Some Classes of Matrix Transformations between the Subsets of Maddox Spaces Defined by Speed

ANTS AASMA[1], PINNANGUDI N. NATARAJAN[2]
[1]Department of Economics and Finance,
Tallinn University of Technology,
Akadeemia tee 3-456, 12618,
ESTONIA

[2]Old no. 2/3, new no.3/3 Second main road, R. A. Puram, Chennai 600028,
INDIA

Abstract: - Let *X, Y* be two sets of sequences with real or complex entries, and (X,Y) the set of all matrices with real or complex entries to map *X* into *Y*. Let λ and μ be speeds of convergence, i.e., monotonically increasing strictly positive sequences. In this paper, some classes of matrix transforms, where *X* is the subset of certain Maddox space defined by λ, and *Y* is the subset of another Maddox space defined by μ, have been characterized. The application possibilities of the main results are demonstrated with two examples.

Key-Words: - Matrix transforms, paranormal spaces, Maddox spaces, boundedness with speed, convergence with speed, absolute convergence with speed, paranormal boundedness with speed, paranormal convergence with speed, paranormal absolute convergence with speed, Zweier matrix.

Received: May 26, 2024. Revised: November 21, 2024. Accepted: December 22, 2024. Published: December 31, 2024.

1 Introduction

Let X,Y be two sequence spaces and $A=(a_{nk})$ be a matrix with real or complex entries. Throughout this paper, we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. If for each $x=(x_k)\in X$ the series

$$A_n x := \sum_k a_{nk} x_k$$

converge and the sequence $Ax=(A_n x)$ belongs to Y, we say that *A* transforms *X* into *Y*. By (X,Y) we denote the set of all matrices, which transform *X* into *Y*. Let ω be the set of all real or complex-valued sequences. Further, we need the following well-known sub-spaces of ω: c - the space of all convergent sequences, c_0 - the space of all sequences converging to zero, l_∞ - the space of all bounded sequences, and

$$l_1 := \left\{ x=(x_k) : \sum_n |x_n| < \infty \right\}.$$

For estimation and comparison of speeds of convergence of sequences are used different methods, [1], [2], [3], [4], [5], [6], [7]. We use the method, introduced in [1] and [6]. Let $\lambda := (\lambda_k)$ be a positive (i.e., $\lambda_k > 0$ for every *k*) monotonically increasing sequence. Following [1] and [6], a convergent sequence $x=(x_k)$ with

$$\lim_k x_k := s \text{ and } v_k := \lambda_k (x_k - s) \tag{1}$$

is called bounded with the speed λ (shortly, λ-bounded) if $v_k = O(1)$ (or $(v_k)\in l_\infty$), and convergent with the speed λ (shortly, λ-convergent) if the finite limit

$$\lim_k v_k := d$$

exists (or $(v_k)\in c$). Following the authors of the current paper ([8], [9]), a convergent sequence $x=(x_k)$ with the finite limit *s* is called absolutely convergent with speed λ (or shortly, absolutely λ-convergent), if $(v_k)\in l_1$. We denote the set of all λ-bounded sequences by l_∞^λ, the set of all λ-convergent sequences by c^λ, and the set of all absolutely λ-convergent sequences by l_1^λ. Moreover, let

$$c_0^\lambda := \left\{ x=(x_k) : x \in c^\lambda \text{ and } \lim_k \lambda_k (x_k - s) = 0 \right\}.$$

It is not difficult to see that:

$$l_\alpha^\lambda \subset c_0^\lambda \subset c^\lambda \subset l_\infty^\lambda \subset c.$$

In addition to it, for unbounded sequence λ these inclusions are strict. For $\lambda_k = O(1)$, we get $d = 0$, and hence

$$c^\lambda = l_\infty^\lambda = c.$$

Therefore, the case $\lambda_k \neq O(1)$ is most important, since in this case relation (1) allows to evaluate the quality of convergence of converging sequences. Indeed, let x^1 and x^2 be two convergent sequences with the finite limit s. If $(v_k) \in c$ (or $v_k = O(1)$) for $x = x^1$, and $(v_k) \notin c$ (or $v_k \neq O(1)$)` for $x = x^2$, then the sequence x^1 converges faster than sequence x^2. Thus, we can say that λ, in the case $\lambda_k \neq O(1)$, accelerates the speed of convergence of x^1.

Matrix transformations, and boundedness and convergence with speed are widely used in approximation theory to transform non-convergent sequences into convergent ones, or to transform convergent sequences into faster convergent sequences, [5], [10], [11], [12]. Besides, in [1] matrix transformations and boundedness with speed are used for the estimation of the order of approximation of Fourier expansions in Banach spaces by one author of the present paper. An interesting task would also be to study the convergence acceleration in wavelet and spline approximation, such as, for example in works [13], [14], [15] and [16], and for numerical solving of integral equations, [17], [18], [19].

In general, the problems of improvement of the quality of convergence of sequences by matrix transformations have been studied by several authors for example, [1], [20], [21], [22], [23], [24], [25], [26] and [27]. Moreover, in [26] and [27], the λ-convergence and the λ-boundedness in abstract spaces, considering instead of a matrix with real or complex entries a matrix, whose elements are bounded linear operators from a Banach space X into a Banach space Y, have been studied. We note that the results connected with convergence, absolute convergence, and boundedness with speed can be used in several applications. For example, in theoretical physics, such results can be used for accelerating the slowly convergent processes, a good overview of such investigations can be found, for example, from the sources [28] and [29].

Let $p = (p_k)$ be a sequence of strictly positive numbers, and let

$$c_0(p) := \left\{ x = (x_k) : \lim_k |x_k|^{p_k} = 0 \right\},$$

$$l_\infty(p) := \left\{ x = (x_k) : |x_k|^{p_k} = O(1) \right\},$$

$$c(p) := \left\{ x = (x_k) : \lim_k |x_k - l|^{p_k} = 0 \text{ for some } l \in C \right\},$$

$$l(p) := \left\{ x = (x_k) : \sum_k |x_k|^{p_k} < \infty \right\}.$$

The sets $c_0(p)$, $l_\infty(p)$, $c(p)$ and $l(p)$ are known as Maddox spaces, [30], [31] and [32]. These sets are also linear paranormal spaces if $p \in l_\infty$, [32]. Therefore, throughout the paper we assume that $p \in l_\infty$. If $0 < \inf p_k \leq \sup p_k < \infty$, then, [33].

$$c_0(p) = c_0, \quad c(p) = c, \quad l_\infty(p) = l_\infty.$$

In [9], we extended the concepts of boundedness, convergence and absolute convergence with speed to the Maddox spaces. Now we recall these concepts. A convergent sequence $x = (x_k)$ with the finite limit s is called paranormally bounded (paranormally convergent or paranormally zero-convergent or paranormally absolutely convergent) with speed λ with respect to p (shortly, paranormally λ-bounded, paranormally λ-convergent, paranormally λ-zero-convergent or paranormally absolutely λ-convergent with respect to p), if $(v_k) \in l_\infty(p)$, $(v_k) \in c(p)$, $(v_k) \in c_0(p)$ or $(v_k) \in l(p)$, respectively). Let

$$(c_0(p))^\lambda := \{ x = (x_k) : (v_k) \in c_0(p) \},$$

$$(l_\infty(p))^\lambda := \{ x = (x_k) : (v_k) \in l_\infty(p) \},$$

$$(c(p))^\lambda := \{ x = (x_k) : (v_k) \in c(p), \},$$

$$(l(p))^\lambda := \{ x = (x_k) : (v_k) \in l(p) \}.$$

The sets $(c_0(p))^\lambda$, $(l_\infty(p))^\lambda$, $(c(p))^\lambda$ and $(l(p))^\lambda$ we call as speed-Maddox spaces. These spaces are the subsets of certain Maddox spaces. First, we notice that

$$l(p) \subset c_0(p) \subset c(p) \subset l_\infty(p)$$

and $l_\infty \subset l_\infty(p)$ (see [29], Remark 2.8 and Corollary 2.11). Hence $(c_0(p))^\lambda$, $(l_\infty(p))^\lambda$, $(c(p))^\lambda$ and $(l(p))^\lambda$ are the subsets of $l_\infty(p)$, since $(c_0(p))^\lambda$, $(l_\infty(p))^\lambda$, $(c(p))^\lambda$ are the subsets of l_∞ by definition. Also, it is easy to see that $(c_0(p))^\lambda \subset c(p)$.

The paranormal boundedness, and the paranormal convergence with speed provide an additional tool for evaluating the quality of convergence of converging sequence. For example, if x^1 and x^2 be two convergent sequences belonging c^{λ} and $x^1 \in (c(p))^{\lambda}$, but $x^2 \notin (c(p))^{\lambda}$, then converge faster than x^2.

Matrix transforms between different Maddox spaces are studied in several works, [30], [31], [32], [33] and [34]. In the present paper we extend these studies investigating matrix transforms between speed-Maddox spaces. Let $\mu := (\mu_k)$ be another speed of convergence, i.e., a monotonically increasing positive sequence, and $q := (q_k)$ another bounded sequence of strictly positive numbers. Matrix transforms from $c_0^{\lambda}, c^{\lambda}$ and l_{∞}^{λ} into $(c_0(q))^{\mu}$, $(c(q))^{\mu}$ or $(l_{\infty}(q))^{\mu}$ have been characterized in [9]. The present paper is the continuation of [9]. We find necessary and sufficient conditions for a matrix A would transform $(l(p))^{\lambda}$ into $(c_0(q))^{\mu}$, $(c(q))^{\mu}$, $(l_{\infty}(q))^{\mu}$ or $(l(q))^{\mu}$, and $(c_0(p))^{\lambda}$, $(c(p))^{\lambda}$, $(l_{\infty}(p))^{\lambda}$ into $(l(q))^{\mu}$. The possibilities of using the main results of the present paper we demonstrate with two examples, where A is the Zweier matrix $Z_{1/2}$ and λ, μ, p, q are suitably selected sequences.

2 Auxiliary Results

For the proof of main results, we need some auxiliary results. For presenting these results throughout this section by $B = (b_{nk})$ and $C = (c_{nk})$ we denote arbitrary matrices with real or complex entries, and by $p := (p_k)$, $q := (q_k)$ bounded sequences with $p_k > 0$, $q_k > 0$ for every k.

Lemma 1 (Corollary in [34]). *A matrix* $B \in (l_{\infty}(p), c)$ *if and only if*

$$\sum_k |b_{nk}| M^{1/p_k} \text{ converges uniformly in } n \text{ for all integers } M > 1, \tag{2}$$

$$\text{there exists finite limits } \lim_n b_{nk} := b_k. \tag{3}$$

Moreover,

$$\lim_n B_n x = \sum_k{}' b_k x_k \tag{4}$$

for every $x = (x_k) \in l_{\infty}(p)$.

Lemma 2 ([33], Corollary 2). $B \in (c_0(p), c)$ *if and only if condition* (3) *holds and*

$$\sum_k |b_{nk}| M^{-1/p_k} = O(1) \text{ for some } M > 1. \tag{5}$$

Moreover, the equation (4) *holds for every* $x = (x_k) \in c_0(p)$.

Lemma 3. $B \in (c(p), c)$ *if and only if conditions* (3), (5) *hold, and*

$$\text{there exists } b \text{ with } \lim_n \sum_k{}' b_{nk} := b. \tag{6}$$

Moreover,

$$\lim_n B_n x = lb + \sum_k{}' (x_k - l) b_k$$

for every $x = (x_k) \in c(p)$, *where* $\lim_k x_k = l$.

Proof. We note that the condition

$$\lim_k |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathrm{C}$$

in the definition of $c(p)$ is equivalent to $(x_k - l) \in c_0(p)$. This implies $\lim_k x_k = l$, since $c_0(p) \subset c_0$ for $p \subset l_{\infty}$ (see [32], Corollary 2.11).

Further proof follows from Theorem 9 in [33].

To present the following lemmas, we denote

$$\mathrm{K}_1 := \{k \in \mathrm{N} : 0 < p_k \le 1\},$$

$$\mathrm{K}_2 := \{k \in \mathrm{N} : p_k > 1\},$$

and by K we denote an arbitrary finite subset of N.

Lemma 4 ([34], Theorem 1 and Corollary in p. 100-101). $B \in (l(p), c)$ *if and only if condition* (3) *holds and*

$$\sup_n \sup_{k \in \mathrm{K}_1} |b_{nk}|^{p_k} < \infty, \text{ for every } k, \tag{7}$$

$$\sup_n \sum_{k \in \mathrm{K}_2} |b_{nk}|^{\frac{p_k}{p_k - 1}} M^{\frac{p_k}{1 - p_k}} < \infty \text{ for some } M > 1. \tag{8}$$

Moreover, (4) *holds for every* $x = (x_k) \in l(p)$.

The following Lemmas 5 - 10 are presented in [30] and [32].

Lemma 5. $C \in (l(p), c_0(q))$ *if and only if*

$$\lim_n |c_{nk}|^{q_n} = 0 \text{ for every } k, \tag{9}$$

$$\sup_n \sup_{k \in \mathrm{K}_1} \left| c_{nk} L^{1/q_n} \right|^{p_k} < \infty \text{ for every } L > 0 \tag{10}$$

$$\text{for every } L > 0 \text{ there exists } M > 0, \text{ such that } \sup_n \sum_{k \in \mathrm{K}_2} \left| c_{nk} L^{1/q_n} M^{-1} \right|^{p_k/(p_k - 1)} < \infty. \tag{11}$$

Lemma 6. $C \in (l(p), c(q))$ *if and only if*

$$\sup_n \sup_{k \in \mathrm{K}_1} |c_{nk}|^{p_k} < \infty, \text{ for every } k, \tag{12}$$

$$\sup_n \sum_{k \in \mathrm{K}_2} \left| c_{nk} M^{-1} \right|^{p_k/(p_k - 1)} < \infty \text{ for some } M > 0, \tag{13}$$

and there exists a sequence (c_k) *such that*

$$\lim_n |c_{nk} - c_k|^{q_n} = 0 \text{ for every } k, \quad (14)$$

$$\sup_n \sup_{k \in K_1} \left(|c_{nk} - c_k| L^{1/q_n} \right)^{p_k} < \infty \text{ for all } L > 0 \quad (15)$$

for every $L > 0$ *there exists* $M > 0$, *such that*

$$\sup_n \sum_{k \in K_2} \left(|c_{nk} - c_k| L^{1/q_n} M^{-1} \right)^{p_k/(p_k - 1)} < \infty. \quad (16)$$

Lemma 7. $C \in (l(p), l_\infty(q))$ *if and only if*

$$\sup_n \sup_{k \in K_1} \left| c_{nk} L^{-1/q_n} \right|^{p_k} < \infty \text{ for some } L > 0 \quad (17)$$

$$\sup_n \sum_{k \in K_2} \left| c_{nk} L^{-1/q_n} \right|^{p_k/(p_k - 1)} < \infty \text{ for some } L > 0. \quad (18)$$

Lemma 8. *Let* $0 < p_k \le 1$ *and* $1 \le q_k \le H$ *for every k for some positive number H. Then* $C \in (l(p), l(q))$ *if and only if*

$$\sup_k \sum_n \left| c_{nk} L^{-1/p_k} \right|^{g_n} < \infty \text{ for some } L > 0. \quad (19)$$

Lemma 9. *Let* $1 \le q_k \le H$ *for every k for some positive number H. Then* $C \in (c_0(p), l(q))$ *if and only if*

$$\sup_K \sum_n \left| \sum_{k \in K} c_{nk} L^{-1/p_k} \right|^{g_n} < \infty \text{ for some } L > 0. \quad (20)$$

Lemma 10. *Let* $1 \le q_k \le H$ *for every k for some positive number H. Then* $C \in (l_\infty(p), l(q))$ *if and only if*

$$\sup_K \sum_n \left| \sum_{k \in K} c_{nk} L^{1/p_k} \right|^{g_n} < \infty \text{ for every } L > 0. \quad (21)$$

3 Matrix Transforms from $(l(p))^\lambda$ Into $(c_0(q))^\mu$, $(c(q))^\mu$, $(l_\infty(q))^\mu$, $(l(q))^\mu$

Let further throughout the paper $p := (p_k)$, $q := (q_k)$ be bounded sequences with $p_k > 0$, $q_k > 0$ for every k, and $B = (b_{nk})$, $C = (c_{nk})$ matrices defined by

$$b_{nk} := \frac{a_{nk}}{\lambda_k}$$

and

$$c_{nk} := \frac{\mu_n (a_{nk} - a_k)}{\lambda_k},$$

provided that

there exists finite limits $\lim_m a_{nk} := a_k$. (22)

Also, we need the sequences

$e := (1,1,...)$ and $e^k := (1,0,...,1,0,...)$,

where 1 is in the position number k. We note that.

$$e, e^k \in (l(p))^\lambda.$$

Before proving the main results, we prove the following lemma.

Lemma 11. *For each* $v := (v_k) \in l(p)$ *there exists a sequence* $x := (x_k) \in (l(p))^\lambda$, *such that relation* (1) *holds.*

Proof. Let $v \in l(p)$. As in this case $l(p) \subset c_0(p)$ and $c_0(p) \subset c_0$, then

$$\lim_k \frac{v_k}{\lambda_k} = 0. \quad (23)$$

Denoting

$$x_k := \frac{v_k}{\lambda_k} + s \quad (24)$$

for some $s \in \mathbb{C}$, we obtain that relation (1) holds with $x := (x_k) \in (l(p))^\lambda$.

Theorem 1. $A = (a_{nk}) \in ((l(p))^\lambda, (c_0(q))^\mu)$ *if and only if conditions* (7) – (11), (22) *are satisfied, and*

$$Ae = (\tau_n) \in (c_0(q))^\mu, \tau_n := A_n e = \sum_k a_{nk}. \quad (25)$$

Proof. Necessity. Assume that $((l(p))^\lambda, (c_0(q))^\mu)$. As $e^k, e \in (l(p))^\lambda$, then conditions (22) and (25) hold. Besides, (22) (22) implies the validity of (3). Since, from (1) we get (24), where $s = \lim_k x_k$ and $(v_k) \in l(p)$ for every $x = (x_k) \in (l(p))^\lambda$, it follows that

$$A_n x = \sum_k{}' b_{nk} v_k + s\tau_n. \quad (26)$$

As $(\tau_n) \in (c_0(q))^\mu$ by (25), then the finite limit

$$\tau := \lim_n \tau_n \quad (27)$$

exists. Hence, from (26) we obtain that the matrix B transforms this sequence $(v_k) \in l(p)$ into c. Thus, by Lemma 11, $B \in (l(p), c)$. This implies, by Lemma 4, that conditions (7) and (8) are satisfied, and the finite limit

$$\phi := \lim_n A_n x = \sum_k{}' b_k v_k + s\tau$$

exists for every $x \in (l(p))^\lambda$. Writing

$$\mu_n (A_n x - \phi) = \mu_n \sum_k{}' c_{nk} v_k + \mu_n (\tau_n - \tau) s, \quad (28)$$

we conclude, by (25) that $C \in (l(p), c_0(q))$. Hence conditions (9) – (11) are satisfied by Lemma 5.

Sufficiency. Assume that all conditions of Theorem 1 are satisfied. Then relation (26) also holds for every $x\in(l(p))^{\lambda}$ and $(\tau_n)\in(c_0(q))^{\mu}$ by (25). (16), In addition, $B\in(l(p),c)$ and the finite limit ϕ exists for every $x\in(l(p))^{\lambda}$ by Lemma 4, since (7) and (8) hold and (3) is satisfied due to (22). Hence relation (28) holds for every $x\in(l(p))^{\lambda}$. As (9) - (11) are valid, then $C\in(l(p),c_0(q))$ by Lemma 5. Therefore, due to (25), $A\in((l(p))^{\lambda},(c_0(q))^{\mu})$.

Theorem 2. $A=(a_{nk})\in((l(p))^{\lambda},(c(q))^{\mu})$ *if and only if conditions* (7), (8), (12), (13), (22) *hold, there exists a sequence* (c_k) *such that conditions* (14) - (16) *hold, and*

$$Ae=(\tau_n)\in(c(q))^{\mu},\ \tau_n:=A_ne=\sum_k a_{nk}. \qquad (29)$$

Proof is similar to the proof of Theorem 1. The only difference is that now instead of Lemma 5 we use Lemma 6.

Theorem 3. $A=(a_{nk})\in((l(p))^{\lambda},(l_\infty(q))^{\mu})$ *if and only if conditions* (7), (8), (17), (18), (22) *hold, and*

$$Ae=(\tau_n)\in(l_\infty(q))^{\mu},\ \tau_n=A_ne=\sum_k a_{nk}. \qquad (30)$$

Proof is similar to the proof of Theorem 1. The only difference is that now instead of Lemma 5 we use Lemma 7.

Theorem 4. *Let* $0<p_k\le 1$ *and* $1\le q_k\le H$ *for every k for some positive number H. Then* $A=(a_{nk})\in((l(p))^{\lambda},(l(q))^{\mu})$ *if and only if condition* (7), (19), (22) *hold, and*

$$Ae=(\tau_n)\in(l(q))^{\mu},\ \tau_n=A_ne=\sum_k a_{nk}. \qquad (31)$$

Proof is similar to the proof of Theorem 1. The only difference is that now instead of Lemma 5 we use Lemma 8.

We now give an example of Theorems 1–4 when A is a Zweier matrix, i.e., $A=Z_{1/2}=(a_{nk})$, where (see [1], p.3) $a_{00}=1/2$ and

$$a_{nk}=\begin{cases}1/2, \text{if } k=n-1 \text{ or } k=n,\\ 0, \text{ if } k<n-1 \text{ or } k>n\end{cases}$$

for $n\ge 1$.

Example 1. Let $\lambda=(\lambda_k)$, $\mu=(\mu_k)$, $p=(p_k)$ and $q=(q_k)$ be defined as follows:

$$\lambda_k:=(k+1)^{1+\alpha}, \alpha>0,\ \mu_k:=k+1, \qquad (32)$$

$$p_k:=\frac{1}{k+2},\ q_k:=r+\frac{1}{k+1}, r>0. \qquad (33)$$

We show that then

$$Z_{1/2}:(l(p))^{\lambda}\to(c_0(q))^{\mu}\cap(c(q))^{\mu}\cap(l_\infty(q))^{\mu}\cap(l(q))^{\mu}.$$

As in this case $K_1=N$, it is sufficient to prove that all conditions of Theorems 1 – 4 connected with K_1 are satisfied.

First, we see that condition (22) holds with $a_k\equiv 0$. Also conditions (25) and (29) - (31) hold, since

$$Ae=(1/2,1,1,...)$$

with limit 1.

For $B=(b_{nk})$ and $C=(c_{nk})$ we obtain $b_{00}=c_{00}=1/2$, and

$$b_{nk}=\begin{cases}\dfrac{1}{2(n+1)^{1+\alpha}}, \text{if } k=n,\\ \dfrac{1}{2n^{1+\alpha}}, \quad \text{if } k=n-1,\\ 0, \quad \text{if } k<n-1 \text{ or } k>n,\end{cases} \qquad (34)$$

$$c_{nk}=\begin{cases}\dfrac{1}{2(n+1)^{\alpha}}, \quad \text{if } k=n,\\ \dfrac{1}{2}\left(1+\dfrac{1}{n}\right)\dfrac{1}{n^{\alpha}}, \text{if } k=n-1,\\ 0, \quad \text{if } k<n-1 \text{ or } k>n,\end{cases} \qquad (35)$$

for $n\ge 1$. As $\lim_n c_{nk}=0$ for every k, then we can take $c_k\equiv 0$, and so conditions (9) and (14) hold.

Let us denote

$$R_{nk}:=\left|b_{nk}\right|^{p_k},\ S_{nk}:=\left|c_{nk}\right|^{p_k},\ T_{nk}(L):=\left|c_{nk}L^{1/q_n}\right|^{p_k},$$
$$U_{nk}(L):=\left|c_{nk}L^{-1/q_n}\right|^{p_k},\ V_{nk}(L):=\left|c_{nk}L^{-1/p_k}\right|^{q_n}$$

for $L>0$. Then R_{00}, S_{00}, $T_{00}(L)$, $U_{00}(L)$, $V_{00}(L)$ are constants for every $L>0$, and for $n\ge 1$ we obtain:

$$R_{nk}=S_{nk}=T_{nk}(L)=U_{nk}(L)=V_{nk}(L)=0,$$

if $k<n-1$ or $k>n$ for every $L>0$. Also

$$R_{nn}=2^{-\frac{1}{n+2}}(n+1)^{-\frac{1+\alpha}{n+2}},\ R_{n,n-1}=2^{-\frac{1}{n+1}}n^{-\frac{1+\alpha}{n+2}},$$

$$T_{nn}(L)=2^{-\frac{1}{n+2}}(n+1)^{-\frac{\alpha}{n+2}}L^{\frac{n+1}{(n+2)(r(n+1)+1)}},$$

$$T_{n,n-1}(L)=2^{-\frac{1}{n+1}}\left(1+\frac{1}{n}\right)^{-\frac{1}{n+1}}L^{\frac{1}{r(n+1)+1}}$$

for $n\ge 1$, from which we obtain with the help of L'Hôpital's rule that

$$\lim_n R_{nn}=\lim_n R_{n,n-1}=\lim_n T_{nn}(L)=\lim_n T_{n,n-1}(L)=1$$

for every $L>0$. Similarly, using L'Hôpital's rule, it is possible to show that

$$\lim_n S_{nn} = \lim_n S_{n,n-1} = \lim_n U_{nn}(L) = \lim_n U_{n,n-1}(L) = 1$$

and

$$\lim_n V_{nn}(L) = \lim_n V_{n,n-1}(L) = 0$$

for every $L > 0$. Therefore conditions (7), (9), (12), (15) and (17) are satisfied. Thus, all conditions of Theorems 1 – 4 connected with K_1 hold.

4 Matrix Transforms from $(l_\infty(p))^\lambda$, $(c(p))^\lambda$ and $(c_0(p))^\lambda$ into $(l(q))^\mu$

Before proving the main results of this section, we prove two preliminary lemmas.

Lemma 12. *If*

$$\lim_k \lambda_k^{p_k} = \infty \qquad (36)$$

then for each $v := (v_k) \in l_\infty(p)$ *there exists a sequence* $x := (x_k) \in (l_\infty(p))^\lambda$, *such that relation* (1) *holds.*

Proof. If $v \in l_\infty(p)$, then $|v_k|^{p_k} = O(1)$, and

$$\lim_k \left|\frac{v_k}{\lambda_k}\right|^{p_k} = 0$$

by (36). Hence (23) holds. Defining now $x = (x_k)$ by (24) for some $s \in \mathbb{C}$, we obtain that relation (1) holds with $x \in (l_\infty(p))^\lambda$.

Lemma 13. *For every* $v := (v_k) \in c_0(p)$ ($v \in c(p)$) *there exists* $x \in (c_0(p))^\lambda$ ($x \in (c(p))^\lambda$, *respectively*), *such that relation* (1) *holds.*

Proof. 1. Let $v \in c_0(p)$. As in this case $c_0(p) \subset c_0$ (see [29], Corollary 2.11), then $v \in c_0$, and hence relation (23) holds, since $\lambda_k \ge \lambda_0 > 0$ for every *k*. Defining now $x = (x_k)$ by (24) for some $s \in \mathbb{C}$, we obtain that relation (1) holds with $x \in (c_0(p))^\lambda$.

2. Let $v \in c(p)$. Then

$$\lim_k |v_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{C} \qquad (37)$$

by definition. It follows from (37) that $\lim_k v_k = l$, i.e., $v \in c$. But every $v \in c$ with limit l can be represented in the form

$$v = v^0 + le, \ v^0 := (v_k^0) \in c_0.$$

From (37) we obtain that $\lim_k |v_k^0|^{p_k} = 0$, i.e., $v^0 \in c_0(p)$. Further proof is similar to the proof in point 1.

Theorem 5. *Let condition* (36) *be satisfied, and* $1 \le q_k \le H$ *for every k for some positive number H. Then* $A = (a_{nk}) \in ((l_\infty(p))^\lambda, (l(q))^\mu)$ *if and only if conditions* (2), (21), (22) *and* (31) *hold.*

Proof is similar to the proof of Theorem 1. The only difference is that now instead of Lemmas 4 and 5 we use correspondingly Lemma 1 and 10.

Theorem 6. *Let* $1 \le q_k \le H$ *for every k for some positive number H. Then* $A = (a_{nk}) \in ((c(p))^\lambda, (l(q))^\mu)$ *if and only if conditions* (5), (6), (20), (22), (31) *hold, and*

$$A\lambda^{-1} \in (l(q))^\mu, \ \lambda^{-1} := \left(\frac{1}{\lambda_k}\right). \qquad (38)$$

Proof. Necessity. Assume that $A \in ((c(p))^\lambda, (l(q))^\mu)$. Similar to the proof of Theorem 1, we obtain that relation (26) holds for every $x \in (c(p))^\lambda$. As $e^k, e, \lambda^{-1} \in (c(p))^\lambda$, then conditions (22), (31) and (38) hold. Moreover, (31) implies the validity of (27). Hence, from (26) it follows that *B* transforms this sequence $(v_k) \in c(p)$ into *c*. Thus, by Lemma 13, $B \in (c(p), c)$. Consequently, by Lemma 3, conditions (5), (6) are satisfied, and for every $x \in (c(p))^\lambda$ the finite limit

$$\psi := \lim_n A_n x = bd + \sum_k b_k(v_k - d) + s\tau \qquad (39)$$

exists, where $d := \lim_k v_k$. We note here that condition (3) holds by (22). Now, using (26) and (39), we get

$$\begin{aligned} \mu_n(A_n x - \psi) &= \mu_n \sum_k c_{nk}(v_k - d) + \\ &\mu_n(\tau_n - \tau)s + \mu_n\left(\sum_k b_{nk} - b\right)d \end{aligned} \qquad (40)$$

for every $x \in (c(p))^\lambda$. As $Ae \in (l(q))^\mu$ and $A\lambda^{-1} \in (l(q))^\mu$ correspondingly by (31) and (38), then $C \in (c_0(p), l(q))$. Therefore condition (20) holds by Lemma 9.

Sufficiency. Assume that all conditions of Theorem 6 are satisfied. Then relation (26) also holds for every $x \in (c(p))^\lambda$ and $(\tau_n) \in (l(q))^\mu$ by (31). Hence, $B \in (c(p), c)$ and the finite limit ψ exists for every $x \in (c(p))^\lambda$ by Lemma 3. This implies that relation (40) holds for every $x \in (c(p))^\lambda$. As condition (20) holds, then $C \in (c_0(p), l(q))$. by Lemma 9. Thus, due to (31) and (38), $A \in ((c(p))^\lambda, (l(q))^\mu)$.

Theorem 7. *Let* $1 \le q_k \le H$ *for every k for some positive number H. Then* $A = (a_{nk}) \in ((c_0(p))^\lambda, (l(q))^\mu)$ *if and only if conditions* (5), (20), (22) *and* (31) *hold.*

Proof is similar to the proof of Theorem 6. We only note that in this case $\lambda^{-1} \notin (c_0(p))^\lambda$, and instead of Lemma 3 we use Lemma 2.

Example 2. Let $\lambda=(\lambda_k)$, $\mu=(\mu_k)$ and $q=(q_k)$ be defined as in (32) and (33), and $p=(p_k)$ by

$$p_k := t+\frac{1}{k+2},\ t>1.$$

We prove that then

$$Z_{1/2} : (l_\infty(p))^\lambda \cup (c(p))^\lambda \cup (c_0(p))^\lambda \to (l(q))^\mu,$$

if $\alpha>1$ and $r>1$. For this purpose it is sufficient to show that all conditions of Theorems 5 – 7 are satisfied.

First, we note that conditions (22) and (31) hold (see Example 1) Also condition (36) holds because $\inf p_k > t > 0$. Denoting

$$W_n(M) := \sum_k |b_{nk}| M^{1/p_k},$$

we can write $W_0(M)=(1/2)\cdot M^{2/(2t+1)}$ and

$$W_n(M)=\frac{1}{2(n+1)^{1+\alpha}} M^{\frac{n+2}{t(n+2)+1}} + \frac{1}{2}\left(1+\frac{1}{n}\right)\frac{1}{n^\alpha} M^{\frac{n+1}{t(n+1)+1}}$$

for $n>1$ by (34). Hence $\lim_n W_n(M)=0$ for every $M>0;$ so condition (2) holds. Denoting

$$D_n(M) := \sum_k |b_{nk}| M^{-1/p_k},$$

we, using (34), obtain $D_0(M)=(1/2)\cdot M^{-2/(2t+1)}$ and

$$D_n(M)=\frac{1}{2(n+1)^{1+\alpha}} M^{-\frac{n+2}{t(n+2)+1}} + \frac{1}{2}\left(1+\frac{1}{n}\right)\frac{1}{n^\alpha} M^{-\frac{n+1}{t(n+1)+1}}$$

for $n>1$. This implies $\lim_n D_n(M)=0;$ for every $M>0;$ thus condition (5) holds.

As

$$A_n\lambda^{-1} = \sum_k b_{nk} = \frac{1}{2}\left(\frac{1}{(n+1)^{1+\alpha}} + \frac{1}{n^{1+\alpha}}\right),$$

Then $\lim_n A_n\lambda^{-1}=0,$ i.e., condition (6) holds. Moreover,

$$\sum_n \left|\mu_n\left(A_n\lambda^{-1} - \lim_n A_n\lambda^{-1}\right)\right|^{q_n} =$$

$$\frac{1}{2^{r+1}} + \sum_{n=1}^{\infty}\left|\frac{1}{2}\left(\frac{1}{(n+1)^\alpha} + \left(1+\frac{1}{n}\right)\frac{1}{n^\alpha}\right)\right|^{\frac{r(n+1)+1}{n+1}} \le$$

$$\frac{1}{2^{r+1}} + \sum_{n=1}^{\infty}\left|\left(\left(1+\frac{1}{n}\right)\frac{1}{n^\alpha}\right)\right|^{\frac{r(n+1)+1}{n+1}} =$$

$$\frac{1}{2^{r+1}} + O(1)\sum_{n=1}^{\infty}\left(\frac{1}{n^\alpha}\right)^{\frac{r(n+1)+1}{n+1}} = \frac{1}{2^{r+1}} + O(1)\sum_{n=1}^{\infty}\frac{1}{n^\alpha} < \infty$$

for $\alpha>1$ and $r>1$. Thus, condition (38) holds.

As, using (35), we obtain:

$$\sup_K \sum_n \left|\sum_{k\in K} c_{nk} L^{-1/p_k}\right|^{g_n} \le \sum_n \left|\sum_k c_{nk} L^{-1/p_k}\right|^{g_n} = \frac{1}{2} L^{-(r+1)} +$$

$$\sum_{n=1}^{\infty}\left|\frac{1}{2}\left(\frac{1}{(n+1)^\alpha} L^{-\frac{t(n+2)+1}{n+2}} + \left(1+\frac{1}{n}\right)\frac{1}{n^\alpha} L^{-\frac{t(n+1)+1}{n+1}}\right)\right|^{\frac{r(n+1)+1}{n+1}} \le$$

$$\frac{1}{2} L^{-(r+1)} + \sum_{n=1}^{\infty}\left(\left(1+\frac{1}{n}\right)\frac{1}{n^\alpha} L^{-\frac{t(n+2)+1}{n+2}}\right)^{\frac{r(n+1)+1}{n+1}} =$$

$$\frac{1}{2} L^{-(r+1)} + O(1)\sum_{n=1}^{\infty}\left(\frac{1}{n^\alpha}\right)^{\frac{r(n+1)+1}{n+1}} =$$

$$\frac{1}{2} L^{-(r+1)} + O(1)\sum_{n=1}^{\infty}\frac{1}{n^\alpha} < \infty$$

for $\alpha>1$ and $r>1$, then condition (20) holds. The validity of condition (21) can be proved similarly to (20). Thus, all conditions of Theorems 5 – 7 are satisfied.

5 Conclusion

In [9] we started to study the matrix transforms connected with the subspaces of Maddox spaces defined by speeds of convergence, i.e., by monotonically increasing strictly positive sequences. In this paper, we continued these studies. Let (X,Y) be the set of matrices (with real or complex entries) to map a sequence space *X* into another sequence space *Y*. We characterized the matrix classes (X,Y), where at least one of the spaces *X*, *Y* is the set of paranormally absolutely convergent sequences with speed, i.e., the subset of certain Maddox space, and another space is the subset of another Maddox space defined by another speed. These Maddox spaces are defined here by bounded strictly positive sequences *p* and *q*.

As an application, we presented two examples, where we proved that the Zweier matrix satisfies all conditions of the main theorems of the paper for certain speeds and certain sequences *p* and *q*.

In the future, we intend to extend our results to the case where we consider matrix transformations in abstract spaces, for example, in the ultrametric space. We note that the mentioned two examples for the Zweier matrix have mainly theoretical character. Therefore, in the future, we wish to study the possibilities of applying the findings of the present paper and [9] for solving problems like the estimation of the order of Fourier expansion, numerical solving of integral equations, etc.

References:

[1] A. Aasma, H. Dutta and P. N. Natarajan, *An Introductory Course in Summability Theory*, John Wiley and Sons, 2017, [Online]. https://onlinelibrary.wiley.com/doi/book/10.1002/9781119397786 (Accessed Date: September 10, 2025).

[2] P. Amore, Convergence acceleration of series through a variational approach, *J. Math. Anal. Appl.*, Vol.323, No.1, 2006, pp. 63-77. https://doi.org/10.1016/j.jmaa.2005.09.091.

[3] C. Brezinski and M. Redivo-Zaglia, *Extrapolation and rational approximation—the works of the main contributors*, Springer, 2020. https://doi.org/10.1007/978-3-030-58418-4.

[4] C. Brezinski, Convergence acceleration during the 20th century, *J. Comput. Appl. Math.*, Vol. 122, No.1-2, 2000, pp. 1-21. https://doi.org/10.1016/S0377-0427(00)00360-5.

[5] J.P. Delahaye, *Sequence Transformations*, Springer, 1988.

[6] G. Kangro, Summability factors for the series λ-bounded by the methods of Riesz and Cesàro (Množiteli summirujemosti dlya ryadov, λ-ogranitšennõh metodami Rica i Cezaro), *Acta Comment. Univ. Tartu. Math.*, No. 277, 1971, pp. 136-154. MR0399696, Zbl 0316.40013.

[7] A. Sidi, *Practical Extrapolation Methods.*, Cambridge monographs on applied and computational mathematics 10., *Cambridge University Press*, 2003. https://doi.org/10.1017/CBO9780511546815.

[8] A. Aasma, P. N. Natarajan, Absolute convergence with speed and matrix transforms, *TWMS J. App. and Eng. Math.*, Vol. 14, No. 3, 2024, pp. 1109-1120, [Online]. http://jaem.isikun.edu.tr/web/images/articles/vol.14.no.3/19.pdf (Accessed Date: September 10, 2025).

[9] P. N. Natarajan and A. Aasma, Matrix Transforms Into the Set of a-absolutely Convergent Sequences with Speed and the Regularity of Matrices on the Sub-spaces of C, *WSEAS Transactions on Mathematics*, Vol.23, 2024, pp. 60-67, https://doi.org/10.37394/23206.2024.23.7

[10] P.L.Butzer and R.I. Nessel, *Fourier analysis and approximation: one-dimensional theory*, Birkhäuser Verlag, 1971.

[11] A. Zygmund, *Trigonometric series, 3rd edition*, Cambridge Univ. Press, 2003.

[12] W. Trebels, *Multipliers for* (C,α)*-bounded Fourier expansions in Banach spaces and approximation theory,* Lecture Notes in Math., Vol. 329, Springer Verlag, 1973.

[13] I. G. Burova, Local Splines and the Least Squares Method, *WSEAS Transactions on Systems*, Vol.23, 2024, pp. 188-195. https://doi.org/10.37394/23202.2024.23.21

[14] A. Iftar, Robust Controller Design for Descriptor-type Time-delay Systems, *WSEAS Transactions on Systems*, Vol.20, 2021, pp. 289-294. https://doi.org/10.37394/23202.2021.20.32

[15] C. Abdelaali, A. Bouklata, M. Beniassi and A. Brouri, Identification of Linear Systems Having Time Delay Connected in Series, *WSEAS Transactions on Systems and Control*, Vol.19, 2024, pp. 234-239. https://doi.org/10.37394/23203.2024.19.25

[16] Y. Dem'yanovich and L. T. N. Bich, Discrete and Continuous Wavelet Expansions, *WSEAS Transactions on Mathematics*, Vol.21, 2022, pp. 475-488. https://doi.org/10.37394/23206.2022.21.9

[17] V. M. Ryabov and I. G. Burova, About the Tikhonov Regularization Method for the Solution of Incorrect Problems, *WSEAS Transactions on Systems*, Vol.22, 2023, pp. 656-662. https://doi.org/10.37394/23202.2023.22.66

[18] W. Peerajit, Gauss–Legendre Numerical Integrations for Average Run Length Running on EWMA Control Chart with Fractionally Integrated MAX Process, *WSEAS Transactions on Mathematics*, Vol.23, 2024, pp. 579-590. https://doi.org/10.37394/23206.2024.23.61

[19] D. Albayrak, F. Uçar and S. D. Purohit, Evaluating complex inverse formulas for *q*-Sumudu transforms, *WSEAS Transactions on Mathematics*, Vol.23, 2024, pp. 16-25.https://doi.org/10.37394/23206.2024.23.2

[20] S. Das and H. Dutta, Characterization of some matrix classes involving some sets with speed, *Miskolc Math. Notes*, Vol.19, No. 2, 2018, pp. 813–821. https://doi.org/10.18514/MMN.2018.2495

[21] I. Kornfeld, Nonexistence of universally accelerating linear summability methods, *J. Comput. Appl. Math.*, Vol.53, No.3, 1994, pp. 309-321. https://doi.org/10.1016/0377-0427(94)90059-0

[22] A. Šeletski and A. Tali, Comparison of speeds of convergence in Riesz-Type families of summability methods II, *Math. Model. Anal.*, Vol. 15, No.1, 2010, pp. 103-112. https://doi.org/10.3846/1392-6292.2010.15.103-112

[23] A. Šeletski and A. Tali, Comparison of speeds of convergence in Riesz-Type families of summability methods, *Proc. Estonian Acad. Sci. Phys. Math.*, Vol.57, No. 2, 2008, pp. 70-80. https://doi.org/10.3176/proc.2008.2.02

[24] U. Stadtmüller and A. Tali, Comparison of certain summability methods by speeds of convergence, *Anal. Math.*, Vol.29, No.3, 2003, pp. 227- 242. https://doi.org/10.1023/a:1025419305735

[25] O. Meronen and I. Tammeraid, Several theorems on λ-summable series, *Math. Model. Anal.*, Vol.15, No.1, 2010, pp. 97–102.

[26] I. Tammeraid, Generalized linear methods and convergence acceleration, *Math. Model. Anal.*, Vol.8, No.4, 2003, pp. 329-335.

[27] I. Tammeraid, Convergence acceleration and linear methods, *Math. Model. Anal.*, Vol.8, No.1, 2003, pp. 87-92. https://doi.org/10.3846/13926292.2003.9637213

[28] E. Caliceti, M. Meyer-Hermann, P. Ribeca, A. Surzhykov and U.D. Jentschura, From useful algorithms for slowly convergent series to physical predictions based on divergent perturbative expansions, *Physics Reports-Review Section of Physics Letters*, Vol. 446, No. 1-3, 2007, pp. 1-96. https://doi.org/10.1016/j.physrep.2007.03.003

[29] C.M. Bender, C. Heissenberg, *Convergent and Divergent Series in Physics. A short course by Carl Bender*, in: C. Heissenberg (Ed.), *Lectures of the 22nd "Saalburg" Summer School*, Saalburg, 2016, [Online]. https://saalburg.aei.mpg.de/wp-content/uploads/sites/25/2017/03/bender.pdf (Accessed Date: November 13, 2024).

[30] K-G. Grosse-Erdmann, Matrix transformations between the sequence spaces of Maddox, *J. Math. Anal. Appl.*, Vol. 180, 1993, pp. 223-238. https://doi.org/10.1006/jmaa.1993.1398.

[31] I.J. Maddox, Paranormed sequence spaces generated by infinite matrices, *Proc. Camb. Phil. Soc.*, Vol. 64, No. 2, 1968, pp. 335–340. https://doi.org/10.1017/S0305004100042894.

[32] E. Malkowsky and F. Başar, A Survey On Some Paranormed Sequence Spaces, *Filomat*, Vol. 31, No. 4, 2017, pp. 1099–1122. https://doi.org/10.2298/FIL1704099M.

[33] C.G. Lascarides, A study of certain sequence spaces of Maddox and generalization of a theorem of Iyer, *Pacific. J. Math.*, Vol. 38, No. 2, 1971, pp. 487-500, [Online]. https://projecteuclid.org/journals/pacific-journal-of-mathematics/volume-38/issue-2/A-study-of-certain-sequence-spaces-of-Maddox-and-a/pjm/1102970060.full (Accessed Date: November 13, 2024).

[34] C.G. Lascarides, I.J. Maddox, Matrix transformations between some classes of sequences, *Proc. Camb. Phil. Soc.*, Vol. 68, 1970, pp. 99-104. https://doi.org/10.1017/S0305004100001109.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The authors have no conflicts of interest to declare.