Gain Computation for Batch H_2-FIR Filtering of Predictive Uncertain Disturbed Models using LMI

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Abstract: – The gain for the receding horizon (RH) H_2 finite impulse response (FIR) filter is derived using linear matrix inequality (LMI) under uncertainties, disturbances, initial, and measurement errors. The RH H_2-FIR filter is developed by minimizing the squared Frobenius norm of the weighted error-to-error transfer function, where the weights are related to errors. The filter is tested by a harmonic model with an uncertain system matrix, and its higher accuracy is shown against the OFIR, Kalman, maximum likelihood FIR, and unbiased FIR (UFIR) filters.

Key-Words: - RH H_2-FIR filter, OFIR filter, UFIR filter, Kalman filter, uncertainty, disturbance, robustness.

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1 Introduction

Receding horizon (RH) finite impulse response (FIR) filtering was developed in [1], and generalized in [2], for model predictive control, [3], to produce a predicted estimate at the discrete time index k over a finite horizon $[m-1, k-1]$ of N points, where $m = k-N+1$. The following advantages have been noticed: 1) bounded input bounded output (BIBO) stability, [4], 2) insensitivity to errors beyond $[m-1, k-1]$, [5], 3) round-off errors reduction, [6], and 4) higher robustness, [7]. The H_2 filter, [8], [9], has attracted attention due to the ability to operate as the robust H_∞ and energy-to-peak filters, [10], Kalman filter (KF) in white Gaussian environments, [11], and optimal FIR (OFIR) filter, [12], [13]. The H_2 filter minimizes the squared Frobenius norm of the error-to-error transfer function $\mathcal{T}$ and has closed form solutions, [14], [15]. The gain for the H_2 filter can also be computed numerically using a linear matrix inequality (LMI), [16], [17], [18], [19], [20], [21], [22].

First robust RH H_2-FIR filters for disturbed systems were developed in [23], [24], and other early designs can be found in [25], [26], [27]. Later, the RH H_2-FIR approach has resulted in various robust RH FIR structures, [28], [29], [30]. A serious drawback of the early results is that the FIR filter gain is obtained by minimizing the unweighted $\mathcal{T}$. A novel approach developed in [31], [32], suggests minimizing the squared Frobenius norm of the weighted transfer function $\mathcal{T}$. For disturbed systems, it gave the following efficient solutions: RH bias constrained H_2-FIR filter, [31], RH H_2-FIR predictor, [33], *a posteriori* optimal unbiased H_2-FIR filter, [34], and *a posteriori* H_2-FIR filter, [35]. In this paper, we apply the approach, [31], to uncertain systems under disturbances and other errors.

2 Model and Problem Formulation

Consider a linear system represented in discrete-time state-space with the following equations,

$$\begin{aligned} x_{k+1} &= (F+\Delta F_k)x_k + (E+\Delta E_k)u_k \\ &\quad +(B+\Delta B_k)w_k \,, \end{aligned} \tag{1}$$

$$y_k = (H+\Delta H_k)x_k + (D+\Delta D_k)w_k + v_k \tag{2}$$

where $x_k \in \mathbb{R}^K$, $u_k \in \mathbb{R}^L$, $y_k \in \mathbb{R}^P$, $F \in \mathbb{R}^{K\times K}$, $H \in \mathbb{R}^{P\times K}$, $E \in \mathbb{R}^{K\times L}$, $B \in \mathbb{R}^{K\times M}$, and $D \in \mathbb{R}^{P\times M}$. The uncertain matrices ΔF_k, ΔE_k, ΔB_k, ΔH_k, and ΔD_k are zero mean, norm-bounded, and mutually uncorrelated, [36]. The disturbance $w_k \in \mathbb{R}^M$ and data error $v_k \in \mathbb{R}^P$ are zero mean and mutually uncorrelated with norm-bounded error matrices $Q = E\{w_k w_k^T\}$ and $R = E\{v_k v_k^T\}$, where $E\{z\}$ means averaging of z. By reorganizing the terms, we represent (1) and (2) as

$$x_{k+1} = Fx_k + Eu_k + \xi_k \,, \tag{3}$$

$$y_k = Hx_k + \zeta_k \,, \tag{4}$$

where the vectors ξ_k and ζ_k unite the uncertainties, disturbances, and errors as

$$\xi_k = \Delta F_k x_k + \Delta E_k u_k + (B+\Delta B_k)w_k \,, \tag{5}$$

$$\zeta_k = \Delta H_k x_k + (D+\Delta D_k)w_k + v_k \,. \tag{6}$$

To derive the RH H_2-FIR filter, we will first follow, [13], and extend (3) and (4) to the horizon

$[m, k]$ to have a prediction at $k+1$. Then we will change a time variable and arrive at the RH estimate at k over $[m-1, k-1]$.

Given the state space equations (3) and (4), their extensions to $[m, k]$ are the following,

$$\begin{aligned} X_{m+1,k+1} &= (F_N + \tilde{F}_{m,k})x_m \\ &\quad +(S_N + \tilde{S}_{m,k})U_{m,k} \\ &\quad +(D_N + \tilde{D}_{m,k})W_{m,k}\,, \quad (7)\\ Y_{m,k} &= (H_N + \tilde{H}_{m,k})x_m \\ &\quad +(L_k + \tilde{L}_{m,k})U_{m,k} \\ &\quad +(T_N + \tilde{T}_{m,k})W_{m,k} + V_{m,k}\,, \quad (8) \end{aligned}$$

where all of the block vectors and matrices are defined in Appendix A.

To justify the above model, use the forward-in-time solutions and extend (3) to $[m, k]$ as

$$X_{m+1,k+1} = F_N x_m + S_N U_{m,k} + \hat{F}_N \Xi_{m,k}\,, \quad (9)$$

and similarly extend the uncertain matrix $\Xi_{m,k}$ as

$$\Xi_{m,k} = F^{\Delta}_{m,k} x_m + S^{\Delta}_{m,k} U_{m,k} + (\bar{B}_N + D^{\Delta}_{m,k})W_{m,k}\,. \quad (10)$$

Combine (9) and (10) and arrive at (7). Note that setting the uncertain terms to zero makes (7) the standard extended equation, [13].

Reasoning similarly, extend (4) to $[m, k]$ as

$$Y_{m,k} = H_N x_m + L_k U_{m,k} + M_N \Xi_{m,k} + \Pi_{m,k} \quad (11)$$

and represent the block vector $\Pi_{m,k}$ as

$$\begin{aligned} \Pi_{m,k} &= N^{\Delta}_{m,k} x_m + L^{\Delta}_{m,k} U_{m,k} + M^{\Delta}_{m,k} \Xi_{m,k} \\ &\quad (\bar{T}_N + \bar{T}^{\Delta}_{m,k})W_{m,k} + V_{m,k}\,. \quad (12) \end{aligned}$$

Combine (11) and (12), obtain (8), and complete the proof. Note that zero uncertainties makes (7) the standard extended equation, [13].

From (9), extract the predicted state as

$$\begin{aligned} x_{k+1} &= (F^N + \bar{\tilde{F}}_{m,k})x_m + (\bar{S}_N + \bar{\tilde{S}}_{m,k})U_{m,k} \\ &\quad +(\bar{D}_N + \bar{\tilde{D}}_{m,k})W_{m,k}\,, \quad (13) \end{aligned}$$

Having (8) and (13), we proceed with the one-step H_2-OFIR predictor and then will obtain the RH H_2-OFIR filter.

3 RH H_2-FIR Filter

Using the definition given in [1], and taking into account (8), we define the one-step ahead predicted FIR estimate as

$$\begin{aligned} \tilde{x}_{k+1} &= \mathcal{H}_N Y_{m,k} + \mathcal{H}^{\mathrm{f}}_N U_{m,k} \\ &= \mathcal{H}_N (H_N + \tilde{H}_{m,k})x_m \\ &\quad +\mathcal{H}_N (L_N + \tilde{L}_{m,k})U_{m,k} \\ &\quad +\mathcal{H}_N (G_N + \tilde{T}_{m,k})W_{m,k} \\ &\quad +\mathcal{H}^{\mathrm{f}}_N U_{m,k} + \mathcal{H}_N V_{m,k}\,, \quad (14) \end{aligned}$$

where $\mathcal{H}_N$ is the fundamental gain and $\mathcal{H}^{\mathrm{f}}_N$ is the forced gain.

The unbiasedness condition $E\{\tilde{x}_{k+1}\} = E\{x_{k+1}\}$ applied to (13) and (14) gives two unbiasedness constraints,

$$\begin{aligned} F^N &= \mathcal{H}_N H_N\,, \quad (15)\\ \mathcal{H}^{\mathrm{f}}_N &= \bar{S}_N - \mathcal{H}_N L_N\,. \quad (16) \end{aligned}$$

The estimation error $\varepsilon_{k+1} = x_{k+1} - \tilde{x}_{k+1}$ becomes

$$\begin{aligned} \varepsilon_{k+1} &= (F^N - \mathcal{H}_N H_N + \bar{\tilde{F}}_{m,k} - \mathcal{H}_N \tilde{H}_{m,k})x_m \\ &\quad +(\bar{S}_N - \mathcal{H}_N L_N - \mathcal{H}^{\mathrm{f}}_N + \bar{\tilde{S}}_{m,k} \\ &\quad -\mathcal{H}_N \tilde{L}_{m,k})U_{m,k} + (\bar{D}_N - \mathcal{H}_N T_N + \bar{\tilde{D}}_{m,k} \\ &\quad -\mathcal{H}_N \tilde{T}_{m,k})W_{m,k} - \mathcal{H}_N V_{m,k} \quad (17) \end{aligned}$$

and can further be generalized as

$$\begin{aligned} \varepsilon_{k+1} &= (\mathcal{B}_N + \tilde{\mathcal{B}}_{m,k})x_m + \tilde{\mathcal{U}}_{m,k} U_{m,k} \\ &\quad +(\mathcal{W}_N + \tilde{\mathcal{W}}_{m,k})W_{m,k} - \mathcal{V}_N V_{m,k}\,, \quad (18) \end{aligned}$$

where the regular error residual matrices $\mathcal{B}_N$, $\mathcal{W}_N$, and $\mathcal{V}_N$ are given by

$$\begin{aligned} \mathcal{B}_N &= F^N - \mathcal{H}_N H_N\,, \quad \mathcal{W}_N = \bar{D}_N - \mathcal{H}_N T_N\,, \\ \mathcal{V}_N &= \mathcal{H}_N\,, \quad (19) \end{aligned}$$

and the uncertain error residual matrices as

$$\begin{aligned} \tilde{\mathcal{B}}_{m,k} &= \bar{\tilde{F}}_{m,k} - \mathcal{H}_N \tilde{H}_{m,k}\,, \\ \tilde{\mathcal{U}}_{m,k} &= \bar{\tilde{S}}_{m,k} - \mathcal{H}_N \tilde{L}_{m,k}\,, \\ \tilde{\mathcal{W}}_{m,k} &= \bar{\tilde{D}}_{m,k} - \mathcal{H}_N \tilde{T}_{m,k}\,. \quad (20) \end{aligned}$$

We next introduce the sub errors

$$\begin{aligned} \bar{\varepsilon}_{x(k+1)} &= \mathcal{B}_N x_m\,,\ \bar{\varepsilon}_{w(k+1)} = \mathcal{W}_N W_{m,k}\,, \\ \bar{\varepsilon}_{v(k+1)} &= \mathcal{V}_N V_{m,k},\ \tilde{\varepsilon}_{x(k+1)} = \tilde{\mathcal{B}}_{m,k} x_m\,, \\ \tilde{\varepsilon}_{w(k+1)} &= \tilde{\mathcal{W}}_{m,k} W_{m,k}\,,\ \tilde{\varepsilon}_{u(k+1)} = \tilde{\mathcal{U}}_{m,k} U_{m,k}\,, \quad (21) \end{aligned}$$

and represent the error model as

$$\begin{aligned} \varepsilon_{k+1} &= \bar{\varepsilon}_{x(k+1)} + \bar{\varepsilon}_{w(k+1)} + \bar{\varepsilon}_{v(k+1)} + \tilde{\varepsilon}_{x(k+1)} \\ &\quad +\tilde{\varepsilon}_{w(k+1)} + \tilde{\varepsilon}_{u(k+1)}\,, \quad (22) \end{aligned}$$

which will be further used to derive the H_2-FIR predictor.

3.1 H_2-FIR Predictor

To derive the H_2-FIR predictor for uncertain systems, we will need the following definitions.

Given a block column matrix $Z_{m,k} = [\, z_m^T \; z_{m+1}^T \; \ldots \; z_k^T \,]^T$ specified on $[m,k]$. Its recursive form is [28],

$$Z_{m,k} = A_w Z_{m-1,k-1} + B_w z_k \,, \tag{23}$$

where A_w and B_w are strictly sparse matrices,

$$A_w = \begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & I \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}. \tag{24}$$

Given the system $\left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & 0 \end{array}\right]$, where the sparse matrices A_w and B_w are defined by (24) and C_w is a real matrix, the transfer function $\mathcal{T}(z) = C_w(Iz - A_w)^{-1} z B_w$, and a symmetric positive definite weighting matrix Ξ. Then the squared Frobenius norm of the weighted transfer function $\bar{\mathcal{T}}(z)$ is [31],

$$\begin{aligned} \|\bar{\mathcal{T}}(z)\|_F^2 &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{tr}\,[\mathcal{T}(e^{j\omega T})\Xi\mathcal{T}^*(e^{j\omega T})]\,\mathrm{d}\omega T \\ &= \mathrm{tr}(C_w \Xi C_w^T)\,. \end{aligned} \tag{25}$$

Using the above definitions, the trace of the error matrix of the H_2-FIR predictor can be written as

$$\begin{aligned} \mathrm{tr}\,P &= \mathcal{E}\{(\bar{\varepsilon}_{x(k+1)} + \bar{\varepsilon}_{w(k+1)} + \bar{\varepsilon}_{v(k+1)} + \tilde{\varepsilon}_{x(k+1)} \\ &\quad + \tilde{\varepsilon}_{w(k+1)} + \tilde{\varepsilon}_{u(k+1)})^T(\ldots)\} \\ &= \|\bar{\mathcal{T}}_{\bar{x}}(z)\|_F^2 + \|\bar{\mathcal{T}}_{\bar{w}}(z)\|_F^2 + \|\bar{\mathcal{T}}_{\bar{v}}(z)\|_F^2 \\ &\quad + \|\bar{\mathcal{T}}_{\tilde{x}}(z)\|_F^2 + \|\bar{\mathcal{T}}_{\tilde{w}}(z)\|_F^2 + \|\bar{\mathcal{T}}_{\tilde{u}}(z)\|_F^2\,, \end{aligned} \tag{26}$$

where $(\ldots)$ means the term that is equal to the relevant preceding term and the squared Frobenius norms are defined by

$$\|\bar{\mathcal{T}}(z)\|_F^2 = \mathrm{tr}\mathcal{E}\{\mathcal{T}\varpi_k\varpi_k^T\mathcal{T}^T\}\,. \tag{27}$$

which gives

$$\begin{aligned} \|\bar{\mathcal{T}}_{\bar{x}}(z)\|_F^2 &= \mathrm{tr}(\mathcal{B}_N\chi_m\mathcal{B}_N^T)\,, & (28) \\ \|\bar{\mathcal{T}}_{\bar{w}}(z)\|_F^2 &= \mathrm{tr}(\mathcal{W}_N\mathcal{Q}_N\mathcal{W}_N^T)\,, & (29) \\ \|\bar{\mathcal{T}}_{\bar{v}}(z)\|_F^2 &= \mathrm{tr}(\mathcal{V}_N\mathcal{R}_N\mathcal{V}_N^T)\,. & (30) \end{aligned}$$

Using (25), the $\|\bar{\mathcal{T}}_{\tilde{x}}(z)\|_F^2$ can be written as

$$\begin{aligned} \|\bar{\mathcal{T}}_{\tilde{x}}(z)\|_F^2 &= \mathrm{tr}\mathcal{E}\{\tilde{\mathcal{B}}_{m,k}x_m x_m^T\tilde{\mathcal{B}}_{m,k}^T\} \\ &= \mathrm{tr}\mathcal{E}\{(\bar{\tilde{F}}_{m,k} - \mathcal{H}_N\tilde{H}_{m,k})x_m x_m^T \\ &\quad \times(\bar{\tilde{F}}_{m,k} - \mathcal{H}_N\tilde{H}_{m,k})^T\} \\ &= \mathrm{tr}(\tilde{\chi}_m^F - \tilde{\chi}_m^{FH}\mathcal{H}_N^T - \mathcal{H}_N\tilde{\chi}_m^{HF} \\ &\quad + \mathcal{H}_N\tilde{\chi}_m^H\mathcal{H}_N^T)\,, \end{aligned} \tag{31}$$

where two uncertain matrices are given by

$$\begin{aligned} \tilde{\chi}_m^{FH} &= \mathcal{E}\{\bar{\tilde{F}}_{m,k}x_m x_m^T\tilde{H}_{m,k}^T\}\,, & (32) \\ \tilde{\chi}_m^H &= \mathcal{E}\{\tilde{H}_{m,k}x_m x_m^T\tilde{H}_{m,k}^T\}\,, & (33) \end{aligned}$$

and two others, $\tilde{\chi}_m^F$ and $\tilde{\chi}_m^{HF}$, are ignored by the filter gain.

The $\|\bar{\mathcal{T}}_{\tilde{w}}(z)\|_F^2$ can be transformed to

$$\begin{aligned} \|\bar{\mathcal{T}}_{\tilde{w}}(z)\|_F^2 &= \mathrm{tr}\mathcal{E}\{\tilde{\mathcal{W}}_{m,k}W_{m,k}W_{m,k}^T\tilde{\mathcal{W}}_{m,k}^T\} \\ &= \mathrm{tr}\mathcal{E}\{(\bar{\tilde{D}}_{m,k} - \mathcal{H}_N\tilde{T}_{m,k})W_{m,k}W_{m,k}^T \\ &\quad \times(\bar{\tilde{D}}_{m,k} - \mathcal{H}_N\tilde{T}_{m,k})^T\} \\ &= \mathrm{tr}(\tilde{Q}_N^D - \tilde{Q}_N^{DT}\mathcal{H}_N^T - \mathcal{H}_N\tilde{Q}_N^{TD} \\ &\quad + \mathcal{H}_N\tilde{Q}_N^T\mathcal{H}_N^T)\,, \end{aligned} \tag{34}$$

where two uncertain matrices are taken into account

$$\begin{aligned} \tilde{Q}_N^{DT} &= \mathcal{E}\{\bar{\tilde{D}}_{m,k}W_{m,k}W_{m,k}^T\tilde{T}_{m,k}^T\}\,, & (35) \\ \tilde{Q}_N^T &= \mathcal{E}\{\tilde{T}_{m,k}W_{m,k}W_{m,k}^T\tilde{T}_{m,k}^T\}\,, & (36) \end{aligned}$$

and $\tilde{Q}_N^D$ and $\tilde{Q}_N^{TD}$ are ignored by the filter gain.

The $\|\bar{\mathcal{T}}_{\tilde{u}}(z)\|_F^2$ becomes

$$\begin{aligned} \|\bar{\mathcal{T}}_{\tilde{u}}(z)\|_F^2 &= \mathrm{tr}\mathcal{E}\{\tilde{\mathcal{U}}_{m,k}U_{m,k}U_{m,k}^T\tilde{\mathcal{U}}_{m,k}^T\} \\ &= \mathrm{tr}\mathcal{E}\{(\bar{\tilde{S}}_{m,k} - \mathcal{H}_N\tilde{L}_{m,k})U_{m,k}U_{m,k}^T \\ &\quad \times(\bar{\tilde{S}}_{m,k} - \mathcal{H}_N\tilde{L}_{m,k})^T\} \\ &= \mathrm{tr}(\tilde{M}_N^S - \tilde{M}_N^{SL}\mathcal{H}_N^T - \mathcal{H}_N\tilde{M}_N^{LS} \\ &\quad + \mathcal{H}_N\tilde{M}_N^L\mathcal{H}_N^T)\,, \end{aligned} \tag{37}$$

where two uncertain error matrices have the value,

$$\begin{aligned} \tilde{M}_N^{SL} &= \mathcal{E}\{\bar{\tilde{S}}_{m,k}U_{m,k}U_{m,k}^T\tilde{L}_{m,k}^T\}\,, & (38) \\ \tilde{M}_N^L &= \mathcal{E}\{\tilde{L}_{m,k}U_{m,k}U_{m,k}^T\tilde{L}_{m,k}^T\}\,. & (39) \end{aligned}$$

and $\tilde{M}_N^S$ and $\tilde{M}_N^{LS}$ are not used in the filter gain.

3.2 Gain for the RH H_2-FIR Filter using LMI

By changing the time variable, the batch RH H_2-FIR filter can now be defined by

$$\tilde{x}_k = \mathcal{H}_N Y_{m-1,k-1} + (\bar{S}_N - \mathcal{H}_N L_N)U_{m-1,k-1}\,, \tag{40}$$

and the error covariance matrix can be written as

$$P = \mathcal{B}_N \chi_m \mathcal{B}_N^T + \mathcal{W}_N \mathcal{Q}_N \mathcal{W}_N^T + \mathcal{V}_N \mathcal{R}_N \mathcal{V}_N^T + \tilde{P}_x + \tilde{P}_w + \tilde{P}_u \,, \quad (41)$$

where the uncertain error matrices are defined by

$$\begin{aligned} \tilde{P}_x &= \mathcal{E}\{\tilde{\mathcal{B}}_{m,k} x_m x_m^T \tilde{\mathcal{B}}_{m,k}^T\} \\ &= \tilde{\chi}_m^F - \tilde{\chi}_m^{FH} \mathcal{H}_N^T - \mathcal{H}_N \tilde{\chi}_m^{HF} \\ &\quad + \mathcal{H}_N \tilde{\chi}_m^H \mathcal{H}_N^T \,, \quad (42) \\ \tilde{P}_w &= \mathcal{E}\{\tilde{\mathcal{W}}_{m,k} W_{m,k} W_{m,k}^T \tilde{\mathcal{W}}_{m,k}^T\} \\ &= \tilde{Q}_N^D - \tilde{Q}_N^{DT} \mathcal{H}_N^T - \mathcal{H}_N \tilde{Q}_N^{TD} \\ &\quad + \mathcal{H}_N \tilde{Q}_N^T \mathcal{H}_N^T \,, \quad (43) \\ \tilde{P}_u &= \mathcal{E}\{\tilde{\mathcal{U}}_{m,k} U_{m,k} U_{m,k}^T \tilde{\mathcal{U}}_{m,k}^T\} \\ &= \tilde{M}_N^S - \tilde{M}_N^{SL} \mathcal{H}_N^T - \mathcal{H}_N \tilde{M}_N^{LS} \\ &\quad + \mathcal{H}_N \tilde{M}_N^L \mathcal{H}_N^T . \quad (44) \end{aligned}$$

The gain for the suboptimal bias-constrained RH H_2 FIR filter can also be computed numerically using LMI. To this end, we introduce a positive definite matrix $\mathcal{Z}$ such that

$$\mathcal{Z} > \mathcal{W}_N \mathcal{Q}_N \mathcal{W}_N^T + \mathcal{V}_N \mathcal{R}_N \mathcal{V}_N^T + \tilde{P}_x + \tilde{P}_w + \tilde{P}_u \,. \quad (45)$$

Then, we substitute (19) and (20), transform (45) to

$$\begin{aligned} &\mathcal{Z} - (\mathcal{H}_N G_N - \bar{D}_N) \mathcal{Q}_N (\mathcal{H}_N G_N - \bar{D}_N)^T \\ &- \mathcal{H}_N \mathcal{R}_N \mathcal{H}_N^T - \tilde{\chi}_m^F + \tilde{\chi}_m^{FH} \mathcal{H}_N^T + \mathcal{H}_N \tilde{\chi}_m^{HF} \\ &- \mathcal{H}_N \tilde{\chi}_m^H \mathcal{H}_N^T - \tilde{Q}_N^D + \tilde{Q}_N^{DT} \mathcal{H}_N^T + \mathcal{H}_N \tilde{Q}_N^{TD} \\ &- \mathcal{H}_N \tilde{Q}_N^T \mathcal{H}_N^T - \tilde{M}_N^S + \tilde{M}_N^{SL} \mathcal{H}_N^T \\ &+ \mathcal{H}_N \tilde{M}_N^{LS} - \mathcal{H}_N \tilde{M}_N^L \mathcal{H}_N^T > 0 \,, \end{aligned}$$

and generalize as

$$\mathcal{Z} - \mathcal{A} + \mathcal{B} \mathcal{H}_N^T + \mathcal{H}_N \mathcal{C} - \mathcal{H}_N \mathcal{D} \mathcal{H}_N^T > 0 \,, \quad (46)$$

where auxiliary matrices are defined as $\mathcal{A} = \bar{D}_N \mathcal{Q}_N \bar{D}_N^T + \tilde{\chi}_m^F + \tilde{Q}_N^D + \tilde{M}_N^S$, $\mathcal{B} = G_N^p \mathcal{Q}_N \bar{D}_N^T + \tilde{\chi}_m^{FH} + \tilde{Q}_N^{DT} + \tilde{M}_N^{SL}$, $\mathcal{C} = \bar{D}_N \mathcal{Q}_N G_N^{p^T} + \tilde{\chi}_m^{HF} + \tilde{Q}_N^{TD} + \tilde{M}_N^{LS}$, and $\mathcal{D} = \Omega_N - \tilde{\chi}_m^H - \tilde{Q}_N^T - \tilde{M}_N^L$. Using the Schur complement, we further represent (46) in the LMI form of

$$\begin{bmatrix} \mathcal{Z} - \mathcal{A} + \mathcal{B} \mathcal{H}_N^T + \mathcal{H}_N \mathcal{C} & \mathcal{H}_N \\ \mathcal{H}_N^T & \mathcal{D}^{-1} \end{bmatrix} > 0 \,. \quad (47)$$

Finally, the gain for the suboptimal RH H_2-FIR filter can be determined numerically by solving the minimization problem

$$\mathcal{H}_N = \min_{\mathcal{H}_N, \mathcal{Z}} \operatorname{tr} \mathcal{Z} \,, \quad \text{subject to (47)} \quad (48)$$

where the minimization should be started with the UFIR filter gain $\hat{\mathcal{H}}_N = (C_N^T C_N)^{-1} C_N^T$. Next, we consider an example of a quasi harmonic model.

4 Numerical Example

A quasi harmonic system is represented by the following state space equations,

$$x_{k+1} = \begin{bmatrix} 0.6 & 0.4 \\ -0.4 & 0.6 + \delta \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_k \,, \quad (49)$$

$$y_k = \begin{bmatrix} 1 & 1 \end{bmatrix} x_k + v_k \,, \quad (50)$$

where $\delta \geqslant 0$ is the uncertain constant, [37]. We represent the uncertain system matrix as $F^u = F + \Delta F = F + \delta \bar{F} = \begin{bmatrix} 0.6 & 0.4 \\ -0.4 & 0.6 \end{bmatrix} + \delta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and the uncertain vector (5) as $\xi_k = \delta \bar{F} \hat{x}_k + B w_k$. The disturbance w_k is assumed to be Gauss-Markov $w_{k+1} = \phi w_k + \zeta_k$, where $\zeta_k \in \mathcal{N}(0,1)$, and the colored measurement noise v_k to be $v_{k+1} = \psi v_k + \xi_k$, where $\xi_k \in \mathcal{N}(0,1)$. The block error matrix $\mathcal{Q}_N$ of w_k and $\mathcal{R}_N$ of v_k are computed numerically. The initial state is assumed to be known.

For the model (49) and (50), only the uncertain matrices $\tilde{\mathcal{Q}}_N^{DT}$ and $\tilde{\mathcal{Q}}_N^T$ should be specified for the filter gain. To transform matrix $\tilde{\mathcal{Q}}_N^{DT}$ (35), we start with $\tilde{D}_N = \hat{F}_N D_N^\Delta$ and represent matrix D_N^Δ as $D_N^\Delta = \delta \mathcal{Z}_N$, where

$$\mathcal{Z}_N = \begin{bmatrix} 0 & & \dots & 0 & 0 \\ \bar{F}B & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{F}F^{N-3}B & \bar{F}F^{N-4}B & \dots & 0 & 0 \\ \bar{F}F^{N-2}B & \bar{F}F^{N-3}B & \dots & \bar{F}B & 0 \end{bmatrix} .$$

Matrix $\bar{\tilde{D}}_N$ can now be written as $\bar{\tilde{D}}_N = \bar{\hat{F}}_N D_N^\Delta$, where $\bar{\hat{F}}_N = [\, F^{N-1} \; F^{N-2} \; \dots \; F \; I \,]$ is the last row vector in $\hat{F}_N$. Next, matrix $\tilde{T}_N = \bar{H}_N \hat{F}_N D_V^\Delta$ gives $\tilde{\mathcal{Q}}_N^{DT} = \bar{\hat{F}}_N \mathcal{Q}_N^\Delta \hat{F}_N^T \bar{H}_N^T$, where $\mathcal{Q}_N^\Delta$ is determined by averaging as

$$\mathcal{Q}_N^\Delta = D_N^\Delta \mathcal{E}\{W_{m,k} W_{m,k}^T\} D_N^{\Delta^T} = \delta^2 \mathcal{Z}_N \mathcal{Q}_N \mathcal{Z}_N^T \,.$$

Similarly, we obtain $\tilde{\mathcal{Q}}_N^T = \bar{H}_N \hat{F}_N \mathcal{Q}_N^\Delta \hat{F}_N^T \bar{H}_N^T$.

The RH H_2-FIR filter gain can now be determined numerically by solving the minimization problem (48) and the estimate computed as $\hat{x}_k = \mathcal{H}_N Y_{m-1,k-1}$. It is worth noting that the gain $\mathcal{H}_N$ is obtained by (48) for full block error matrices $\mathcal{Q}_N$ and $\mathcal{R}_N$ that makes it more accurate than the Kalman-like recursive schemes when w_k and v_k are not white and thus $\mathcal{Q}_N$ and $\mathcal{R}_N$ are not diagonal. We next assume that the uncertainty can

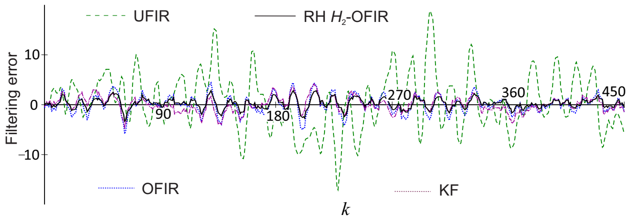

Figure 1: Filtering errors produced by the filters for uncertain system with $\delta = 0.4$ under the heavy disturbance with $\phi = 0.95$.

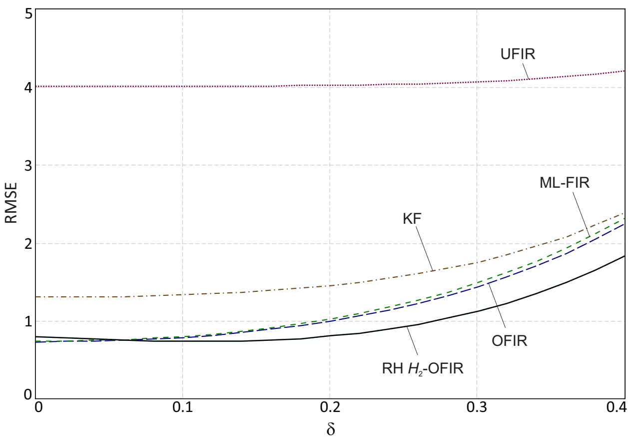

Figure 2: RMSEs produced by the filters as functions of δ for uncertain system with $\delta = 0.4$ under heavy disturbance with $\phi = 0.95$.

take values from $\delta \in [0 \ldots 0.4]$ and investigate filtering errors using the UFIR filter, [38], OFIR filter, [13], ML-FIR filter, [39], and KF, [38], as benchmarks.

In the first case, we set $\psi = 0$ and $\phi = 0.95$, and tune the filters to $\delta = 0.4$. Typical filtering errors are shown in Fig. 1, and we infer that the UFIR filter ($N_{\text{opt}} = 4$) fails to give accurate estimates, while the RH H_2-FIR filter looks the best. The effect of δ on the RMSEs is shown in Fig. 2, and we deduce that the UFIR filter is the most robust and the less accurate. The most accurate RH H_2-FIR filter demonstrates a sufficient robustness, and the remaining filters give in-between estimates (Table 1).

Table 1: Case 1: RMSEs Produced by the filters for various δ

Filter	0	0.1	0.2	0.3	0.4
UFIR	4.005	4.005	4.018	4.066	4.199
KF	1.302	1.331	1.452	1.752	2.390
OFIR	**0.730**	0.786	0.996	1.440	2.249
ML-FIR	0.735	**0.737**	1.026	1.486	2.316
H_2-FIR	0.793	**0.737**	**0.803**	**1.122**	**1.831**

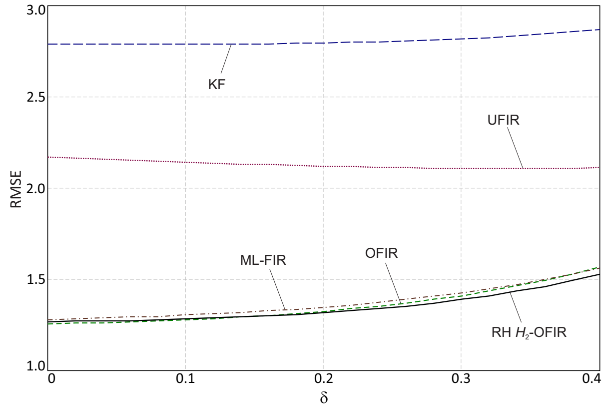

Figure 3: RMSEs produced by the filters as functions of δ for uncertain system with $\delta = 0.4$ and system disturbance with $\phi = 0.95$.

Table 2: Case 2: RMSEs Produced by the filters for various δ

Filter	0	0.1	0.2	0.3	0.4
UFIR	2.169	2.143	2.121	2.107	2.112
KF	2.790	2.789	2.796	2.818	2.871
OFIR	**1.258**	**1.279**	1.325	1.412	1.569
ML-FIR	1.280	1.304	1.348	1.426	1.565
H_2-FIR	1.269	1.282	**1.317**	**1.390**	**1.529**

In the second extreme case we set $\psi = 0.95$ and $\phi = 0$. The RMSEs are sketched in Fig. 3. What we can see is that the KF is the worst here, the UFIR filter ($N_{\text{opt}} = 5$) gives better estimates, and the ML-FIR, OFIR, and RH H_2-FIR filters produce the smallest and consistent estimates, although the latter still increases errors at a lower rate (Table 2).

5 Conclusions

The robust RH H_2-FIR filter developed in this paper for uncertain and disturbed systems operating under initial errors and data errors has demonstrated a better performance than other filters. This was achieved by by minimizing the squared Frobenius norm of the weighted error-to-error transfer function with weights related to errors. An example of a harmonic model has shown that the RH H_2-FIR filter has a better accuracy than the OFIR, ML-FIR, and Kalman filters and is almost as robust as the UFIR filter.

A Partitioned Vectors and Matrices

The block vectors are defined as

$$\begin{aligned}
X_{m,k} &= \begin{bmatrix} x_m^T & x_{m+1}^T & \dots & x_k^T \end{bmatrix}^T , \\
Y_{m,k} &= \begin{bmatrix} y_m^T & y_{m+1}^T & \dots & y_k^T \end{bmatrix}^T , \\
U_{m,k} &= \begin{bmatrix} u_m^T & u_{m+1}^T & \dots & u_k^T \end{bmatrix}^T , \\
W_{m,k} &= \begin{bmatrix} w_m^T & w_{m+1}^T & \dots & w_k^T \end{bmatrix}^T \\
V_{m,k} &= \begin{bmatrix} v_m^T & v_{m+1}^T & \dots & v_k^T \end{bmatrix}^T , \\
\Xi_{m,k} &= \begin{bmatrix} \xi_m^T & \xi_{m+1}^T & \dots & \xi_k^T \end{bmatrix}^T , \\
\Pi_{m,k} &= \begin{bmatrix} \zeta_m^T & \zeta_{m+1}^T & \dots & \zeta_k^T \end{bmatrix}^T ,
\end{aligned}$$

and the block matrices as

$$\begin{aligned}
F_N &= \begin{bmatrix} F^T & F^{2^T} & \dots F^{N-1^T} & F^{N^T} \end{bmatrix}^T , \\
S_N &= \begin{bmatrix} E & 0 & \dots & 0 & 0 \\ FE & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F^{N-2}E & F^{N-3}E & \dots & E & 0 \\ F^{N-1}E & F^{N-2}E & \dots & FE & E \end{bmatrix} ,
\end{aligned}$$

matrix D_N becomes matrix S_N if we replace E with B, $H_N = \bar{H}_N F^{-1} F_N$, $L_N = \bar{H}_N S_N$, $T_N = G_N + \bar{T}_N$, $G_N = \bar{H}_N D_N$, $\bar{H}_N = \mathrm{diag}(\underbrace{H\,, H\,\dots\,H}_{N})$,

$\bar{T}_N = \mathrm{diag}(\underbrace{D\,, D\,\dots\,D}_{N})$,

$$\begin{aligned}
F_{m,k}^{\Delta} &= \begin{bmatrix} \Delta F_m \\ \Delta F_{m+1} F_m^u \\ \vdots \\ \Delta F_{k-1} \tilde{\mathcal{F}}_{k-2}^m \\ \Delta F_k \tilde{\mathcal{F}}_{k-1}^m \end{bmatrix} , \\
\tilde{\mathcal{F}}_r^g &= \begin{cases} F_r^u F_{r-1}^u \dots F_g^u\,, & g < r+1\,, \\ I\,, & g = r+1 \\ 0\,, & g > r+1 \end{cases} ,
\end{aligned}$$

where $F_j^u = F + \Delta F_j$, $j \in [g, r]$, matrix $S_{m,k}^{\Delta}$ is given at the top of the next page, matrix $D_{m,k}^{\Delta}$ becomes $S_{m,k}^{\Delta}$ if we replace ΔE_i with ΔB_i, $i \in [m, k]$, and E with B, $\tilde{F}_{m,k} = \hat{F}_N F_{m,k}^{\Delta}$, $\tilde{S}_{m,k} = \hat{F}_N S_{m,k}^{\Delta}$, $\tilde{D}_{m,k} = \hat{F}_N \tilde{D}_{m,k}^{\Delta}$,

$$\hat{F}_N = \begin{bmatrix} I & 0 & \dots & 0 & 0 \\ F & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F^{N-2} & F^{N-3} & \dots & I & 0 \\ F^{N-1} & F^{N-2} & \dots & F & I \end{bmatrix} ,$$

$\bar{\tilde{F}}_{m,k}$, $\bar{\tilde{S}}_{m,k}$, and $\bar{\tilde{D}}_{m,k}$ are the last row vectors in $\tilde{F}_{m,k}$, $\tilde{S}_{m,k}$, and $\tilde{D}_{m,k}$, respectively, $H_N = \bar{H}_N F^{-1} F_N$, $L_N = M_N \bar{E}_N$,

$$\begin{aligned}
M_N &= \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ H & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ HF^{N-3} & HF^{N-4} & \dots & 0 & 0 \\ HF^{N-2} & HF^{N-3} & \dots & H & 0 \end{bmatrix} , \\
N_{m,k}^{\Delta} &= \begin{bmatrix} \Delta H_m \\ \Delta H_{m+1} F \\ \vdots \\ \Delta H_{k-1} F^{N-2} \\ \Delta H_k F^{N-1} \end{bmatrix} ,
\end{aligned}$$

$L_{m,k}^{\Delta} = M_{m,k}^{\Delta} \bar{E}_N$, $\bar{T}_{m,k}^{\Delta} = \mathrm{diag}(\,\Delta D_m\ \Delta D_{m-1}\ \dots\ \Delta D_k\,)$, and

$$\begin{aligned}
\tilde{H}_{m,k} &= N_{m,k}^{\Delta} + (M_N + M_{m,k}^{\Delta}) F_{m,k}^{\Delta}\,, \\
\tilde{L}_{m,k} &= L_{m,k}^{\Delta} + (M_N + M_{m,k}^{\Delta}) S_{m,k}^{\Delta}\,, \\
\tilde{T}_{m,k} &= M_{m,k}^{\Delta} \bar{B}_N + (M_N + M_{m,k}^{\Delta}) D_{m,k}^{\Delta} + \bar{T}_{m,k}^{\Delta}\,.
\end{aligned}$$

References:

[1] W. H. Kwon and S. Han, *Receding horizon control: model predictive control for state models*. Springer, London, 2005.

[2] Y. S. Shmaliy and S. Zhao, *Optimal and Robust State Estimation: Finite Impulse Response (FIR) and Kalman Approaches*. New York: Wiley & Sons, 2022.

[3] M. S. Darup, A. Redder, I. Shames, F. Farokhi, and D. Quevedo, “Towards encrypted MPC for linear constrained systems,” *IEEE Contr. Syst. Lett.*, vol. 2, no. 2, pp. 195–2000, Apr. 2018.

[4] D. Wilson, “Convolution and Hankel operator norms for linear systems,” *IEEE Trans. Autom. Contr.*, vol. 34, no. 1, pp. 94–97, 1989.

[5] O. K. Kwon, W. H. Kwon, and K. S. Lee, “FIR filters and recursive forms for discrete-time state-space models,” *Automatica*, vol. 25, no. 5, pp. 715–728, Sep. 1989.

[6] O. K. Kwon, P. S. Kim, and P. Park, “A receding horizon Kalman FIR filter for discrete time-invariant systems,” *IEEE Trans. Autom. Control*, vol. 44, no. 9, pp. 1787–1791, Sep. 1999.

[7] Y. S. Shmaliy, F. Lehmann, and S. Zhao, “Comparing robustness of the Kalman, H_∞, and UFIR filters,” *IEEE Trans. Signal Process.*, vol. 66, no. 13, pp. 3447–3458, Jul. 2018.

$$S^{\Delta}_{m,k} = \begin{bmatrix} \Delta E_m & 0 & \dots & 0 & 0 \\ \Delta F_{m+1}(E+\Delta E_m) & \Delta E_{m+1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta F_{k-1}\tilde{\mathcal{F}}^{m+1}_{k-2}(E+\Delta E_m) & \Delta F_{k-1}\tilde{\mathcal{F}}^{m+2}_{k-2}(E+\Delta E_{m+1}) & \dots & \Delta E_{k-1} & 0 \\ \Delta F_k\tilde{\mathcal{F}}^{m+1}_{k-1}(E+\Delta E_m) & \Delta F_k\tilde{\mathcal{F}}^{m+2}_{k-1}(E+\Delta E_{m+1}) & \dots & \Delta F_k(E+\Delta E_{k-1}) & \Delta E_k \end{bmatrix}$$

$$M^{\Delta}_{m,k} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ \Delta H_{m+1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta H_{k-1}F^{N-3} & \Delta H_{k-1}F^{N-4} & \dots & 0 & 0 \\ \Delta H_k F^{N-2} & \Delta H_k F^{N-3} & \dots & \Delta H_k & 0 \end{bmatrix},$$

[8] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Prentice-Hall, Upper Saddle River, NJ, 1996.

[9] B. Hassibi, A. H. Sayed, and T. Kailath, *Indefinite-quadratic estimation and control: a unified approach to H^2 and H^∞ theories*. SIAM, Philadelphia, 1999.

[10] C. K. Ahn, S. Zhao, Y. S. Shmaliy, and H. Li, "On the $\ell_2 - \ell_\infty$ and H_∞ performance of the continuous-time deadbeat H_2 FIR filter," *IEEE Trans. Circ. Syst. II Express Briefs*, vol. 65, no. 11, pp. 1798–1802, 2018.

[11] J. C. Doyle, K. Glover, P. P. Khargonerar, and B. A. Francis, "Solutions to Standard H_2 and H_∞ control problems," *IEEE Trans. Autom. Contr.*, vol. 34, no. 8, pp. 831–847, Aug. 1989.

[12] Y. S. Shmaliy, "Linear optimal FIR estimation of discrete time-invariant state-space models," *IEEE Trans. Signal Process.*, vol. 58, no. 6, pp. 3086–3096, 2010.

[13] Y. S. Shmaliy and O. Ibarra-Manzano, "Time-variant linear optimal finite impulse response estimator for discrete state-space models," *Int. J. Adapt. Contr. Signal Process.*, vol. 26, no. 2, pp. 95–104, 2012.

[14] P. L. Rawicz, P. R. Kalata, K. M. Murphy, and T. A. Chmielewski, "Explicit formiula for two state Kalman, H_2 and H_∞ Target tracking," *IEEE Trans. Aero. Electr. Syst.*, vol. 39, no. 1, pp. 53–69, 2003.

[15] T. Gaspar and P. Oliveira, "Model-based H_2 adaptive filter for 3D positioning and tracking systems," *Automatica*, vol. 50, no. 1, pp. 225–232, Jan. 2014.

[16] L. Xie, Y. C. Soh, C. Du, and Y. Zou, "Robust H_2 estimation and control," *J. Contr. Theory Appl.*, vol. 2, pp. 20–26, 2004.

[17] L. Xie, "On robust H_2 estimation," *Acta Automat. Sinica*, vol. 31, no. 1, pp. 1–12, 2005.

[18] M. Souza, A. R. Fioravanti, and J. C. Geromel, "H_2 sampled-data filtering of linear systems," *IEEE Trans. Signal Process.*, vol. 62, no. 18, pp. 4839–4846, 2014.

[19] W. M. Haddad, D. S. Bernstein, and D. Mustafa, "Mixed-norm H_2/H_∞ regulation and estimation: The discrete-time case," *Syst. & Contr. Lett.*, vol. 16, pp. 235–247, 1991.

[20] M. D. S. Aliyu and E. K. Boukas, "Discrete-time mixed H_2/H_∞ nonlinear filtering," *Int. J. Robust Nonlinear Cont.*, vol. 21, pp. 1257–1282, 2011.

[21] E. Gershon, U. Shaked, and I. Yaesh, *H_∞ Control and Estimaiton of State-multiplicative Linear Systems*. Springer-Verlag, London, 2005.

[22] V. M. Deshpande and R. Bhattacharya, "Sparse sensing and optimal precision: an integrated framework for H_2/H_∞ pptimal observer design," *IEEE Contr. Syst. Lett.*, vol. 5, no. 2, pp. 481–486, Apr. 2022.

[23] K. Z. Liu and T. Sato, "LMI solution to singular H_2 suboptimal control problems," in *14th World Congress of IFAC*. Beijing, China, July 1999, 1999, pp. 3011–3016.

[24] Z. Tan, Y. C. Soh, and L. Xie, "Envelope-constrained H_2 FIR filter design," *Circ. Syst. Signal Process.*, vol. 18, no. 6, pp. 539–551, 1999.

[25] B. S. Chen and J. C. Hung, "Fixed-order H_2 and H_∞ optimal deconvolution filter designs," *Signal Process.*, vol. 80, no. 2, pp. 311–331, 2000.

[26] S. Wang, L. Xie, and C. Zhang, "H_2 optimal inverse of periodic FIR digital filters," *IEEE Trans. Signal Process.*, vol. 48, no. 9, pp. 2696–2700, 2000.

[27] ——, "Mixed H_2/H_∞ deconvolution of uncertain periodic FIR channels," *Signal Process.*, vol. 81, no. 10, pp. 2089–2103, 2001.

[28] Y. S. Lee, S. H. Han, and W. H. Kwon, "H_2/H_∞ FIR filters for discrete-time state space models," *Int. J. of Contr. Autom. Syst.*, vol. 4, no. 5, pp. 645–652, Oct. 2006.

[29] C. K. Ahn, Y. S. Shmaliy, S. Zhao, and H. Li, "Continuous-time deadbeat H_2 FIR filter," *IEEE Trans. Circuits Syst. II Express Briefs*, vol. 64, no. 8, pp. 987–991, Aug. 2017.

[30] C. K. Ahn, Y. S. Shmaliy, and S. Zhao, "A new unbiased FIR filter with improved robustness based on Frobenius norm with exponential weight," *IEEE Trans. Circuits Syst. II Express Briefs*, vol. 65, no. 4, pp. 521–525, Apr. 2018.

[31] J. Ortega-Contreras, E. Pale-Ramon, Y. S. Shmaliy, and Y. Xu, "A novel approach to H_2 FIR prediction under disturbances and measurement errors," *IEEE Signal Process. Lett.*, vol. 28, pp. 150–154, 2021.

[32] Y. S. Shmaliy, "Framework for robust H_2-OFIR filtering of uncertain and disturbed systems with data errors," *IEEE Systems Journal*, vol. 17, no. 2, pp. 2923–2926, 2023.

[33] Y. S. Shmaliy, Y. Xu, J. A. Andrade-Lucio, and O. G. Ibarra-Manzano, "Predictive tracking under persistent disturbances and data errors using H_2 FIR approach," *IEEE Trans. Ind. Electr.*, (to be published).

[34] E. G. Pale-Ramon, Y. S. Shmaliy, J. A. Andrade-Lucio, and L. J. Morales-Mendoza, "Bias-constrained optimal H_2 finite impulse response filter for object tracking under disturbances and data errors," *IEEE Trans. Contr. Syst. Techn.*, (to be published).

[35] J. A. Ortega-Contreras, Y. S. Shmaliy, J. A. Andrade-Lucio, and O. G. Ibarra-Manzano, "Robust H_2-OFIR filtering: improving tracking of disturbed systems under initial and data errors," *IEEE Trans. Aero. Electron, Syst.*, (to be published).

[36] A. Spagolla, C. F. Morais, R. C. L. F. Oliveira, and P. L. D. Peres, "Reduced order positive filter design for positive uncertain discrete-time linear systems," *IEEE Contr. Syst. Lett.*, vol. 6, pp. 1148–1153, 2022.

[37] S. Kim, V. M. Deshpande, and R. Bhattacharya, "Robust Kalman filtering with probabilistic uncertainty in system parameters," *IEEE Contr. Syst. Lett.*, vol. 5, no. 1, pp. 295–300, Jan. 2021.

[38] Y. S. Shmaliy, S. Zhao, and C. K. Ahn, "Unbiased FIR filtering: an iterative alternative to Kalman filtering ignoring noise and initial conditions," *IEEE Contr. Syst. Mag.*, vol. 37, no. 5, pp. 70–89, 2017.

[39] S. Zhao and Y. S. Shmaliy, "Unified maximum likelihood form for bias constrained FIR filters," *IEEE Signal Process. Lett.*, vol. 23, no. 12, pp. 1848–1852, Dec. 2016.

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