Analytical Explicit Formulas of Average Run Length of DEWMA Control Chart based on Seasonal Moving Average Process

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Abstract: - The primary objective of this research is to propose explicit formulas for the Average Run Length (ARL) of the Double Exponentially Weighted Moving Average control chart (DEWMA) for the Seasonal Moving Average process (SMA $(Q)_L$) with exponential white noise. The Numerical Integral Equation by the midpoint rule is employed to compare the results derived from the formulas and evaluate their accuracy using the percentage of accuracy (%Acc). The DEWMA control chart's efficacy is measured by calculating the average run length (ARL), median run length (MRL), and standard deviation of run length (SDRL). Significant agreement was observed between the numerical approximations and the explicit formulations for $SMA(2)_4$ and $SMA(3)_{12}$ processes. This finding indicates the formulations are sufficiently precise. A comparison of Exponentially Weighted Moving Average (EWMA) and DEWMA control charts relating to mean process variations is performed. For practical data, WTI oil prices are used to determine the efficacy of the explicit formula approach.

Key-Words: - Average Run Length, Seasonal Moving Average Process, Double Exponentially Weighted Moving Average, Exponential white noise, Numerical Integral Equation, Midpoint rule.

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1 Introduction

A statistical process control method is a quality control approach that employs statistical methods to control and monitor a process. It is extensively employed in numerous areas such as telecommunications, education, energy production, healthcare, finance, and software development to guarantee that processes remain consistent, predictable, and within specified limits, [1], [2], [3]. The Shewhart control chart effectively identifies significant changes in the process's mean, as described by [4]. However, the Cumulative Sum (CUSUM) and the exponential weighted Moving Average (EWMA) control schemes are more effective in identifying minor to moderate changes. The study, [5], presented the CUSUM control chart, which is particularly suitable for monitoring processes for small changes. The CUSUM chart exhibits more sensitivity to small changes than the Shewhart control chart, which makes it a valuable tool for proactive quality management. Subsequently, the EWMA control chart was introduced into the study, [6]. In statistical process control, an EWMA control chart is utilized to monitor and identify shifts in the process mean. This control chart gives more weight to recent observations, making them more responsive to changes than the traditional Shewhart control chart. The advantages of utilizing the EWMA control chart are supported by the study, [7], [8]. Recently, an alternative to the traditional EWMA control chart was developed and called the Double Exponentially Weighted Moving Average (DEWMA) control chart, [9]. The DEWMA control chart benefits processes that exhibit both short-term and long-term shifts. The main idea is to obtain two levels of smoothing parameters. The DEWMA control chart is suitable for detecting small changes in the mean of a process with normally distributed observations. It is more effective than the EWMA control chart in detecting minor shifts, [10], [11].

Autocorrelation in time series data refers to the correlation of a series with its past and future values. The Box-Jenkins methodology, specifically the Autoregressive Integrated Moving Average (ARIMA) model, is commonly used to handle autocorrelated data. A Seasonal Moving Average process ($SMA(Q)_L$) is a time series analysis technique that combines moving averages with a seasonal component. It is designed to capture both the trend and the seasonal patterns in a time series data set. The seasonal moving average process is beneficial when dealing with data that exhibits regular and repeating patterns at specific intervals. The proper control charts must be applied to these data. Furthermore, residuals probably consist of some type of white noise. However, alternative displays, such as exponential white noise, could occur in particular datasets, [12], [13].

The average run length (ARL) is the primary metric to assess control chart approaches. It is computed for both in-control (ARL0) and out-of-control (ARL1) scenarios. Before a process exceeds the control limit, its average run length is denoted as ARL0. In general, the ARL0 value should be sufficiently substantial. The mean of the observations collected from the beginning of the change process until it exceeds the control threshold is denoted by ARL1. Consequently, it is essential to minimize the ARL1 value.

ARL can be generally assessed using various techniques, including Monte Carlo simulation, the Markov Chain Approach (MCA), numerical integral equations (NIE), and explicit formulations. For example, the study [14] utilized Monte Carlo simulation to examine ARL for nonparametric double EWMA control charts based on the mood statistic for process variability. The study [15] investigated the ARL of EWMA and CUSUM control charts using the Markov Chain approach when the observation follows a zero-inflated negative binomial distribution. The numerical integral equation (NIE) method was employed to evaluate the effectiveness of the CUSUM control chart in approximating the ARL for a long-memory fractionally integrated autoregressive model with an exogenous variable in the study [16]. In addition, the ARL was determined through the research study [17] which employed the NIE technique for including Simpson's rules, Gaussian, midpoint, and trapezoidal rules into the extended exponentially weighted moving average control chart. This approach was utilized when the observations followed continuous distributions, specifically the exponential, Weibull, and Gamma distributions. In addition, for the explicit formula method, which is a method that gives accurate value and high precision, the study [18] employed the integral equation approach to prove the ARL of the EWMA procedure for autocorrelated data.

From the research that has been done, no researcher has investigated the exact formula for the Average Run Length (ARL) of the DEWMA control chart for $SMA(Q)_L$ models. Consequently, developing an explicit formula for the ARL on the DEWMA control chart for the $SMA(Q)_L$ model is the objective of this research. Likewise, an evaluation of the effectiveness of the EWMA control chart is included. Furthermore, to evaluate the effectiveness of the DEWMA control chart, the WTI crude oil price dataset was utilized. Based on the information provided, studies have yet to be carried out with the exact formula for the Average Run Length (ARL) of the DEWMA control chart.

2 Process and Control Charts

2.1 Process

Given Y_t be a sequence of Seasonal Moving Average random processes. The $SMA(Q)_L$ process is the process of the random errors that occurred in past periods, $\zeta_{t-L},\zeta_{t-2L},\ldots,\zeta_{t-QL}$.

The general Seasonal Moving Average processes, denoted by the SMA $(Q)_L$ process, can be written as in Eq. (1):

$$Y_t = \theta_0 + \zeta_t - \theta_1\zeta_{t-L} - \theta_2\zeta_{t-2L} - \ldots - \theta_Q\zeta_{t-QL} \qquad (1)$$

where ζ_t is an exponential white noise process. A seasonal moving average coefficient, $-1 \le \theta_i \le 1$.

The primary focus of this research is the one-sided control chart, which is used to examine the case of a positive change. The sequence of independent identical distribution random variables with exponential parameters is denoted as (β). It is normally assumed that the known parameter for in-control process given $\beta = \beta_0$. The parameter β could be changed to the out-of-control value $\beta = \beta_1$ where $\theta < \infty$, and θ is the change-point time.

2.2 The Double Exponentially Weighted Moving Average Control Chart

The DEWMA control chart was initially introduced by [9]. Later, the study [11] improved the control chart's robustness by adjusting smoothing parameters. The DEWMA statistic is presented below in Eq. (2).

$$DE_t = \lambda_2 E_t + (1-\lambda_2) DE_{t-1} \quad (2)$$

where $E_t = \lambda_1 Y_t + (1-\lambda_1)E_{t-1}$ is EWMA statistic, Y_t is a sequence of SMA(Q)$_L$ model, λ_1 and λ_2 are an exponential smoothing parameters, $0 < \lambda_1, \lambda_2 \le 1$. The upper control limit (UCL) and lower control limit (LCL) of the DEWMA control chart are given by

$$\mu \pm F_{DE}\sigma\sqrt{\frac{\lambda_1^2\lambda_2^2}{(\lambda_1-\lambda_2)^2}\left[\frac{(1-\lambda_2)^2}{1-(1-\lambda_2)^2}+\frac{(1-\lambda_1)^2}{1-(1-\lambda_1)^2}-2\frac{(1-\lambda_1)(1-\lambda_2)}{1-(1-\lambda_1)(1-\lambda_2)}\right]}$$

where μ is the target mean, σ is the process standard deviation, and F_{DE} is width of the control limits. The stopping time of the DEWMA control chart (τ_b) is given by

$$\tau_b = \{t > 0;\, DE_t \ge b\},$$

where τ_b is the stopping time and b is a *UCL*.

2.3 The Exponentially Weighted Moving Average Control Chart

The design scheme for the EWMA control chart was initially proposed by [6]. It is frequently utilized to monitor processes and detect change. The following mathematical expression defines the EWMA control chart's statistical characteristics:

$$E_t = \lambda Y_t + (1-\lambda)E_{t-1} \quad (3)$$

where Y_t is a sequence of SMA(Q)$_L$ model with exponential white noise, and λ is an exponential smoothing parameter $(0 < \lambda \le 1)$.

The UCL and LCL of the EWMA control chart are given by

$$\mu \pm F_E\sigma\sqrt{\frac{\lambda}{2-\lambda}},$$

where μ is the target mean, σ is the process standard deviation, and F_E is width of the control limits. The stopping time of the EWMA control chart (τ_h) is represented as:

$$\tau_h = \{t \ge 0 : E_t > h\}$$

where τ_h is the stopping time, and h is a *UCL*.

3 The Method for Evaluation ARL

3.1 The Explicit Formula of ARL on DWMA Control Chart

This section demonstrates proving the explicit formula of average run length (ARL) on the DEWMA control charts using the SMA (Q)$_L$ model. The proof begins by substituting Eq. (3) into Eq. (2) as follows:

$$DE_t = \lambda_1\lambda_2\left(\theta_0 + \zeta_t - \theta_1\zeta_{t-L} - \theta_2\zeta_{t-2L} - \ldots - \theta_Q\zeta_{t-QL}\right) + \lambda_2(1-\lambda_1)E_{t-1} + (1-\lambda_2)DE_{t-1}$$

when the first time $t = 1$ such that $DE_0 = \psi$ is determined, then the initial values $E_0 = \gamma$ and $\zeta_{t-L}, \zeta_{t-2L}, \ldots, \zeta_{t-QL}$ equal to 1. The DEWMA statistics with SMA(Q)$_L$ can be defined as follows:

$$DE_1 = \lambda_1\lambda_2\left(\theta_0 + \zeta_t - \theta_1\zeta_{t-L} - \theta_2\zeta_{t-2L} - \ldots - \theta_Q\zeta_{t-QL}\right) + \lambda_2(1-\lambda_1)\gamma + (1-\lambda_2)\psi$$

For in control process, the interval of DE_1 that is in between the lower and upper control limits or from 0 to b can be written as follows.

$$0 < \lambda_1\lambda_2(\theta_0 - \theta_1\zeta_{1-L} - \theta_2\zeta_{1-2L} - \ldots - \theta_Q\zeta_{1-QL}) + \lambda_2(1-\lambda_1)\gamma + (1-\lambda_2)\psi + \lambda_1\lambda_2\zeta_1 < b.$$

Thus, the interval of ζ_1 can be expressed as:

$$\left[\begin{array}{l}\frac{-(1-\lambda_2)\psi}{\lambda_1\lambda_2} - \frac{\lambda_2(1-\lambda_1)\gamma}{\lambda_1\lambda_2} \\ -\frac{\lambda_1\lambda_2\left(\theta_0 - \theta_1\zeta_{1-L} - \ldots - \theta_Q\zeta_{1-QL}\right)}{\lambda_1\lambda_2}\end{array}\right] < \zeta_1 < \left[\begin{array}{l}\frac{b-(1-\lambda_2)\psi}{\lambda_1\lambda_2} - \frac{\lambda_2(1-\lambda_1)\gamma}{\lambda_1\lambda_2} \\ -\frac{\lambda_1\lambda_2\left(\theta_0 - \theta_1\zeta_{1-L} - \ldots - \theta_Q\zeta_{1-QL}\right)}{\lambda_1\lambda_2}\end{array}\right].$$

Using the Fredholm integral equation, the ARL integral equation for the SMA(Q)L model with an initial value of $DE_0 = \psi$ is determined on the DEWMA control chart. The expression for the equation appears as follows:

$$H(\psi) = \int_{\left[\frac{-(1-\lambda_2)\psi}{\lambda_1\lambda_2} - \frac{\lambda_2(1-\lambda_1)\gamma}{\lambda_1\lambda_2} - \frac{\lambda_1\lambda_2\left(\theta_0 - \theta_1\zeta_{1-L} - \ldots - \theta_Q\zeta_{1-QL}\right)}{\lambda_1\lambda_2}\right]}^{\left[\frac{b-(1-\lambda_2)\psi}{\lambda_1\lambda_2} - \frac{\lambda_2(1-\lambda_1)\gamma}{\lambda_1\lambda_2} - \frac{\lambda_1\lambda_2\left(\theta_0 - \theta_1\zeta_{1-L} - \ldots - \theta_Q\zeta_{1-QL}\right)}{\lambda_1\lambda_2}\right]} H\left(\begin{array}{l}\lambda_1\lambda_2\theta_0 - \theta_1\zeta_{1-L} - \ldots - \theta_P\zeta_{1-PL} \\ +\lambda_2(1-\lambda_1)\gamma + (1-\lambda_2)\psi + \lambda_1\lambda_2\zeta_1\end{array}\right) \times f(\upsilon)d\upsilon.$$

Given

$$\upsilon = \lambda_1\lambda_2\left(\theta_0 - \theta_1\zeta_{1-L} - \theta_2\zeta_{1-2L} - \ldots - \theta_Q\zeta_{1-QL}\right) + \lambda_2(1-\lambda_1)\gamma + (1-\lambda_2)\psi + \lambda_1\lambda_2\zeta_1$$

Then, $H(\psi)$ can be rewritten as follows:

$$H(\psi) = 1 + \frac{1}{\lambda_1\lambda_2} \times \int_0^b H(\upsilon) f\left(\frac{\eta - (1-\lambda_2)\psi - \lambda_1\lambda_2\left(\theta_0 - \theta_1\zeta_{1-L} - \ldots - \theta_Q\zeta_{1-QL}\right) - \lambda_2(1-\lambda_1)\gamma}{\lambda_1\lambda_2}\right) d\upsilon \quad (4)$$

From Eq. (4), the function of ζ_1 has an exponential distribution. Consequently, $H(\psi)$ can be represented as follows:

$$H(\psi)=1+\frac{e^{\frac{(1-\lambda_2)\psi}{\beta\lambda_1\lambda_2}}\cdot e^{\frac{\lambda_2(1-\lambda_1)\gamma}{\beta\lambda_1\lambda_2}}\cdot e^{\frac{\lambda_1\lambda_2\left(\theta_0-\theta_1\zeta_{1-L}-\ldots-\theta_Q\zeta_{1-QL}\right)}{\beta}}}{\beta\lambda_1\lambda_2}\times\int_0^b H(\upsilon)\cdot e^{\frac{-\upsilon}{\beta\lambda_1\lambda_2}}d\upsilon. \quad (5)$$

The ARL solution is verified by Banach's fixed-point theorem. This holds for the ARL solution that the solution exists and has uniqueness. From Eq. (5), we suppose that

$$h(\psi)=\frac{e^{\frac{(1-\lambda_2)\psi}{\beta\lambda_1\lambda_2}}\cdot e^{\frac{\lambda_2(1-\lambda_1)\gamma}{\beta\lambda_1\lambda_2}}\cdot e^{\frac{\lambda_1\lambda_2\left(\theta_0-\theta_1\zeta_{1-L}-\ldots-\theta_Q\zeta_{1-QL}\right)}{\beta}}}{\beta\lambda_1\lambda_2}$$

and $o=\int_0^b H(\upsilon)\cdot e^{\frac{-\upsilon}{\beta\lambda_1\lambda_2}}d\upsilon$.

Therefore, the ARL solution contained in Eq. (5) can be rewritten by reformatting the variables as follows:

$$H(\psi)=1+h(\psi)\cdot o \quad (6)$$

The integral equation of o, which can be expressed as follows:

$$o=\int_0^b e^{\frac{-\upsilon}{\beta\lambda_1\lambda_2}}\left(1+\frac{e^{\frac{(1-\lambda_2)\psi}{\beta\lambda_1\lambda_2}}\cdot e^{\frac{\lambda_2(1-\lambda_1)\gamma}{\beta\lambda_1\lambda_2}}\cdot e^{\frac{\lambda_1\lambda_2\left(\theta_0-\theta_1\zeta_{1-L}-\ldots-\theta_Q\zeta_{1-QL}\right)}{\beta}}}{\beta\lambda_1\lambda_2}\cdot o\right)d\upsilon$$

$$o=\frac{-\beta\lambda_1\lambda_2\left[e^{\frac{-b}{\beta\lambda_1\lambda_2}}-1\right]}{1+\frac{e^{\frac{\lambda_1\lambda_2\left(\theta_0-\theta_1\zeta_{1-L}-\theta_2\zeta_{1-2L}-\ldots-\theta_Q\zeta_{1-QL}\right)+\lambda_2(1-\lambda_1)\gamma}{\beta\lambda_1\lambda_2}}}{\lambda_2}\cdot\left[e^{\frac{-b}{\beta\lambda_1}}-1\right]}. \quad (7)$$

Subsequently, the solution of o is substituted into Eq.(6), as illustrated in Eq.(7), which is obtained as:

$$H(\psi)=1-\frac{\lambda_2 e^{\frac{(1-\lambda_2)\psi}{\beta\lambda_1\lambda_2}}\cdot\left[e^{\frac{-b}{\beta\lambda_1\lambda_2}}-1\right]}{\lambda_2 e^{\frac{\lambda_1\lambda_2\left(\theta_0-\theta_1\zeta_{1-L}-\ldots-\theta_Q\zeta_{1-QL}\right)+\lambda_2(1-\lambda_1)\gamma}{\beta\lambda_1\lambda_2}}+\left[e^{\frac{-b}{\beta\lambda_1}}-1\right]}. \quad (8)$$

The explicit Average Run Length (ARL) formula for the SMA(Q)$_L$ model is provided in Eq. (8), which is used for the DEWMA control chart. Furthermore, the process that is in control is substituted β with β_0, whereas the process that is out of control is substituted β with β_1.

3.2 The Numerical Integral Equation of ARL on DEWMA control chart

The average run length results are compared using the explicit formula method with the numerical integral equation or NIE method. Let $J(\psi)$ refer to the ARL of the DEWMA control chart for the SMA(Q)$_L$ model. The approximation of the ARL is performed using the midpoint rule. The interval $[0,b]$ is divided into $0\le v_1\le v_2\le\ldots\le v_m\le b$ using a set of constant weights $w_j=b/m$.

The evaluation of the integral approximation is conducted as follows:

$$\int_0^b H(\upsilon)f(\upsilon)d\upsilon\approx\sum_{j=1}^m w_j f(v_j) \quad (9)$$

Denote the integral equation using a numerical approximation in the form of $J(v_i)$. It is obtained through the solution of the linear equations:

$$J(v_i)=1+\frac{1}{\lambda_1\lambda_2}\sum_{j=1}^m J(v_j)\times f\left(\frac{\upsilon_j-(1-\lambda_2)\psi-\lambda_1\lambda_2\left(\theta_0-\theta_1\zeta_{1-L}-\ldots-\theta_Q\zeta_{1-QL}\right)-\lambda_2(1-\lambda_1)\gamma}{\lambda_1\lambda_2}\right)$$

$$i=1,2,\ldots,m. \quad (10)$$

The m linear equation system is expressed as $H_{m\times1}+R_{m\times m}H_{m\times1}$. If the inverse $(I_m-R_{m\times m})^{-1}$ exists, the unique solution is presented as:

$$H_{m\times1}=(I_m-R_{m\times m})^{-1}\cdot 1_{m\times1},$$

where $H_{m\times1}=[J(v_1),J(v_2),\ldots,J(v_m)]'$, $\mathrm{diag}(1,1,\ldots,1)$ is the unit matrix order m, $1_{m\times1}=[1,1,\ldots,1]'$ is a column vector of $J(v_i)$, and $R_{m\times m}$ is a matrix, the (m,m^{th}) elements of R matrix are defined as

$$[R_{ij}]\approx\frac{1}{\lambda_1\lambda_2}w_j\times f\left(\frac{\upsilon_j-(1-\lambda_2)\psi-\lambda_1\lambda_2\left(\theta_0-\theta_1\zeta_{1-L}-\ldots-\theta_Q\zeta_{1-QL}\right)-\lambda_2(1-\lambda_1)\gamma}{\lambda_1\lambda_2}\right)$$

Finally, v_i is instead of ψ into $J(v_i)$, The ARL estimation using the NIE method is shown as follows:

$$J(\psi) \approx 1 + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{m} J(\psi_j) \times f\left(\frac{\upsilon_j - (1-\lambda_2)\psi - \lambda_1\lambda_2\left(\theta_0 - \theta_1\zeta_{1-L} - \ldots - \theta_Q\zeta_{1-QL}\right) - \lambda_2(1-\lambda_1)\gamma}{\lambda_1\lambda_2}\right) \quad (11)$$

where υ_j represents a set of interval division points $\upsilon_j = (j-0.5)w_j$ for $j = 1,2,...,m$ and w_j is a weight of the composite midpoint formula; $w_j = b/m.$

3.3 The Uniqueness and Existence of the Explicit ARL Formula

In this section, the Banach fixed-point theorem is used to prove that the explicit formula of the ARL has an existence and gives a unique solution.

Theorem 1 Banach's Fixed-point Theorem: Let (Y,d) be a complete metric space and let $T:Y \to Y$ be a contraction mapping on *Y*. Then, *T* has a unique fixed point $y \in Y$ (such that $T(y) = y$) with contraction constant $s \in [0,1)$ such that $\|T(H(\psi)_1) - T(H(\psi)_2)\| \le s\|H(\psi)_1 - H(\psi)_2\|$, for all $H(\psi)_1, H(\psi)_2 \in Y$. There exists a unique $H(\psi) \in Y$ such that $T(H(\psi)) = H(\psi)$, i.e., a unique fixed-point in Y.

Proof: Specify T in Eq. (9) is a contraction mapping for $H(\psi)_1, H(\psi)_2 \in u[0,b]$.

To illustrate that
$\|T(H(\psi)_1) - T(H(\psi)_2)\| \le s\|H(\psi)_1 - (H(\psi)_2\|$ for all $H(\psi)_1, H(\psi)_2 \in Y$ with $s \in [0,1)$.

Consider: $\|T(H(\psi)_1) - T(H(\psi)_2)\|_\infty$

$$= \sup_{\psi\in[0,b]} \left|(H(\psi)_1 - (H(\psi)_2\right|$$

$$= \sup_{\psi\in[0,b]} \left| \frac{e^{\frac{(1-\lambda_2)\psi}{\beta\lambda_1\lambda_2}} \cdot e^{\frac{\lambda_2(1-\lambda_1)\gamma}{\beta\lambda_1\lambda_2}} \cdot e^{\frac{\lambda_1\lambda_2\left(\theta_0 - \theta_1\zeta_{1-L} - \ldots - \theta_Q\zeta_{1-QL}\right)}{\beta}}}{\beta\lambda_1\lambda_2} \int_0^b (H_1(g) - H_2(g)) \cdot e^{\frac{-g}{\beta\lambda_1\lambda_2}} dg \right|$$

$$\le \sup_{\psi\in[0,b]} \left| \|H(\psi)_1 - H(\psi)_2\|_\infty e^{\frac{(1-\lambda_2)\psi}{\beta\lambda_1\lambda_2}} \cdot e^{\frac{\lambda_2(1-\lambda_1)\gamma}{\beta\lambda_1\lambda_2}} \cdot e^{\frac{\lambda_1\lambda_2\left(\theta_0 - \theta_1\zeta_{1-L} - \ldots - \theta_Q\zeta_{1-QL}\right)}{\beta}} \left(1 - e^{\frac{-b}{\beta\lambda_1\lambda_2}}\right) \right|$$

$$= \|H(\psi)_1 - H(\psi)_2\|_\infty \sup_{\psi\in[0,b]} \left| e^{\frac{(1-\lambda_2)\psi}{\beta\lambda_1\lambda_2}} \cdot e^{\frac{\lambda_2(1-\lambda_1)\gamma}{\beta\lambda_1\lambda_2}} \cdot e^{\frac{\lambda_1\lambda_2\left(\theta_0 - \theta_1\zeta_{1-L} - \ldots - \theta_Q\zeta_{1-QL}\right)}{\beta}} \right| . \left|1 - e^{\frac{-b}{\beta\lambda_1\lambda_2}}\right|$$

$$\le s\|H(\psi)_1 - H(\psi)_2\|_\infty$$

in which

$$s = \sup_{\psi\in[0,b]} \left| e^{\frac{(1-\lambda_2)\psi}{\beta\lambda_1\lambda_2}} \cdot e^{\frac{\lambda_2(1-\lambda_1)\gamma}{\beta\lambda_1\lambda_2}} \cdot e^{\frac{\lambda_1\lambda_2\left(\theta_0 - \theta_1\zeta_{1-L} - \theta_2\zeta_{1-2L} - \ldots - \theta_Q\zeta_{1-QL}\right)}{\beta}} \right| \times \left|1 - e^{\frac{-b}{\beta\lambda_1\lambda_2}}\right|; \quad s \in [0,1).$$

This study evaluates the results acquired for ARL_0 and ARL_1 by employing explicit formulations and the NIE method to involve the $SMA(Q)_L$ process run on the DEWMA control chart. Comparing the accuracy of the ARL to the percentage accuracy that can be derived from

$$\%Accuracy = 100 - \left|\frac{S(\psi) - \tilde{S}(\psi)}{S(\psi)}\right| \times 100\%.$$

Plus, performance metrics including the SDRL and MRL are employed to evaluate the efficacy of control charts, [19]. Following are the computations of the SDRL and MRL values for the in-control process.

$$ARL_0 = \frac{1}{\beta}, SDRL_0 = \sqrt{\frac{1-\beta}{\beta^2}}, MRL_0 = \frac{\log(0.5)}{\log(1-\beta)}, \quad (12)$$

where β represents a type I error. The current study determined the value of ARL0 to be 370. To determine ARL0, apply Eq. (12) to SDRL0 and MRL0 at approximations 370 and 256, respectively. On the contrary, SDRL1 and MRL1 are calculated by substituting ρ for β, where ρ represents a type II error. The enhanced capability to detect shifts in the process mean is indicated by the minimum values of the ARL1, SDRL1, and MRL1.A comparison of the control charts' performances was conducted utilizing the relative mean index (RMI), [20]. For a given scenario, the control chart that exhibits the highest performance has the lowest ARL and RMI values. RMI is calculated as

$$RMI = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{ARL_{shift,i} - Min[ARL_{shift,i}]}{Min[ARL_{shift,i}]}\right)$$

$Min[ARL_{shift,i}]$ represents the smallest ARL at the same level of shift for all of the control charts, while $ARL_{shift,i}$ represents the ARL of the control chart corresponding to the shift size of row *i*.

Likewise, metrics for performance can be employed to evaluate the efficacy of a control chart such as the average additional quadratic loss

(AEQL), the Performance Comparison Index (PCI). AEQL can be calculated as follows, [21],

$$AEQL = \frac{1}{\Delta}\sum_{\delta_i=\delta_{\min}}^{\delta_{\max}} \left(\delta_i^2 \times ARL(\delta_i)\right) \quad (13)$$

where δ represents the specific change in the process, and Δ represents the aggregate of number of divisions from $\delta_{\min}$ to $\delta_{\max}$. In this study, $\Delta = 9$ is determined from $\delta_{\min} = 0.001$ to $\delta_{\max} = 0.10.$ with the lowest AEQL value, a control chart is considered to be the most effective.

The PCI value is determined by dividing the AEQL of the control chart by that of the control chart with the lowest AEQL, which represents the control chart with the highest level of effectiveness. The mathematical representation of the PCI is as an expression.

$$PCI = \frac{AEQL}{AEQL_{lowest}} \quad (14)$$

4 Numerical Results

The accuracy measurements for ARL as determined by explicit formulas and the numerical integral equation method in the comparative study are presented in Table 1 and Table 2 in Appendix. The in-control process $\beta_0 = 1$, out-of-control process $\beta_1 = (1+\delta)\beta_0$, shift sizes (δ) = 0.001, 0.003, 0.005, 0.01, 0.03, 0.05, and 0.10 , $ARL_0 = 370.$ The ARL of explicit formulas and NIE method with SMA$(2)_4$ process for different choices of λ_1 with $\lambda_2 = 0.10$ $\theta_0 = 1$ is presented in Table 1 (Appendix). Table 2 (Appendix) shows the ARL of explicit formulas and NIE method with SMA$(3)_{12}$ process for different choices of λ_1 with $\lambda_2 = 0.10$ $\theta_0 = 1$ The results from The ARLs of both approaches are remarkably comparable with a 100% accuracy rate, as demonstrated in Table 1 and Table 2 in Appendix. Moreover, the explicit formula requires a CPU time of approximately 0.1 seconds, a significant decrease in comparison to the NIE method. Furthermore, an analysis of the efficacy of the EWMA and DEWMA control charts was presented. The explicit formulations were employed to assess the ARL outcomes through the λ_1 of the values 0.1 and 0.2 on the DEWMA control chart while maintaining the exponential smoothing parameter at 1. The values obtained for SMA$(2)_4$ and SMA$(3)_{12}$ are correspondingly displayed in Table 3 and Table 4 in Appendix. In all cases, the ARL of the DEWMA control chart is significantly lower than that of the EWMA control chart. Furthermore, it is observed that the MRL and SDRL outcomes correspond with the ARL. Particularly, the DEWMA control chart displays the lowest MRL and SDRL values at all shift levels when λ_1 is set to 0.10. Consequently, the DEWMA control chart exhibits superior efficacy in comparison to the EWMA control chart.

4.1 Application

In this section, the dataset of monthly WTI oil price from January 2016 to December 2022 is studied to determine the performance of the explicit formula for ARL on the DEWMA control chart and compare the performance with the EWMA control chart. Initially, we started by finding a model for WTI oil price and estimating parameters using maximum likelihood. The data has proved that it is a SMA$(Q)_L$ model. The estimated parameters are as follows: $\theta_1 = 0.951$, and the in-control parameter of exponential white noise equal to 26.1741, as shown in Table 5 (Appendix). By employing the parameters of this predictive model, it is able to illustrate the following: $\hat{Y}_t = -0.951\zeta_{t-12}$

The explicit formula method is employed to compare the efficacy of the SMA$(1)_{12}$ model's ARL values on the DEWMA and EWMA control charts with respect to ARL, SDRL, and MRL. The findings are simply presented in Table 6 (Appendix). It is apparent that the results correspond with the information presented in Table 3 and Table 4 in Appendix. According to the results presented in Table 6 and Figure 1 (Appendix), for ARL_0=370 and 500, the DEWMA control chart when $\lambda_1 = 0.1$ exhibits the least RMI and AEQL across all levels. Likewise, the PCI of the DEWMA control chart is 1. In addition, the DEWMA (D_t) statistics with $\lambda_1 = 0.05$ $\lambda_2 = 0.10$ for the WTI oil price dataset fitted to SMA$(1)_{12}$ model are presented in Figure 2 (Appendix). The results indicate that the DEWMA chart can detect a shift at the 5th observation. In brief, practical applications involving the utilization of the DEWMA control chart to identify variations in the process consistently demonstrate the efficacy of the explicit formula approach.

5 Conclusion

The explicit ARL formula for the SMA $(Q)_L$ model on DEWMA control charts is investigated in this study. The explicit formula is accurate and reduces processing time. When comparing the ARL using percentage accuracy criteria between the explicit formula and the numerical integral equation (NIE) method, the research found that the ARL value was not significantly different. Moreover, considering the effectiveness of DEWMA and EWMA control charts for detecting changes in process mean, the results indicate that the DEWMA control chart shows superior performance compared to the EWMA control chart. Moreover, when considering the RMI, AEQL, and PCI criteria for the DEWMA control chart, the RMI and AEQL values are the lowest as well as the PCI value of 1. In addition, the study found that the results from the model with known parameters and the results from real data with approximated parameters were consistent, i.e., the DEWMA control chart gave the best performance. In future research, the ARL explicit formulas on DEWMA control charts can also be developed for other interesting models, and the integral equation method can be applied to express the formula for average run length values on other types of control charts.

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During the preparation of this work the authors used QuillBot in order to modify a few phrases in the research work citation section to enhance academic quality. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

- Rapin Sunthornwat has organized the conceptualization and simulation.
- Yupaporn Areepong has organized the conceptualization, writing-original draft and validation
- Saowanit Sukparungsee has implemented the methodology,software, and validation.

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Conflicts of Interest

The authors declare no conflict of interest.

APPENDIX

Table 1. *ARL* of explicit formulas and NIE method with SMA(2)4 process for different choices of λ_1 with $\lambda_2 = 0.10$ $\theta_0 = 1, ARL_0 = 370$

λ_1	Coefficients of process			Methods	Shift size (δ)						
	θ_1	θ_2	b		0.001	0.003	0.005	0.01	0.03	0.05	0.1
0.05	0.1	0.1	0.000225156	Explicit	145.81900	66.23820	43.03390	23.16610	8.52882	5.45640	3.12670
				CPU_{Exp}	(0.109)	(0.109)	(0.110)	(0.078)	(0.109)	(0.062)	(0.078)
				NIE	145.81900	66.23820	43.03390	23.16610	8.52882	5.45640	3.12670
				CPU_{NIE}	(2.594)	2.578)	(2.594)	(2.562)	(2.578)	(2.578)	(2.594)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		0.2	0.000203685	Explicit	143.87800	65.04740	42.20470	22.69740	8.35598	5.34913	3.07039
				CPU_{Exp}	(0.078)	(0.125)	(0.109)	(0.078)	(0.094)	(0.078)	(0.110)
				NIE	143.87800	65.04740	42.20470	22.69740	8.35598	5.34913	3.07039
				CPU_{NIE}	(2.594)	(2.610)	(2.547)	(2.562)	(2.546)	(2.563)	(2.515)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		0.3	0.000184265	Explicit	141.94200	63.88640	41.40070	22.24500	8.18970	5.24602	3.01634
				CPU_{Exp}	(0.093)	(0.078)	(0.094)	(0.125)	(0.094)	(0.078)	(0.110)
				NIE	141.94200	63.88640	41.40070	22.24500	8.18970	5.24602	3.01634
				CPU_{NIE}	(2.563)	(2.594)	(2.531)	(2.563)	(2.577)	(2.562)	(2.561)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
0.10	0.2	0.1	0.00111134	Explicit	165.70400	78.99960	52.05010	28.32890	10.45420	6.65591	3.76031
				CPU_{Exp}	(0.078)	(0.093)	(0.094)	(0.078)	(0.094)	(0.094)	(0.109)
				NIE	165.70400	78.99960	52.05010	28.32890	10.45420	6.65591	3.76031
				CPU_{NIE}	(2.594)	(2.563)	(2.530)	(2.563)	(2.547)	(2.577)	(2.547)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		0.2	0.001005034	Explicit	163.23200	77.36820	50.88600	27.65620	10.20130	6.49790	3.67646
				CPU_{Exp}	(0.093)	(0.110)	(0.078)	(0.109)	(0.078)	(0.110)	(0.093)
				NIE	163.23200	77.36820	50.88600	27.65620	10.20130	6.49790	3.67646
				CPU_{NIE}	(2.546)	(2.593)	(2.578)	(2.561)	(2.562)	(2.578)	(2.578)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		0.3	0.000908945	Explicit	160.84500	75.79950	49.76890	27.01190	9.95952	6.34697	3.59646
				CPU_{Exp}	(0.094)	(0.078)	(0.093)	(0.093)	(0.078)	(0.109)	(0.078)
				NIE	160.84500	75.79950	49.76890	27.01190	9.95952	6.34697	3.59646
				CPU_{NIE}	(2.594)	(2.593)	(2.594)	(2.594)	(2.610)	(2.578)	(2.563)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
0.30	0.3	0.1	0.0059015	Explicit	180.36000	89.32170	59.55260	32.73470	12.13240	7.70853	4.32208
				CPU_{Exp}	(0.093)	(0.094)	(0.110)	(0.125)	(0.094)	(0.093)	(0.078)
				NIE	180.36000	89.32170	59.55260	32.73470	12.13240	7.70853	4.32208
				CPU_{NIE}	(2.578)	(2.563)	(2.578)	(2.547)	(2.610)	(2.531)	(2.594)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		0.2	0.00533475	Explicit	177.54200	87.31870	58.08870	31.87020	11.80130	7.50045	4.21068
				CPU_{Exp}	(0.063)	(0.093)	(0.109)	(0.093)	(0.078)	(0.109)	(0.109)
				NIE	177.54200	87.31870	58.08870	31.87020	11.80130	7.50045	4.21068
				CPU_{NIE}	(2.579)	(2.593)	(2.579)	(2.578)	(2.546)	(2.516)	(2.562)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
		0.3	0.00482289	Explicit	174.89400	85.41480	56.69630	31.04810	11.48670	7.30289	4.10504
				CPU_{Exp}	(0.094)	(0.093)	(0.094)	(0.109)	(0.109)	(0.110)	(0.094)
				NIE	174.89400	85.41480	56.69630	31.04810	11.48670	7.30289	4.10504
				CPU_{NIE}	(2.578)	(2.562)	(2.594)	(2.578)	(2.547)	(2.594)	(2.547)
				%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Note: The numerical results in parentheses are computational times in seconds.

Table 2. *ARL* of explicit formulas and NIE method with SMA(3)$_{12}$ process for different choices of λ_1 with $\lambda_2 = 0.10\ \ \theta_0 = 1, ARL_0 = 370$

λ_1	Coefficients of process				Methods	Shift size (δ)						
	θ_1	θ_2	θ_3	b		0.004	0.008	0.01	0.04	0.08	0.1	0.4
0.05	0.1	-0.1	0.1	0.000248896	Explicit	53.15080	28.94420	23.65270	6.75154	3.78219	3.18540	1.42691
					CPU_{Exp}	(0.093)	(0.109)	(0.094)	(0.078)	(0.078)	(0.110)	(0.078)
					NIE	53.15080	28.94420	23.65270	6.75154	3.78219	3.18540	1.42691
					CPU_{NIE}	(2.437)	(2.468)	(2.469)	(2.421)	(2.469)	(2.469)	(2.469)
					%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
			0.2	0.000225157	Explicit	52.15110	28.35920	23.16840	6.61415	3.71010	3.12672	1.41046
					CPU_{Exp}	(0.109)	(0.063)	(0.109)	(0.062)	(0.093)	(0.078)	(0.094)
					NIE	52.15110	28.35920	23.16840	6.61415	3.71010	3.12672	1.41046
					CPU_{NIE}	(2.499)	(2.453)	(2.499)	(2.422)	(2.437)	(2.469)	(2.485)
					%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
			0.3	0.000203685	Explicit	51.16020	27.78830	22.69740	6.48184	3.64082	3.07039	1.39477
					CPU_{Exp}	(0.078)	(0.110)	(0.078)	(0.094)	(0.109)	(0.078)	(0.110)
					NIE	51.16020	27.78830	22.69740	6.48184	3.64082	3.07039	1.39477
					CPU_{NIE}	(2.484)	(2.500)	(2.438)	(2.469)	(2.484)	(2.438)	(2.500)
					%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
0.10	0.2	-0.1	0.1	0.001228952	Explicit	64.10160	35.41860	29.02460	8.29192	4.59482	3.84820	1.61916
					CPU_{Exp}	(0.125)	(2.562)	(0.094)	(0.109)	(0.110)	(0.078)	(0.094)
					NIE	64.10160	35.41860	29.02460	8.29192	4.59482	3.84820	1.61916
					CPU_{NIE}	(2.500)	(2.562)	(2.469)	(2.484)	(2.470)	(2.437)	(2.500)
					%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
			0.2	0.00111134	Explicit	62.71990	34.58400	28.32890	8.08917	4.48728	3.76031	1.59304
					CPU_{Exp}	(0.094)	(0.062)	(0.094)	(0.094)	(0.110)	(0.078)	(0.125)
					NIE	62.71990	34.58400	28.32890	8.08917	4.48728	3.76031	1.59304
					CPU_{NIE}	(2.532)	(2.452)	(2.625)	(2.485)	(2.500)	(2.469)	(2.468)
					%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
			0.3	0.001005035	Explicit	61.36310	33.77560	27.65700	7.89515	4.38461	3.67647	1.56829
					CPU_{Exp}	(0.094)	(0.078)	(0.110)	(0.094)	(0.110)	(0.094)	(0.078)
					NIE	61.36310	33.77560	27.65700	7.89515	4.38461	3.67647	1.56829
					CPU_{NIE}	(2.485)	(2.484)	(2.500)	(2.453)	(2.501)	(2.454)	(2.468)
					%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
0.30	0.3	-0.1	0.1	0.0065291	Explicit	73.16630	40.93890	33.63600	9.64808	5.31717	4.43959	1.79894
					CPU_{Exp}	(0.078)	(0.125)	(0.109)	(0.094)	(0.078)	(0.078)	(0.110)
					NIE	73.16630	40.93890	33.63600	9.64808	5.31717	4.43959	1.79894
					CPU_{NIE}	(2.453)	(2.500)	(2.484)	(2.453)	(2.453)	(2.438)	(2.500)
					%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
			0.2	0.0059015	Explicit	71.42620	39.86350	32.73470	9.37994	5.17383	4.32208	1.76269
					CPU_{Exp}	(0.125)	(0.094)	(0.078)	(0.094)	(0.094)	(0.125)	(0.109)
					NIE	71.42620	39.86350	32.73470	9.37994	5.17383	4.32208	1.76269
					CPU_{NIE}	(2.531)	(2.484)	(2.468)	(2.500)	(2.469)	(2.500)	(2.499)
					%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00
			0.3	0.00533475	Explicit	69.73130	38.82910	31.87020	9.12500	5.03784	4.21068	1.72857
					CPU_{Exp}	(0.078)	(0.094)	(0.078)	(0.110)	(0.078)	(0.093)	(0.109)
					NIE	69.73130	38.82910	31.87020	9.12500	5.03784	4.21068	1.72857
					CPU_{NIE}	(2.484)	(2.501)	(2.453)	(2.501)	(2.453)	(2.483)	(2.468)
					%Acc	100.00	100.00	100.00	100.00	100.00	100.00	100.00

Note: The numerical results in parentheses are computational times in seconds

Table 3. The ARL of DEWMA control chart for SMA(2)$_4$ using explicit formula against EWMA control chart given $\lambda_2 = 0.2$

$\theta_0 = 1, \theta_1 = -0.1,$ and $\beta_0 = 1$.

		$\theta_2 = 0.1$			$\theta_2 = 0.2$			$\theta_2 = 0.3$		
δ	Control Chart	DEWMA $\lambda_1 = 0.1$	DEWMA $\lambda_1 = 0.2$	EWMA $\lambda_1 = 1$	DEWMA $\lambda_1 = 0.1$	DEWMA $\lambda_1 = 0.2$	EWMA $\lambda_1 = 1$	DEWMA $\lambda_1 = 0.1$	DEWMA $\lambda_1 = 0.2$	EWMA $\lambda_1 = 1$
	UCL	0.00302919	0.01009005	0.076238	0.0027369	0.0091074	0.0687253	0.00247316	0.0082225	0.0619762
0.00	ARL_0	370.73600	370.66600	370.54000	370.66900	370.28400	370.70800	370.22800	370.92400	370.83800
	$SDRL_0$	370.23570	370.16570	370.03970	370.16870	369.78370	370.20770	369.72770	370.42370	370.33770
	MRL_0	256.62790	256.57940	256.49200	256.58140	256.31460	256.60850	256.27580	256.75820	256.69860
0.001	ARL_1	174.12200	188.52700	201.65900	171.44400	185.38700	198.24900	168.75700	182.58500	194.94400
	$SDRL_1$	173.62130	188.02630	201.15840	170.94330	184.88630	197.74840	168.25630	182.08430	194.44340
	MRL_1	120.34530	130.33010	139.43250	118.48900	128.15360	137.06890	116.62650	126.21140	134.77800
0.003	ARL_1	84.82340	95.40960	105.82300	82.95170	93.08090	103.02500	81.13420	90.91580	100.36900
	$SDRL_1$	84.32192	94.90828	105.32180	82.45018	92.57955	102.52380	80.63265	90.41442	99.86775
	MRL_1	58.44784	65.78571	73.00379	57.15046	64.17157	71.06435	55.89065	62.67082	69.22334
0.005	ARL_1	56.25770	64.05940	71.93230	54.89990	62.33360	69.79390	53.59220	60.71930	67.77890
	$SDRL_1$	55.75546	63.55743	71.43055	54.39760	61.83158	69.29210	53.08985	60.21722	67.27704
	MRL_1	38.64726	44.05511	49.51229	37.70608	42.85885	48.03004	36.79962	41.73988	46.63332
0.007	ARL_1	42.18980	48.32550	54.59710	41.13040	46.96390	52.88460	40.11390	45.68780	51.27670
	$SDRL_1$	41.68680	47.82289	54.09479	40.62732	46.46121	52.38221	39.61074	45.18503	50.77424
	MRL_1	28.89578	33.14890	37.49618	28.16143	32.20508	36.30914	27.45680	31.32052	35.19459
0.01	ARL_1	30.78450	35.42300	40.21480	29.98870	34.39090	38.90070	29.22750	33.42220	37.67050
	$SDRL_1$	30.28037	34.91942	39.71165	29.48446	33.88721	38.39744	28.72315	32.91840	37.16714
	MRL_1	20.98971	24.20512	27.52675	20.43805	23.48968	26.61583	19.91037	22.81818	25.76307
0.03	ARL_1	11.38150	13.16430	15.04160	11.07920	12.76560	14.52290	10.79150	12.39070	14.03980
	$SDRL_1$	10.87001	12.65443	14.53300	10.56738	12.25540	14.01398	10.27935	11.88018	13.53056
	MRL_1	7.53717	8.77366	10.07550	7.32748	8.49716	9.71581	7.12791	8.23715	9.38081
0.05	ARL_1	7.23445	8.35451	9.53979	7.04515	8.10357	9.21170	6.86526	7.86767	8.90654
	$SDRL_1$	6.71586	7.83858	9.02595	6.52602	7.58711	8.69734	6.34559	7.35068	8.39166
	MRL_1	4.65938	5.43697	6.25951	4.52791	5.26279	6.03185	4.40297	5.09903	5.82009
0.07	ARL_1	5.42936	6.25424	7.12966	5.29024	6.06924	6.88712	5.15813	5.89536	6.66167
	$SDRL_1$	4.90394	5.73248	6.61078	4.76407	5.54675	6.36752	4.63122	5.37214	6.14135
	MRL_1	3.40502	3.97848	4.58660	3.30825	3.84991	4.41816	3.21633	3.72905	4.26155
0.10	ARL_1	4.06582	4.66418	5.30117	3.96514	4.52981	5.12454	3.86961	4.40356	4.96043
	$SDRL_1$	3.53059	4.13405	4.77506	3.42888	3.99867	4.59743	3.33231	3.87141	4.43232
	MRL_1	2.45535	2.87247	3.31585	2.38509	2.77886	3.19296	2.31839	2.69088	3.07874
RMI		**0**	0.1392	0.2824	**0**	0.1355	0.2739	**0**	0.1325	0.2662
AEQL		**0.0114**	0.0131	0.0149	**0.0111**	0.0127	0.0144	**0.0108**	0.0124	0.0140
PCL		**1**	1.1503	1.3094	**1**	1.1457	1.2984	**1**	1.1415	1.2883

Table 4. The ARL of DEWMA control chart for SMA(3)$_{12}$ using explicit formula against EWMA control chart given $\lambda_2 = 0.2$

$\theta_0 = 1, \theta_1 = 0.1, \theta_2 = 0.2$ and $\beta_0 = 1$.

		$\theta_3 = 0.1$			$\theta_3 = 0.2$			$\theta_3 = 0.3$		
δ	Control Chart	DEWMA $\lambda_1 = 0.1$	DEWMA $\lambda_1 = 0.2$	EWMA $\lambda_1 = 1$	DEWMA $\lambda_1 = 0.1$	DEWMA $\lambda_1 = 0.2$	EWMA $\lambda_1 = 1$	DEWMA $\lambda_1 = 0.1$	DEWMA $\lambda_1 = 0.2$	EWMA $\lambda_1 = 1$
	UCL	0.02002024	0.00670642	0.0504501	0.00182621	0.00605838	0.0455371	0.001650965	0.0054738	0.0411125
0.00	ARL_0	370.03400	370.07300	370.22900	370.51700	370.73600	370.60500	370.49600	370.66900	370.92400
	$SDRL_0$	369.53370	369.57270	369.72870	370.01670	370.23570	370.10470	369.99570	370.16870	370.42370
	MRL_0	256.14130	256.16830	256.27650	256.47610	256.62790	256.53710	256.46150	256.58140	256.75820
0.001	ARL_1	163.72400	176.70300	188.41400	161.41000	174.12200	185.46700	159.05600	171.44400	182.58500
	$SDRL_1$	163.22320	176.20230	187.91330	160.90920	173.62130	184.96630	158.55520	170.94330	182.08430
	MRL_1	113.13790	122.13430	130.25180	111.53400	120.34530	128.20900	109.90230	118.48900	126.21140
0.003	ARL_1	77.73330	86.73170	95.38080	76.15040	84.82340	93.10100	74.60320	82.95170	90.91580
	$SDRL_1$	77.23168	86.23025	94.87948	75.64875	84.32192	92.59965	74.10151	82.45018	90.41442
	MRL_1	53.53330	59.77059	65.76575	52.43610	58.44784	64.18550	51.36364	57.15046	62.67082
0.005	ARL_1	51.15130	57.65970	64.04660	50.01480	56.25770	62.34250	48.91420	54.89990	60.71930
	$SDRL_1$	50.64883	57.15751	63.54463	49.51228	55.75546	61.84048	48.41162	54.39760	60.21722
	MRL_1	35.10767	39.61907	44.04624	34.31988	38.64726	42.86502	33.55697	37.70608	41.73988
0.007	ARL_1	38.21910	43.29020	48.31820	37.33720	42.18980	46.96900	36.48660	41.13040	45.68780
	$SDRL_1$	37.71579	42.78728	47.81559	36.83381	41.68680	46.46631	35.98313	40.62732	45.18503
	MRL_1	26.14336	29.65856	33.14384	25.53203	28.89578	32.20861	24.94241	28.16143	31.32052
0.01	ARL_1	27.81030	31.61480	35.41910	27.15090	30.78450	34.39360	26.51710	29.98870	33.42220
	$SDRL_1$	27.30572	31.11078	34.91552	26.64621	30.28037	33.88991	26.01230	29.48446	32.91840
	MRL_1	18.92794	21.56528	24.20242	18.47083	20.98971	23.49155	18.03146	20.43805	22.81818
0.03	ARL_1	10.25730	11.69930	13.16380	10.00910	11.38150	12.76590	9.77180	11.07920	12.39070
	$SDRL_1$	9.74448	11.18813	12.65393	9.49595	10.87001	12.25570	9.25831	10.56738	11.88018
	MRL_1	6.75732	7.75760	8.77331	6.58513	7.53717	8.49736	6.42049	7.32748	8.23715
0.05	ARL_1	6.53155	7.43379	8.35432	6.37661	7.23445	8.10371	6.22868	7.04515	7.86767
	$SDRL_1$	6.01079	6.91574	7.83839	5.85530	6.71586	7.58725	5.70682	6.52602	7.35068
	MRL_1	4.17116	4.79780	5.43684	4.06351	4.65938	5.26289	3.96072	4.52791	5.09903
0.07	ARL_1	4.91324	5.57601	6.25414	4.79960	5.42936	6.06931	4.69121	5.29024	5.89536
	$SDRL_1$	4.38483	5.05132	5.73238	4.27043	4.90394	5.54682	4.16128	4.76407	5.37214
	MRL_1	3.04589	3.50701	3.97841	2.96677	3.40502	3.84996	2.89129	3.30825	3.72905
0.10	ARL_1	3.69272	4.17205	4.66413	3.61071	4.06582	4.52985	3.53254	3.96514	4.40356
	$SDRL_1$	3.15333	3.63785	4.13400	3.07026	3.53059	3.99871	2.99104	3.42888	3.87141
	MRL_1	2.19481	2.52946	2.87243	2.13748	2.45535	2.77889	2.08281	2.38509	2.69088
RMI		**0**	0.1261	0.2517	**0**	0.1233	0.2453	**0**	0.1205	0.2394
AEQL		**0.0103**	0.0117	0.0131	**0.0101**	0.0114	0.0127	**0.0099**	0.0111	0.0124
PCL		**1**	1.1336	1.2701	**1**	1.1301	1.2619	**1**	1.1266	1.2542

Table 5. The coefficients for the SMA(1)$_{12}$ model using the real-world dataset.

model parameters	**SMA(1)$_{12}$**	SE	p-value
SMA(1)	0.951	0.036	0.000
RMSE		27.372	
Normalized BIC		6.672	
Residual	**Residual of SMA(1) model**		
Exponential parameter		26.1741	
One-sample Kolmogorov-Smirnov test		1.343	
p-value		0.054	

Table 6. The ARL of DEWMA control chart for $SMA(1)_{12}$ using explicit formula against EWMA control chart given $\lambda_2 = 0.2$

$\theta_1 = 0.951$ and $\beta_0 = 26.1741$.

ARL_0		370			500		
δ	Control Chart	DEWMA $\lambda_1 = 0.1$	DEWMA $\lambda_1 = 0.2$	EWMA $\lambda_1 = 1$	DEWMA $\lambda_1 = 0.1$	DEWMA $\lambda_1 = 0.2$	EWMA $\lambda_1 = 1$
	UCL	0.00541645	0.01119763	0.0574913	0.00542023	0.01120545	0.0575314
0.00	ARL_0	370.80500	370.80300	370.88600	500.72000	500.87000	500.83000
	$SDRL_0$	370.30470	370.30270	370.38570	500.21980	500.36980	500.32980
	MRL_0	256.67570	256.67430	256.73190	346.72600	346.82990	346.80220
0.001	ARL_1	330.02300	330.25200	330.50300	429.07000	429.57100	429.85500
	$SDRL_1$	329.52260	329.75160	330.00260	428.56970	429.07070	429.35470
	MRL_1	228.40780	228.56650	228.74050	297.06200	297.40920	297.60610
0.003	ARL_1	270.55100	271.01600	271.43400	333.65000	334.42500	334.97700
	$SDRL_1$	270.05050	270.51550	270.93350	333.14960	333.92460	334.47660
	MRL_1	187.18490	187.50720	187.79690	230.92180	231.45900	231.84160
0.005	ARL_1	229.27300	229.83000	230.30900	272.99400	273.83000	274.45400
	$SDRL_1$	228.77250	229.32950	229.80850	272.49350	273.32950	273.95350
	MRL_1	158.57310	158.95920	159.29120	188.87820	189.45770	189.89020
0.01	ARL_1	166.04200	166.62500	167.11200	187.78500	188.55400	189.15100
	$SDRL_1$	165.54120	166.12420	166.61120	187.28430	188.05330	188.65030
	MRL_1	114.74460	115.14870	115.48630	129.81580	130.34880	130.76260
0.03	ARL_1	79.18420	79.58010	79.90340	83.77920	84.22710	84.58410
	$SDRL_1$	78.68261	79.07852	79.40183	83.27770	83.72561	84.08261
	MRL_1	54.53900	54.81342	55.03752	57.72405	58.03451	58.28197
0.05	ARL_1	52.15090	52.43500	52.66610	54.09170	54.39940	54.64610
	$SDRL_1$	51.64848	51.93259	52.16370	53.58937	53.89708	54.14379
	MRL_1	35.80056	35.99749	36.15768	37.14586	37.35915	37.53015
0.10	ARL_1	28.34190	28.50650	28.64000	28.89590	29.06770	29.20600
	$SDRL_1$	27.83741	28.00204	28.13556	28.39150	28.56332	28.70165
	MRL_1	19.29646	19.41056	19.50311	19.68050	19.79960	19.89547
0.30	ARL_1	10.39300	10.45450	10.50430	10.46180	10.52430	10.57470
	$SDRL_1$	9.88036	9.94194	9.99180	9.94924	10.01182	10.06229
	MRL_1	6.85146	6.89413	6.92868	6.89919	6.94255	6.97751
0.50	ARL_1	6.57892	6.61700	6.64783	6.60471	6.64315	6.67421
	$SDRL_1$	6.05832	6.09653	6.12746	6.08420	6.12277	6.15393
	MRL_1	4.20407	4.23052	4.25194	4.22198	4.24869	4.27026
1.00	ARL_1	3.67314	3.69293	3.70895	3.68001	3.69990	3.71598
	$SDRL_1$	3.13350	3.15354	3.16976	3.14046	3.16060	3.17687
	MRL_1	2.18113	2.19496	2.20616	2.18593	2.19983	2.21107
3.00	ARL_1	1.74342	1.75044	1.75612	1.74432	1.75136	1.75705
	$SDRL_1$	1.13846	1.14612	1.15232	1.13944	1.14713	1.15333
	MRL_1	0.81323	0.81839	0.82257	0.81389	0.81907	0.82325
RMI		**0**	0.0041	0.0075	**0**	0.0044	0.0079
AEQL		**2.0412**	2.0504	2.0578	**0.6183**	2.0547	2.0621
PCL		**1**	1.0044	1.0081	**1**	3.3231	3.3351

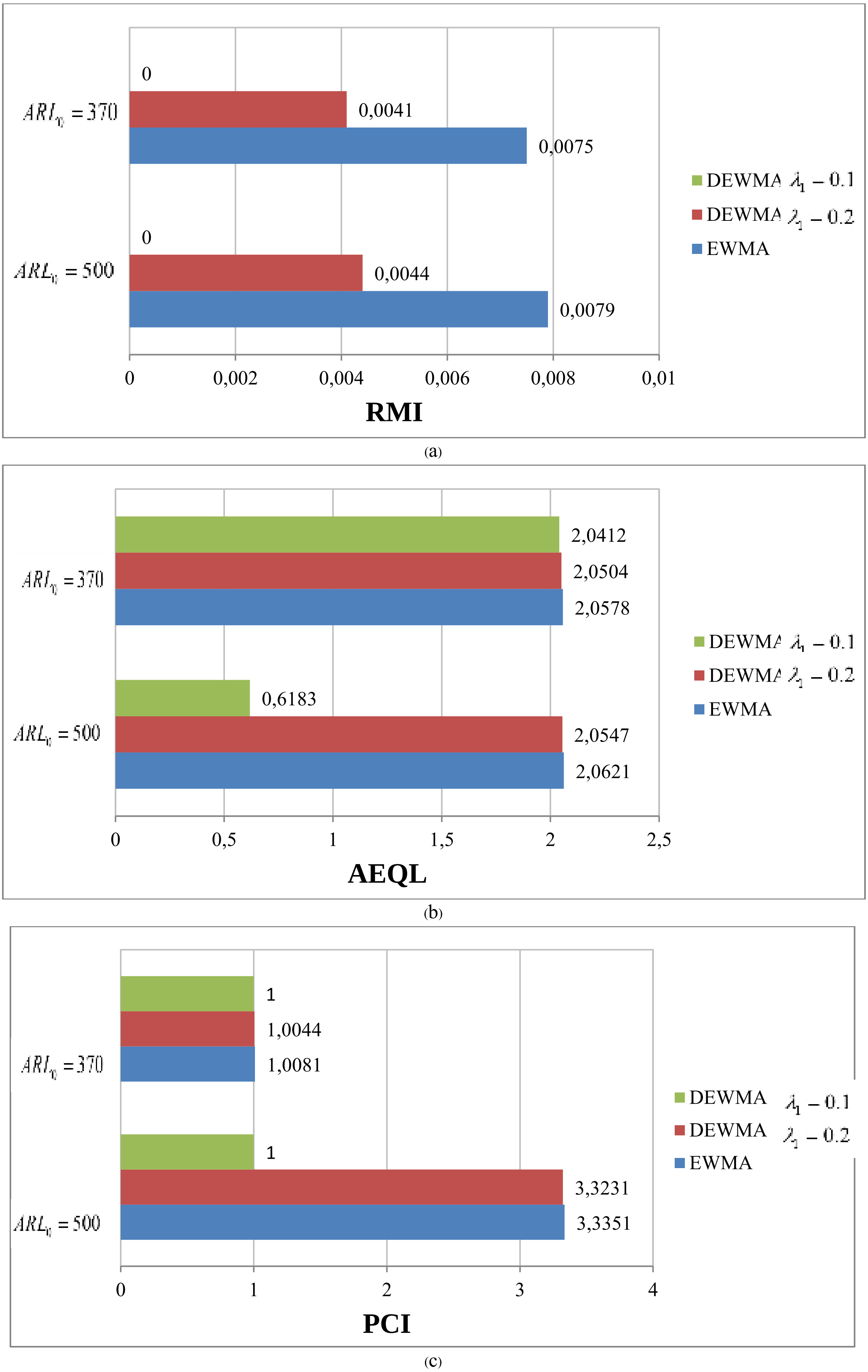

Fig. 1: Comparison the RMI, AEQL and PCI values among DEWMA and EWMA control charts

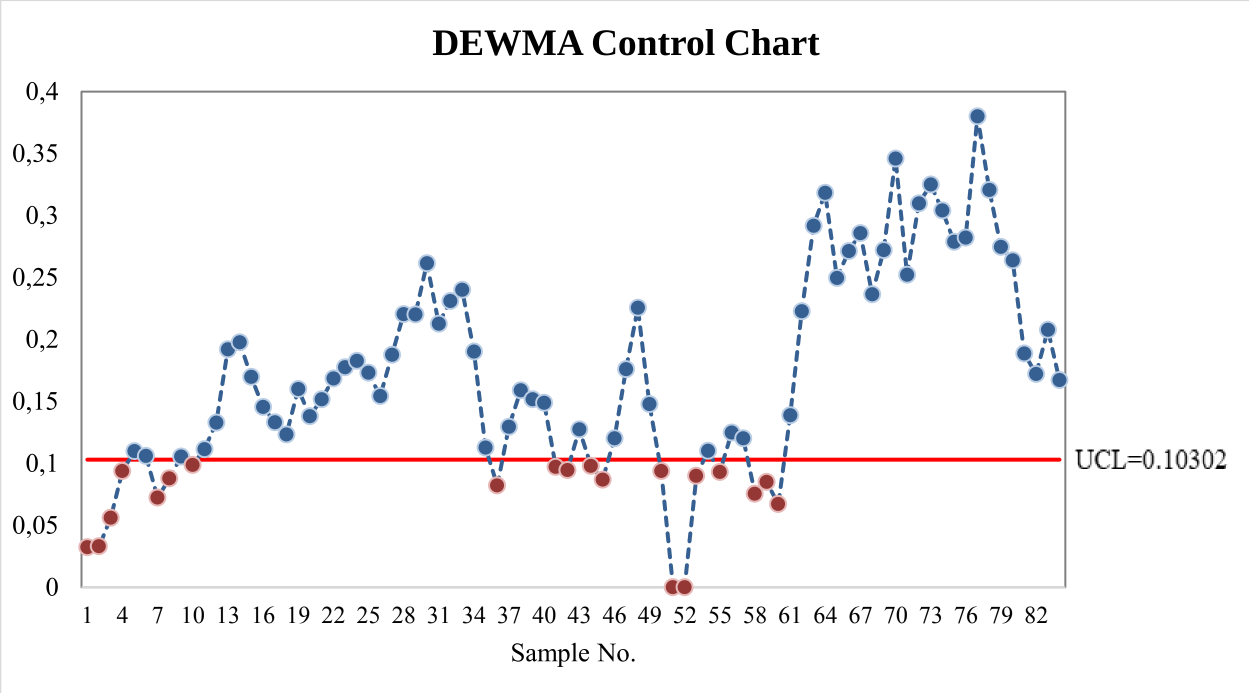

Fig. 2: The dataset fitted to SMA(1)12 process running on DEWMA control chart when $\lambda_1 = 0.05, \lambda_2 = 0.1$