{t-1} + \lambda_1\sum_{i=1}^{q}\theta_1\varepsilon_{t-1}$$

$$\frac{a - \lambda_1\eta - \lambda_1\varepsilon_t - (1-\lambda_1+\lambda_2)E_{t-1} + (\lambda_1\theta_i - \lambda_2)\varepsilon_{t-1} + \lambda_1\sum_{i=1}^{q}\theta_1\varepsilon_{t-1}}{\lambda_1} < \lambda\varepsilon_1 <$$

$$\frac{b - \lambda_1\eta - \lambda_1\varepsilon_t - (1-\lambda_1+\lambda_2)E_{t-1} + (\lambda_1\theta_i - \lambda_2)\varepsilon_{t-1} + \lambda_1\sum_{i=1}^{q}\theta_1\varepsilon_{t-1}}{\lambda_1}$$

$$\frac{a-(1-\lambda_1+\lambda_2)u + (\lambda_1\theta_i - \lambda_2)\varepsilon_{t-1}}{\lambda_1} + \sum_{i=1}^{q}\theta_1\varepsilon_{t-1} - \eta - \varepsilon_t < \lambda\varepsilon_1 <$$

$$\frac{b-(1-\lambda_1+\lambda_2)u + (\lambda_1\theta_i - \lambda_2)\varepsilon_{t-1}}{\lambda_1} + \sum_{i=1}^{q}\theta_1\varepsilon_{t-1} - \eta - \varepsilon_t$$

Let $H_E(u)$ represents the ARL on a two-sided Extended EWMA control chart for the MA(q) process. $H_E(u)$ denotes the ARL for the MA(q) process as follows:

$$ARL = H_E(u) = E_\infty(\tau) \ge T, E_0 = u,$$

where E_∞ is the expectation value.

The function $H_E(u)$ can be derived by Fredholm integral equation of the second kind, [18], $H_E(u)$ is defined as follows:

$$H_E(u) = 1 + \int H(E_1) f(\varepsilon_1) d\varepsilon_1 :$$

$$H_E(u) = 1 + \int_a^b H(\lambda_1\eta + \lambda_1\varepsilon_t + (1-\lambda_1+\lambda_2)u - (\lambda_1\theta_i - \lambda_2)\varepsilon_{t-1} - \lambda_1\sum_{i=1}^{q}\theta_1\varepsilon_{t-1} - \lambda_1\varepsilon_1) f(y)dy \quad (7)$$

Therefore, the function $H_E(u)$ is obtained as follows:

$$H_E(u) = 1 + \frac{1}{\lambda_1}\int_a^b H(y) f\left(\frac{a-(1-\lambda_1+\lambda_2)u + (\lambda_1\theta_i - \lambda_2)\varepsilon_{t-1}}{\lambda_1} + \sum_{i=1}^{q}\theta_1\varepsilon_{t-1} - \eta - \varepsilon_t\right)dy \quad (8)$$

If $\varepsilon_t = Exp(\alpha), y_t = Exp(\alpha)$ then $y = \frac{1}{\alpha}e^{-\frac{y}{\alpha}}, y \ge 0$,

$$H_E(u) = 1 + \frac{e^{\frac{a-(1-\lambda_1+\lambda_2)u+(\lambda_1\theta_i-\lambda_2)\varepsilon_{t-1}}{\lambda_1} + \frac{\sum_{i=1}^{q}\theta_1\varepsilon_{t-1}-\eta-\varepsilon_t}{\alpha}}}{\lambda_1\alpha}\int_a^b H(y)e^{-\frac{y}{\lambda_1\alpha}}dy \quad (9)$$

where

$$J(u) = e^{\frac{a-(1-\lambda_1+\lambda_2)u+(\lambda_1\theta_i-\lambda_2)\varepsilon_{t-1}}{\lambda_1} + \frac{\sum_{i=1}^{q}\theta_1\varepsilon_{t-1}-\eta-\varepsilon_t}{\alpha}},$$

$$Y = \int_a^b H(y)e^{-\frac{y}{\lambda_1\alpha}}dy$$

Consequently, $H_E(u) = 1 + \frac{J(u)}{\lambda_1\alpha}Y$ (10)

Consider the constant *Y* and take run *H(y)*

$$Y = \frac{-\lambda_1\alpha\left(e^{-\frac{b}{\lambda_1\alpha}} - e^{-\frac{a}{\lambda_1\alpha}}\right)}{1 + \frac{1}{\lambda_1-\lambda_2}e^{\frac{-(\lambda_1\theta_1+\lambda_2)v}{\lambda_1\alpha} + \frac{\sum_{i=1}^{q}\theta_1\varepsilon_{t-1}-\eta-\varepsilon_t}{\alpha}}\left(e^{\frac{-(\lambda_1-\lambda_2)b}{\lambda_1\alpha}} - e^{\frac{-(\lambda_1-\lambda_2)a}{\lambda_1\alpha}}\right)} \quad (11)$$

Hence, while the process is in-control, the substitution of α is equivalent to α_0, the explicit formula of the ARL_0 on a two-sided Extended EWMA control chart is as follows;

$$ARL_0 = 1 - \frac{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\lambda_1\alpha_0}}\left(e^{-\frac{b}{\lambda_1\alpha_0}} - e^{-\frac{a}{\lambda_1\alpha_0}}\right)}{(\lambda_1-\lambda_2)e^{\frac{(\lambda_1\theta_1+\lambda_2)v}{\lambda_1\alpha_0} - \frac{\sum_{i=1}^{q}\theta_1\varepsilon_{t-1}-\eta-\varepsilon_t}{\alpha_0}} + \left(e^{\frac{-(\lambda_1-\lambda_2)b}{\lambda_1\alpha_0}} - e^{\frac{-(\lambda_1-\lambda_2)a}{\lambda_1\alpha_0}}\right)} \quad (12)$$

Thus, substituting α is equal α_1 into $H_E(u)$, the explicit formula of the ARL_1 on a two-sided Extended EWMA control chart can be derived as follows:

$$ARL_1 = 1 - \frac{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\lambda_1\alpha_1}}\left(e^{-\frac{b}{\lambda_1\alpha_1}} - e^{-\frac{a}{\lambda_1\alpha_1}}\right)}{(\lambda_1-\lambda_2)e^{\frac{(\lambda_1\theta_1+\lambda_2)v}{\lambda_1\alpha_1} - \frac{\sum_{i=1}^{q}\theta_1\varepsilon_{t-1}-\eta-\varepsilon_t}{\alpha_1}} + \left(e^{\frac{-(\lambda_1-\lambda_2)b}{\lambda_1\alpha_1}} - e^{\frac{-(\lambda_1-\lambda_2)a}{\lambda_1\alpha_1}}\right)}$$

(13)

Finally, the explicit formula for the ARL on a two-sided Extended EWMA control chart for the MA(q) process is given by the solution of Eq. (13). The process is stable and under control with the exponential parameter $\alpha = \alpha_1$, whereas the process is out-of-control with the exponential parameter $\alpha = \alpha_1$, and then $\alpha_1 = (1+\delta)\alpha_0$ where $\alpha_1 > \alpha_0$ and δ is the shift size.

3.2 The NIE method of the ARL on a Two-Sided Extended EWMA Control Chart of the MA(q) Process

The NIE method is estimated the ARL on a two-sided Extended EWMA control chart for the MA(q) process. Let $H_M(u)$ represents the estimated value of the ARL obtained from the m linear equation systems using the composite midpoint quadrature rule.

The evaluation of the ARL approximating NIE method on a two-sided Extended EWMA control chart is conducted as follows;

$$\int_a^b H(y) f(y)dy \approx \sum_{j=1}^{m} w_j f(x_j) \tag{14}$$

The system of m linear equations is represented as:

$L_{m\times1} = 1_{m\times1} + R_{m\times m}L_{m\times1}$ or $L_{m\times1} = (I_m - R_{m\times m})^{-1} 1_{m\times1}$

$L_{m\times1} = \left[L_{NIE}(x_1), L_{NIE}(x_1), ..., L_{NIE}(x_m)\right]^T$,

$I_m = diag(1,1,...,1)$ and $1_{m\times1} = [1,1,...,1]^T$.

Let $R_{m\times m}$ be a matrix. The m to m^{th} element matrix R is defined as follows:

$$[R_{ij}] \approx \frac{1}{\lambda_1} w_j f\left(\frac{y_j - (1-\lambda_1+\lambda_2)u + (\lambda_1\theta_1+\lambda_2)\varepsilon_{t-1}}{\lambda_1} - \sum_{i=1}^{q}\theta_1\varepsilon_{t-1} - \eta - \varepsilon_t\right)$$

The solution of NIE method can be described as

$$H_M(u) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{m} w_j f\left(\frac{y_j - (1-\lambda_1+\lambda_2)u + (\lambda_1\theta_1+\lambda_2)\varepsilon_{t-1}}{\lambda_1} - \sum_{i=1}^{q}\theta_1\varepsilon_{t-1} - \eta - \varepsilon_t\right) \tag{15}$$

where y_j is a set of the division point on the interval $[a,b]$ as $y_j = \left(j - \frac{1}{2}\right)w_j + a, j = 1,2,...,m.$ w_j is a weight of composite midpoint formula $w_j = \frac{b-a}{m}$.

4 Existence and Uniqueness of ARL

The solution derived from the explicit formulations by the ARL provides of the existence a NIE, as established by [19]. In this study, let T represents an operation on the set of all continuous functions that are defined by

$$T(H_E(u)) = 1 + \frac{1}{\lambda_1}\sum_{j=1}^{m} w_j f\left(\frac{y_j - (1-\lambda_1+\lambda_2)u + (\lambda_1\theta_1+\lambda_2)\varepsilon_{t-1}}{\lambda_1} - \sum_{i=1}^{q}\theta_1\varepsilon_{t-1} - \eta - \varepsilon_t\right)dy \tag{16}$$

According, if an operator T satisfies the condition of being a contraction, the equation $T(H_E(u)) = H_E(u)$ has a unique solution, as mentioned. For Eq. (16) to possess a solution that is both existent and unique. The theorem, commonly known as the contraction mapping theorem, was first presented in concrete form in [19].

Typically, it is used to establish the presence of a solution to an integral equation. Subsequently, [20], have widely adopted this tool for solving various existence issues in multiple mathematical areas, owing to its simplicity and usefulness.

Theorem: Assume that $T: X \to X$ is a contraction mapping with contraction constant $0 \le s < 1$, such that $\|T(L_1) - T(L_2)\| \le s\|L_1 - L_2\|$ $\forall L_1, L_2 \in X.$, satisfies this condition. According to [21], there exists a unique $L(.) \in X$ such that $T(L(u)) = L(u)$ has a unique fixed point in X.

Proof: In order to illustrate that T, as defined by the equation $T(H_E(u))$ is a contraction mapping for $H_1, H_2 \in G[a,b]$. that $\|T(H_1) - T(H_2)\| \le s\|H_1 - H_2\|$, $\forall H_1, H_2 \in G[a,b]$. with $0 \le s < 1$ under the norm $\|H_\infty\| \le \sup_{u\in[a,b]}\|H(u)\|$ From $H_E(u)$ and $T(H_E(u))$.

$$\|T(H_1) - T(H_2)\|_\infty$$

$$= \sup_{u\in[a,b]}\left|\frac{1}{\lambda_1\alpha} e^{\left(\frac{(1-\lambda_1+\lambda_2)u+(\lambda_1\theta_1+\lambda_2)\varepsilon_{t-1}}{\lambda_1} - \sum_{i=1}^{q}\theta_1\varepsilon_{t-1}-\eta-\varepsilon_t\right)} \int_a^b (H_1(y) - H_2(y)) e^{-\frac{y}{\lambda_1\alpha}} dy\right|$$

$$\le \sup_{u\in[a,b]}\left|\frac{1}{\lambda_1\alpha} e^{\left(\frac{(1-\lambda_1+\lambda_2)u+(\lambda_1\theta_1+\lambda_2)\varepsilon_{t-1}}{\lambda_1} - \sum_{i=1}^{q}\theta_1\varepsilon_{t-1}-\eta-\varepsilon_t\right)} \int_a^b (H_1(y) - H_2(y)) e^{-\frac{y}{\lambda_1\alpha}} dy\right|$$

$$= \|H_1 - H_2\|_\infty \sup_{u\in[a,b]}\left|e^{\left(\frac{(1-\lambda_1+\lambda_2)u+(\lambda_1\theta_1+\lambda_2)\varepsilon_{t-1}}{\lambda_1} - \sum_{i=1}^{q}\theta_1\varepsilon_{t-1}-\eta-\varepsilon_t\right)}\right| \left|1 - \left[e^{-\frac{b}{\lambda_1\alpha}} - e^{-\frac{a}{\lambda_1\alpha}}\right]\right| \le s\|H_1 - H_2\|_\infty$$

where

$$s=\sup_{u\in[a,b]}\left|e^{\left(\frac{(1-\lambda_1+\lambda_2)u+(\lambda_1\theta_1+\lambda_2)\varepsilon_{t-1}}{\lambda_1}-\sum_{i=1}^{q}\theta_1\varepsilon_{t-1}-\eta-\varepsilon_t\right)}\right|\left|1-\left[e^{-\frac{b}{\lambda_1\alpha}}-e^{-\frac{a}{\lambda_1\alpha}}\right]\right|$$

, $0\le s<1.$

5 Numerical Results

The relative mean index (RMI), [22], is utilized to evaluate the effectiveness of a two-sided Extended EWMA control chart with different bound control limits [a, b]. Additionally, it is utilized to compare the performance of the ARL under different λ conditions. The RMI can be calculated.

$$RMI=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{ARL_i(c)-ARL_i(s)}{ARL_i(s)}\right], \tag{17}$$

where $ARL_i(c)$ is the ARL of row *i* on the control chart being examined and the control chart with the ARL across all control charts is denoted as $ARL_i(s)$ for row *i*. A control chart is considered more successful when it has a lower RMI value, suggesting a superior ability to detect changes.

The ARL was approximated by the NIE method using the composite midpoint rule on the Extended EWMA control chart with a sample size of 1,000 nodes. The absolute percentage difference is used to evaluate the accuracy of ARL. When ARL_0 = 370, η = 0.5, λ_1 = 0.05, 0.10, λ_2 = 0.01, *v* = 0.1, θ_1 = 0.10, -0.10, 0.20, -0.20, θ_2 = 0.10, -0.10 and θ_3 = 0.10. Subsequently, the initial parameter value was examined and found to be $\alpha_0=1$. The out-of-control process, $\alpha_1=(1+\delta)\alpha_0$ is computed by determining shift size (δ) to be 0.01, 0.02, 0.03, 0.05, 0.10, 0.20, 0.30, 0.50, 1.00, 2.00 and 3.00, when a = 0.0001. The upper control limit for a two-sided Extended EWMA control chart and EWMA control charts for the MA(2) process are presented in Table 1 (Appendix) and The MA(3) process will be presented in Table 2 (Appendix). The results of ARL for the a two-sided Extended EWMA control chart for the MA(2) process using an explicit formula with NIE are compared in Tables 3 and Table 4 in Appendix. Both ARL procedures yield equivalent results.

Besides Table 5 and Table 6 in Appendix, the performance comparisons between the Extended EWMA control chart and the EWMA control chart for the MA(2) process are presented. For the case of comparison Extended EWMA control chart and the EWMA control chart for the MA(3) process presented in Table 7 (Appendix). The ARL_1 value of the Extended EWMA control chart is lower than the EWMA control chart.

6 Application to Real-world Data

The ARL was calculated using explicit formulas on a two-sided Extended EWMA control chart with ARL_0 = 370 for λ_1 = 0.05, 0.10 and λ_2 = 0.01, shift sizes (δ) equal to 0.01, 0.02, 0.03, 0.05, 0.07, 0.10, 0.20, 0.30, 0.50, 0.70, 1.00. The performance of the control chart was evaluated by comparing it to the EWMA control chart using actual data on the monthly fuel price in Thailand from January 2019 to May 2023. From the analysis of the autocorrelation function (ACF) and partial autocorrelation function (PACF), it can be concluded that this data corresponds to a stationary time series. The dataset for the MA(3) process was assigned as the significance of the the mean was 1.6156, respective, the observations of the MA(3) process was defined as $X_t=\varepsilon_t+0.544X_{t-1}+0.604X_{t-2}+0.948X_{t-3}$ and the error was exponential white noise, where $\varepsilon_t \sim Exp(1.6156)$. The RMI results show that the Extended EWMA control chart reduced the RMI values more than the EWMA control chart for all shift sizes.

In Table 8 (Appendix), the upper control limits for a two-sided Extended EWMA control chart are presented. In this study, the ARL of the Extended EWMA control chart is evaluated and compared with the EWMA control chart. The results indicate that two-sided Extended EWMA control charts are more effective than the EWMA control chart in small shift sizes, as shown in Table 9 and Figure 1 in Appendix. For the various λ_1 values, the performance of control charts perform better when λ_1 is decreased.

7 Conclusions

In this study, the ARL was used to evaluate the efficacy of control charts. Using the NIE method, the explicit formula is compared. Consequently, both methods show that the ARL values are close. The Extended EWMA control chart with various λ_1 superior performances compared to the EWMA control chart for the MA(3) process in terms of performance. When evaluating the relative effectiveness of the ARL under of the ARL under various smoothing factors, it is advisable to use a smoothing parameter set at 0.05. The simulation studies and the real-world datasets analysis, specifically employing monthly fuel prices in Thailand, ultimately produced identical results. Subsequent studies could assess the most effective variables for the ARMA model while comparing the efficacy of the Extended EWMA control chart with alternative control charts. Furthermore, it is feasible to develop formulas for the ARL values on the Extended EWMA control chart to construct new control charts or other interesting models.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Phunsa Mongkoltawat carried out the writing-original draft preparation and simulation.

Yupaporn Areepong has organized the conceptualization, writing-review and editing, and validation

Saowanit Sukparungsee has implemented the methodology and solfware.

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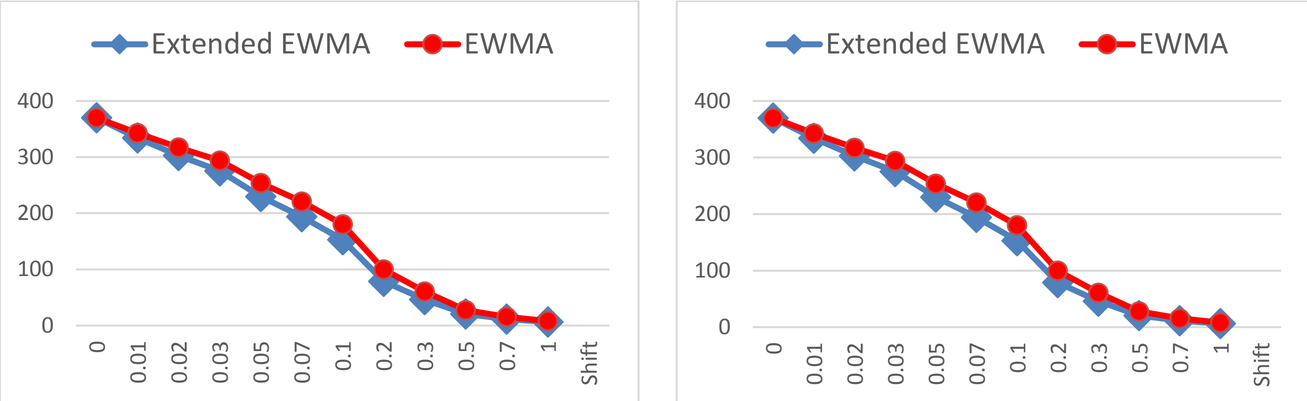
APPENDIX



(a) (b)

Fig. 1: Real-world data of the ARL values on a two-sided Extended EWMA and the EWMA control chart for MA(3) with $ARL_0 = 370$; (a) $\lambda_1 = 0.05$ and (b) $\lambda_1 = 0.10$

Table 1. Upper control limit of the Extended EWMA and the EWMA control charts for MA(2) when $\eta = 0.5$, $v = 0.1$, $a = 0.0001$, $\alpha_0 = 1$ for $ARL_0 = 370$

λ_1	θ_1	$\lambda_2 = 0.01$			
		$\theta_2 = 0.1$		$\theta_2 = -0.1$	
		Extended EWMA	EWMA	Extended EWMA	EWMA
		b	*h*	*b*	*h*
0.05	0.1	1.000525x10^{-4}	1.001058x10^{-4}	1.000536x10^{-4}	1.001038x10^{-4}
	-0.1	1.000515x10^{-4}	1.001038x10^{-4}	1.000525x10^{-4}	1.001018x10^{-4}
	0.2	1.000531x10^{-4}	1.001070x10^{-4}	1.000541x10^{-4}	1.001049x10^{-4}
	-0.2	1.000510x10^{-4}	1.001028x10^{-4}	1.000520x10^{-4}	1.001007x10^{-4}
0.10	0.1	2.5560x10^{-3}	4.650x10^{-3}	2.6060x10^{-3}	4.565x10^{-3}
	-0.1	2.5070x10^{-3}	4.560x10^{-3}	2.5560x10^{-3}	4.470x10^{-3}
	0.2	2.5810x10^{-3}	4.700x10^{-3}	2.6350x10^{-3}	4.610x10^{-3}
	-0.2	2.4830x10^{-3}	4.510x10^{-3}	2.5350x10^{-3}	4.430x10^{-3}

Table 2. Upper control limit of the Extended EWMA and the EWMA control charts for MA(3) when $\eta = 0.5$, $v = 0.1$, $a = 0.0001$, $\theta_3 = 0.1$, $\alpha_0 = 1$ for $ARL_0 = 370$

λ_1	θ_1	$\lambda_2 = 0.01$			
		$\theta_2 = 0.1$		$\theta_2 = 0.2$	
		Extended EWMA	EWMA	Extended EWMA	EWMA
		b	*h*	*b*	*h*
0.05	0.1	1.001413x10^{-4}	1.0001049x10^{-4}	1.001399x10^{-4}	1.001047x10^{-4}
	-0.1	1.001385x10^{-4}	1.0001026x10^{-4}	1.001371x10^{-4}	1.001026x10^{-4}
0.10	0.2	6.9155x10^{-3}	4.646x10^{-3}	6.8460x10^{-3}	4.647x10^{-3}
	-0.2	6.6397x10^{-3}	4.464x10^{-3}	6.5726x10^{-3}	4.465x10^{-3}

Table 3. ARL comparison of two-sided Extended EWMA control chart for MA(2) using explicit formulas against NIE method when η = 0.5 , v = 1, a = 0.0001, α_0 = 1, θ_2 = 0.1 for ARL_0 = 370

			λ_2 = 0.01					
λ_1	θ_1	Shift Size	Explicit	NIE	θ_1	Shift Size	Explicit	NIE
		0.00	370.17869	370.17869		0.00	370.68250	370.68250
		0.01	302.81599	302.81599		0.01	302.28616	302.28616
		0.02	247.07466	247.07466		0.02	247.50712	247.50712
		0.03	203.06234	203.06234		0.03	203.45600	203.45600
		0.05	138.75034	138.75034		0.05	139.06976	139.06976
		0.10	57.09582	57.09582		0.10	57.27460	57.27460
0.05	0.1	0.20	12.58015	12.58015	0.2	0.20	12.63467	12.63467
		0.30	4.02781	4.02781		0.30	4.04597	4.04597
		0.50	1.34871	1.34871		0.50	1.33515	1.33515
		1.00	1.00984	1.00984		1.00	1.00996	1.00996
		2.00	1.00025	1.00025		2.00	1.00025	1.00025
		3.00	1.00004	1.00004		3.00	1.00004	1.00004
		0.00	370.46257	370.46257		0.00	370.55266	370.55266
		0.01	302.28585	302.28585		0.01	302.06109	302.06109
		0.02	247.16732	247.16732		0.02	247.22735	247.22735
		0.03	203.09996	203.09996		0.03	203.14924	203.14924
		0.05	138.72503	138.72503		0.05	138.75862	138.75862
		0.10	57.03698	57.03698		0.10	57.05064	57.05064
0.05	-0.1	0.20	12.55049	12.55049	-0.2	0.20	12.55330	12.55330
		0.30	4.01618	4.01618		0.30	4.01692	4.01692
		0.50	1.34666	1.34666		0.50	1.34674	1.34674
		1.00	1.00975	1.00975		1.00	1.00976	1.00976
		2.00	1.00024	1.00024		2.00	1.00024	1.00024
		3.00	1.00004	1.00004		3.00	1.00004	1.00004
		0.00	370.06564	370.06564		0.00	370.06861	370.06861
		0.01	333.15193	333.15193		0.01	333.22347	333.22347
		0.02	300.52109	300.52109		0.02	300.64644	300.64644
		0.03	271.61438	271.61438		0.03	271.78147	271.78147
		0.05	223.12752	223.12752		0.05	223.35040	223.35040
		0.10	140.71766	140.71766		0.10	140.98373	140.98373
0.10	0.1	0.20	62.73804	62.73804	0.2	0.20	62.95245	62.95245
		0.30	31.74181	31.74181		0.30	31.88915	31.88915
		0.50	10.92773	10.92773		0.50	10.99616	10.99616
		1.00	2.49838	2.49838		1.00	2.51379	2.51379
		2.00	1.20123	1.20123		2.00	1.20397	1.20397
		3.00	1.06776	1.06776		3.00	1.06880	1.06880
		0.00	370.09727	370.09727		0.00	370.13349	370.13349
		0.01	333.11111	333.11111		0.01	333.14335	333.14335
		0.02	300.42319	300.42319		0.02	300.45187	300.45187
		0.03	271.47184	271.47184		0.03	271.49745	271.49745
		0.05	222.99244	222.99244		0.05	222.94495	222.94495
		0.10	140.46368	140.46368		0.10	140.47586	140.47586
0.10	-0.1	0.20	62.52894	62.52894	-0.2	0.20	62.53376	62.53376
		0.30	31.59722	31.59722		0.30	31.59938	31.59938
		0.50	10.86031	10.86031		0.50	10.86089	10.86089
		1.00	2.48319	2.48319		1.00	2.48325	2.48325
		2.00	1.19852	1.19852		2.00	1.19853	1.19853
		3.00	1.06674	1.06674		3.00	1.06674	1.06674

Table 4. ARL comparison of two-sided Extended EWMA control chart for MA(2) using explicit formulas against NIE method when η = 0.5 , v = 0.1, a = 0.0001, α_0 = 1, θ_2 = -0.1 for ARL_0 = 370

λ_2 = 0.01								
λ_1	θ_1	Shift Size	Explicit	NIE	θ_1	Shift Size	Explicit	NIE
		0.00	370.45046	370.45046		0.00	370.18644	370.18644
		0.01	302.06724	302.06724		0.01	301.88188	301.88188
		0.02	247.30410	247.30410		0.02	247.17635	247.17635
		0.03	203.27000	203.27000		0.03	203.18433	203.18433
		0.05	138.91740	138.91740		0.05	138.88449	138.88449
		0.10	57.18817	57.18817		0.10	57.199088	57.199088
0.05	0.1	0.20	12.60800	12.60800	0.2	0.20	12.61906	12.61906
		0.30	4.03704	4.03704		0.30	4.04188	4.04188
		0.50	1.35013	1.35013		0.50	1.35105	1.35105
		1.00	1.00990	1.00990		1.00	1.00994	1.00994
		2.00	1.00026	1.00026		2.00	1.00025	1.00025
		3.00	1.00004	1.00004		3.00	1.00004	1.00004
		0.00	370.17869	370.17869		0.00	370.33767	370.33767
		0.01	301.78621	301.78621		0.01	301.88595	301.88595
		0.02	247.02641	247.02641		0.02	247.08411	247.08411
		0.03	203.00350	203.00350		0.03	203.03164	203.03164
		0.05	138.68476	138.68476		0.05	138.67847	138.67847
		0.10	57.04485	57.04485		0.10	57.01804	57.01804
0.05	-0.1	0.20	12.56086	12.56086	-0.2	0.20	12.54658	12.54658
		0.30	4.02083	4.02083		0.30	4.01516	4.01516
		0.50	1.34755	1.34755		0.50	1.34654	1.34654
		1.00	1.00980	1.00980		1.00	1.00975	1.00975
		2.00	1.00025	1.00025		2.00	1.00024	1.00024
		3.00	1.00004	1.00004		3.00	1.00004	1.00004
		0.00	370.03255	370.03255		0.00	370.55832	370.55832
		0.01	333.15718	333.15718		0.01	333.66486	333.66486
		0.02	300.55677	300.55677		0.02	301.04505	301.04505
		0.03	271.67399	271.67399		0.03	272.14213	272.14213
		0.05	223.22004	223.22004		0.05	223.64729	223.64729
		0.10	140.83984	140.83984		0.10	141.17172	141.17172
0.10	0.1	0.20	62.84123	62.84123	0.2	0.20	63.03650	63.03650
		0.30	31.81380	31.81380		0.30	31.93144	31.93144
		0.50	10.96156	10.96156		0.50	11.01005	11.01005
		1.00	2.50606	2.50606		1.00	2.51594	2.51594
		2.00	1.2.0260	1.2.0260		2.00	1.20427	1.20427
		3.00	1.06828	1.06828		3.00	1.06890	1.06890
		0.00	370.06564	370.06564		0.00	370.64799	370.64799
		0.01	333.11769	333.11769		0.01	333.60681	333.60681
		0.02	300.45600	300.45600		0.02	300.87018	300.87018
		0.03	271.53244	271.53244		0.03	271.87572	271.87572
		0.05	223.01768	223.01768		0.05	223.25598	223.25598
		0.10	140.58612	140.58612		0.10	140.67230	140.67230
0.10	-0.1	0.20	62.63197	62.63197	-0.2	0.20	62.62121	62.62121
		0.30	31.66895	31.66895		0.30	31.64321	31.64321
		0.50	10.89393	10.89393		0.50	10.87520	10.87520
		1.00	2.49079	2.49079		1.00	2.48545	2.48545
		2.00	1.19988	1.19988		2.00	1.19883	1.19883
		3.00	1.06725	1.06725		3.00	1.06684	1.06684

Table 5. ARL comparison of the two-sided Extended EWMA for MA(2) against EWMA control charts when $\eta = 0.5$, $v = 0.1$, a = 0.0001, $\alpha_0 = 1$, $\theta_2 = 0.1$ for $ARL_0 = 370$

$\lambda_2 = 0.01$								
λ_1	θ_1	Shift Size	Extended EWMA	EWMA	θ_1	Shift Size	Extended EWMA	EWMA
		0.00	370.17869	370.45095		0.00	370.68250	370.92352
		0.01	302.81599	304.13148		0.01	302.28616	304.54928
		0.02	247.07466	250.65942		0.02	247.50712	251.02779
		0.03	203.06234	207.37644		0.03	203.45600	207.70061
		0.05	138.75034	143.52444		0.05	139.06976	143.77472
		0.10	57.09582	60.82568		0.10	57.27460	60.95668
0.05	0.1	0.20	12.58015	14.02273	0.2	0.20	12.63467	14.06110
		0.30	4.02781	4.56126		0.30	4.04597	4.57406
		0.50	1.34871	1.44068		0.50	1.33515	1.44271
		1.00	1.00984	1.01398		1.00	1.00996	1.01407
		2.00	1.00025	1.00039		2.00	1.00025	1.00040
		3.00	1.00004	1.00006		3.00	1.00004	1.00006
		RMI	0.0000	0.0417		**RMI**	0.0000	0.0426
		0.00	370.46257	370.78932		0.00	370.55266	370.90745
		0.01	302.28585	304.34904		0.01	302.06109	304.41590
		0.02	247.16732	250.79011		0.02	247.22735	250.82091
		0.03	203.09996	207.44516		0.03	203.14924	207.45097
		0.05	138.72503	143.51918		0.05	138.75862	143.49683
		0.10	57.03698	60.77170		0.10	57.05064	60.73650
0.05	-0.1	0.20	12.55049	13.99128	-0.2	0.20	12.55330	13.97380
		0.30	4.01618	4.54811		0.30	4.01692	4.54106
		0.50	1.34666	1.43815		0.50	1.34674	1.43683
		1.00	1.00975	1.01385		1.00	1.00976	1.01379
		2.00	1.00024	1.00039		2.00	1.00024	1.00039
		3.00	1.00004	1.00006		3.00	1.00004	1.00006
		RMI	0.0000	0.0420		**RMI**	0.0000	0.0415
		0.00	370.06564	370.37293		0.00	370.06861	370.64012
		0.01	333.15193	335.53058		0.01	333.22347	335.80800
		0.02	300.52109	304.53455		0.02	300.64644	304.81762
		0.03	271.61438	276.90469		0.03	271.78147	277.18998
		0.05	223.12752	230.14449		0.05	223.35040	230.42654
		0.10	140.71766	149.13717		0.10	140.98373	149.38762
0.10	0.1	0.20	62.73804	69.65978	0.2	0.20	62.95245	69.83084
		0.30	31.74181	36.59038		0.30	31.88915	36.70304
		0.50	10.92773	13.25218		0.50	10.99616	13.30408
		1.00	2.49838	3.04973		1.00	2.51379	3.06195
		2.00	1.20123	1.30481		2.00	1.20397	1.30715
		3.00	1.06776	1.10797		3.00	1.06880	1.10889
		RMI	0.0000	0.0865		**RMI**	0.0000	0.0858
		0.00	370.09727	370.54704		0.00	370.50878	370.64665
		0.01	333.11111	335.61746		0.01	333.16970	335.21967
		0.02	300.42319	304.55046		0.02	300.20105	304.15831
		0.03	271.47184	276.86313		0.03	271.02817	276.47887
		0.05	222.99244	230.01971		0.05	222.17418	229.65582
		0.10	140.46368	148.92039		0.10	139.42776	148.61770
0.10	0.1	0.20	62.52894	69.44962	-0.2	0.20	61.64139	69.25509
		0.30	31.59722	36.43371		0.30	30.97715	36.30932
		0.50	10.86031	13.17219		0.50	10.56973	13.11655
		1.00	2.48319	3.02938		1.00	2.41779	3.01665
		2.00	1.19852	1.30077		2.00	1.18693	1.29838
		3.00	1.06674	1.10636		3.00	1.06237	1.10543
		RMI	0.0000	0.0865		**RMI**	0.0000	0.0961

Table 6. ARL comparison of the two-sided Extended EWMA for MA(2) against EWMA control charts when $\eta = 0.5$, $v = 1$, a = 0.0001, $\alpha_0 = 1$, $\theta_2 = -0.1$ for $ARL_0 = 370$

$\lambda_2 = 0.01$								
λ_1	θ_1	Shift Size	Extended EWMA	EWMA	θ_1	Shift Size	Extended EWMA	EWMA
		0.00	370.45046	370.78932		0.00	370.18644	370.98963
		0.01	302.06724	304.34904		0.01	301.88188	304.54341
		0.02	247.30410	250.79011		0.02	247.17635	250.97442
		0.03	203.27000	207.45516		0.03	203.18433	207.61716
		0.05	138.91740	143.51918		0.05	138.88449	143.66430
		0.10	57.18817	60.77170		0.10	57.199088	60.85847
0.05	0.1	0.20	12.60800	13.99128	0.2	0.20	12.61906	14.02000
		0.30	4.03704	4.54811		0.30	4.04188	4.55823
		0.50	1.35013	1.43815		0.50	1.35105	1.43985
		1.00	1.00990	1.01385		1.00	1.00994	1.01393
		2.00	1.00026	1.00039		2.00	1.00025	1.00039
		3.00	1.00004	1.00006		3.00	1.00004	1.00006
		RMI	0.0000	0.0403		**RMI**	0.0000	0.0411
		0.00	370.17869	370.99060		0.00	370.33767	370.67096
		0.01	301.78621	304.45406		0.01	301.88595	304.16188
		0.02	247.02641	250.82808		0.02	247.08411	250.56331
		0.03	203.00350	207.43724		0.03	203.03164	207.19883
		0.05	138.68476	143.46101		0.05	138.67847	143.27017
		0.10	57.04485	60.69559		0.10	57.01804	60.58983
0.05	-0.1	0.20	12.56086	13.99510	-0.2	0.20	12.54658	13.92230
		0.30	4.02083	4.53369		0.30	4.01516	4.52250
		0.50	1.34755	1.43547		0.50	1.34654	1.43365
		1.00	1.00980	1.01372		1.00	1.00975	1.01364
		2.00	1.00025	1.00038		2.00	1.00024	1.00038
		3.00	1.00004	1.00006		3.00	1.00004	1.00006
		RMI	0.0000	0.0414		**RMI**	0.0000	0.0402
		0.00	370.03255	370.97142		0.00	370.55832	370.89924
		0.01	333.15718	336.00163		0.01	333.66486	355.97170
		0.02	300.55677	304.89888		0.02	301.04505	304.90322
		0.03	271.67399	277.17968		0.03	272.14213	277.21166
		0.05	223.22004	230.28239		0.05	223.64729	230.35412
		0.10	140.83984	149.08986		0.10	141.17172	149.20432
0.10	0.1	0.20	62.84123	69.52789	0.2	0.20	63.03650	69.63577
		0.30	31.81380	36.47416		0.30	31.93144	36.55393
		0.50	10.96156	13.18605		0.50	11.01005	13.22651
		1.00	2.50606	3.03168		1.00	2.51594	3.04192
		2.00	1.20260	1.30111		2.00	1.20427	1.30314
		3.00	1.06828	1.10648		3.00	1.06890	1.10729
		RMI	0.0000	0.0833		**RMI**	0.0000	0.0875
		0.00	370.06564	370.56369		0.00	370.64799	370.94879
		0.01	333.11769	335.56180		0.01	333.60681	335.87495
		0.02	300.45600	304.43714		0.02	300.87018	304.68964
		0.03	271.53244	276.70421		0.03	271.87572	276.90558
		0.05	223.01768	229.79770		0.05	223.25598	229.91965
		0.10	140.58612	148.64128		0.10	140.67230	148.65197
0.10	0.1	0.20	62.63197	69.21129	-0.2	0.20	62.62121	69.16159
		0.30	31.66895	36.26289		0.30	31.64321	36.21346
		0.50	10.89393	13.08767		0.50	10.87520	13.05782
		1.00	2.49079	3.00839		1.00	2.48545	2.99999
		2.00	1.19988	1.29665		2.00	1.19883	1.29491
		3.00	1.06725	1.0473		3.00	1.06684	1.10403
		RMI	0.0000	0.0771		**RMI**	0.0000	0.0814

Table 7. ARL comparison of the two-sided Extended EWMA for MA(3) against EWMA control charts when $\eta = 0.5$, $v = 0.1$, a = 0.0001, $\alpha_0 = 1$, $\theta_3 = 0.1$ for $ARL_0 = 370$

$\lambda_2 = 0.01$									
λ_1	θ_1	θ_2	Shift Size	Extended EWMA	EWMA	θ_2	Shift Size	Extended EWMA	EWMA
			0.00	370.20498	370.95263		0.00	370.21036	370.22071
			0.01	304.51335	304.86089		0.01	303.90469	304.87383
			0.02	250.94991	252.01436		0.02	250.44891	252.02506
			0.03	207.59709	209.10964		0.03	207.18318	209.11850
			0.05	143.65070	145.54769		0.05	143.36517	145.55384
			0.10	60.85302	62.49469		0.10	60.73348	62.49731
0.05	0.1	0.1	0.20	14.01891	14.70412	0.2	0.20	13.99931	14.70471
			0.30	4.55796	4.82282		0.30	4.55090	4.82298
			0.50	1.43982	1.48832		0.50	1.43895	1.48834
			1.00	1.01393	1.01631		1.00	1.01390	1.01631
			2.00	1.00039	1.00048		2.00	1.00039	1.00048
			3.00	1.00006	1.00008		3.00	1.00006	1.00008
			RMI	0.0000	0.0178		**RMI**	0.0000	0.0191
			0.00	370.15086	370.19947		0.00	370.11395	370.14050
			0.01	303.79561	304.79618		0.01	303.76563	304.74766
			0.02	250.31043	251.91220		0.02	250.28599	251.87212
			0.03	207.02930	208.98534		0.03	207.00930	208.95212
			0.05	143.20605	145.40793		0.05	143.19251	145.38487
			0.10	60.61482	62.38207		0.10	60.60940	62.37227
0.05	-0.1	-0.1	0.20	13.94746	14.65832	-0.2	0.20	13.94639	14.65613
			0.30	4.53390	4.80516		0.30	4.53363	4.80455
			0.50	1.43595	1.48507		0.50	1.43592	1.48499
			1.00	1.01376	1.01615		1.00	1.01376	1.01615
			2.00	1.00039	1.00048		2.00	1.00039	1.00048
			3.00	1.00006	1.00008		3.00	1.00006	1.00008
			RMI	0.0000	0.0191		**RMI**	0.0000	0.0191
			0.00	370.00018	370.00076		0.00	370. 03197	370.04632
			0.01	335.19492	336.67457		0.01	335.23555	336.70185
			0.02	304.23038	306.89485		0.02	304.26756	306.91871
			0.03	276.62855	280.23296		0.03	276.66261	280.25384
			0.05	229.91570	234.82719		0.05	229.94443	234.84322
			0.10	148.99010	155.10102		0.10	149.00933	155.10934
0.10	0.2	0.1	0.20	69.59228	74.85789	0.2	0.20	69.60173	74.86000
			0.30	36.55569	40.37934		0.30	36.56082	40.37961
			0.50	13.24039	15.17707		0.50	13.24228	15.17668
			1.00	3.04781	3.54819		1.00	3.04816	3.54798
			2.00	1.30453	1.40657		2.00	1.30459	1.40652
			3.00	1.10787	1.14909		3.00	1.01789	1.14907
			RMI	0.0000	0.0631		**RMI**	0.0000	0.0713
			0.00	370.00051	370.00267		0.00	370. 00504	370. 05314
			0.01	335.05662	336.53336		0.01	335.10044	336.53444
			0.02	303.97939	306.63857		0.02	304.01943	306.63860
			0.03	276.28868	279.88488		0.03	276.32533	279.88405
			0.05	229.45349	234.35026		0.05	229.48434	234.34817
			0.10	148.42009	154.50139		0.10	148.44065	154.49786
0.10	-0.2	-0.1	0.20	69.10985	74.33251	-0.2	0.20	69.11987	74.32904
			0.30	36.21077	39.99260		0.30	36.21618	39.98994
			0.50	13.06998	14.97698		0.50	13.07196	14.97557
			1.00	3.00549	3.49474		1.00	3.00586	3.49436
			2.00	1.29623	1.39532		2.00	1.29629	1.39524
			3.00	1.10458	1.14448		3.00	1.10460	1.14444
			RMI	0.0000	0.0627		**RMI**	0.0000	0.0625

Table 8. Upper control limit of the two-sided Extended EWMA and the EWMA control charts for MA(3) for real-world data when a = 0.0001, α_0 = 1.6156, θ_1 = -0.544, θ_2 = -0.604, θ_3 = -0.948 for ARL_0 = 370

λ_1	λ_2 = 0.01	
	Extended EWMA	EWMA
	b	*h*
0.05	0.0008880	0.0003374
0.10	0.53400	0.11125

Table 9. ARL comparison of the two-sided Extended EWMA control chart for MA(3) using NIE against EWMA control chart when a = 0.0001, α_0 = 1.6156, θ_1 = -0.544, θ_2 = -0.604, θ_3 = -0.948 for ARL_0 = 370

		λ_2 = 0.01	
λ_1	Shift Size	Extended EWMA	EWMA
0.05	0.00	370.26516	370.84542
	0.01	325.49395	330.13227
	0.02	286.62161	295.04683
	0.03	253.20025	260.19053
	0.05	198.83023	205.47990
	0.07	154.54037	161.42789
	0.10	112.84962	116.65870
	0.20	42.00314	46.54279
	0.30	18.43778	21.93911
	0.50	5.37001	7.92023
	0.70	2.49353	3.72849
	1.00	1.43670	1.52128
	RMI	0.0000	0.0333
0.10	0.00	370.37645	370.76754
	0.01	334.40526	342.62349
	0.02	302.89340	317.01718
	0.03	275.48185	293.95892
	0.05	229.91782	253.85104
	0.07	194.06693	220.54452
	0.10	153.22435	180.43683
	0.20	78.73455	99.89173
	0.30	46.27696	60.86984
	0.50	20.66664	27.75612
	0.70	11.53985	15.35888
	1.00	6.23665	8.02987
	RMI	0.0000	0.1348