Beyond Acceleration: Exploring Higher-Order Time Derivatives of Position in Physical and Engineering Systems

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Abstract: - This article investigates the higher-order derivatives of position concerning time, including jerk, snap, and extending up to the tenth-order derivative. A mathematical model that accurately describes the motion of an object with high precision and smooth control is presented, along with an analysis of its series convergence. Additionally, the article explores the applications of these higher-order derivatives across various fields of physics. Key areas include advanced robotics and mechanical systems, vibration control, aerospace engineering, and seismology, where these derivatives are particularly relevant. Furthermore, the modeling of complex dynamic systems with high precision is discussed, providing extensive insights into the behavior of these systems.

Key-Words: - Higher time derivatives of position, jerk, snap, crackle, pop, jounce, hyper-jerk.

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1 Introduction

We are all familiar with the concepts of position (displacement), velocity, and acceleration. We experience velocity when we move and acceleration when we change our speed. However, when acceleration changes rapidly, we feel jerk and snap. These terms, jerk, and snap, along with crackle and pop, are less commonly understood and represent higher-order derivatives. Velocity, acceleration, jerk, snap, crackle, and pop are mathematically the first, second, third, fourth, fifth, and sixth derivatives of position with respect to time, respectively. There is no consensus on the names of higher-order derivatives. However, in this study, we use the more common terms “snap”, “crackle” and “pop” for the 4th–6th derivatives, named after the pictorial characters of three elves on Kellogg’s Rice Krispies cereal packages from the early 1930s. For the 7th–10th higher derivatives, the terms “lock”, “drop”, “shock”, and “put” have been used informally, as they are largely uncommon in the literature. Position and its various time derivatives define an ordered hierarchy of meaningful concepts.
In everyday contexts, the influences of velocity, acceleration, and perhaps some higher-order derivatives are noticeable in situations where there are sudden changes in motion. For instance, an experienced driver accelerates smoothly, while a novice driver might cause a jerky ride, resulting in a jerk and snap. Additionally, when lifting heavy weights during a deadlift, there is a noticeable jerk when the lifter initially pulls the weight off the ground, indicating a rapid change in acceleration. Similarly, uprooting plants involves a sudden change in force and acceleration, corresponding to a jerk. These examples concretely illustrate how jerk manifests in real-world actions. However, higher-order derivatives, including crackle and pop, are observed in many fields. In engineering and physics, they are evident during vibrations and transitions, especially when multi-resonant modes of vibration occur. For example, in mechanical engineering, they are seen when the cam-follower jumps off the camshaft in automotive systems. They are also observed in civil engineering when switching suddenly between train tracks and roads. It should be emphasized that sudden changes in forces, along with the resulting fluctuations in acceleration, jerk, and higher-order derivatives, can have significant negative impacts beyond the immediate effects of the forces themselves. Jerk and beyond can lead to fatigue cracks in metals and other materials, potentially resulting in structural failures, and can also cause injuries to humans and racing animals. This is why jerk and some higher derivatives are included in standards for limit states, such as those governing amusement rides and elevators. Notably, it has been demonstrated that humans innately attempt to minimize jerk when moving their limbs,

[1]. Additionally, experimental measurements of jerk have been linked to human movement impairment due to stroke, [2], [3], and fatigue, [4].

Despite the importance of higher time derivatives of position in various standards and applications across science and engineering, they are rarely discussed in university textbooks, making them relatively unfamiliar even among engineering professionals. However, these higher time derivatives are crucial for understanding the effects on motion, oscillations, and vibrations in a wide range of applications. Specifically, jerk and snap have numerous applications in motion control, manufacturing, and oscillators, [5], [6]. Higher derivatives such as snap, crackle, and pop have been discussed in a conference presentation, [7]. Additionally, several recent studies discussed these higher time derivatives. For instance, an insightful review of the application of jerk across various fields in science and engineering has been provided in [8]. The authors conducted a thorough systematic review of recent academic articles (2015–2020) and categorized them based on the application of jerk in twenty different categories. They also claimed that, although jerk is often overlooked in secondary and higher education, it is ubiquitous. This review has provided a solid foundation for future research on the significance of jerk in various fields. Also, several authors have explored the first derivative of force (or acceleration) for controls, [9], biomechanics, [10], and robotics, [11]. As well as first and second derivatives of force for robotics, [12]. In [13], the kinematic jerk and snap for multibody dynamics with joint constraints were well studied. In [14], jerk and snap were addressed in the cosmological equation of state. Hyper-jerk (snap) has also been used for arm-reaching tasks, [15] and analysis of handwriting, [16]. In addition, anomalies in the velocity, acceleration, jerk, snap, and crackle of limb segments during daily activities like walking, standing, and sit-to-stand movements have been shown to indicate deficits in neurological and cognitive function, [17].

Among the higher time derivatives of position, jerk has been extensively studied in various contexts. For instance, the jerk vector for Frenet curves in Euclidean 3-space $\mathbb{E}^3$ was resolved in [18]. Subsequently, a new resolution of the jerk vector in the same setting was presented in [19]. Additionally, resolutions of both acceleration and jerk vectors for modified curves in Euclidean 3-space $\mathbb{E}^3$ were examined in [20]. Similarly, resolutions of the jerk vector for Bishop curves in Euclidean 3-space $\mathbb{E}^3$ were investigated in [21]. Also, resolutions of the jerk vector for Darboux curves on regular surfaces in Euclidean 3-space $\mathbb{E}^3$ were explored, in [22]. Furthermore, in [23], the resolutions of the jerk and snap vectors of a point particle moving along a quasi-curve in Euclidean 3-space $\mathbb{E}^3$ were studied. It is also worth noting that jerk has long been considered a design factor for ensuring ride comfort in various applications, such as amusement rides, [24], elevators, [25], ships, [26] and buses, [27]. However, these higher-order time derivatives are not typically covered in physics and engineering courses, except in a few textbooks, [28], [29], [30]. This lack of detailed coverage in educational materials often results in confusion regarding terminology.

Studying the higher-order derivatives of position with respect to time and their applications provides a deeper understanding of how objects move and change over time, describing motion dynamics. This understanding enables better control, analysis, and optimization across a wide range of disciplines and applications, which is the primary motivation for this work. The purpose of this research is to study the higher derivatives of the position vector with respect to time and explore the applications of these higher-order derivatives across various fields of physics. In Section 2, the basic concepts and applicable formulas are presented. In Section 3, a series expansion of the position model is proposed. Section 4 is devoted to applications and illustrations of higher derivatives of the position vector with respect to time. In subsection 4.1, we study the higher-order derivatives in the context of spacecraft trajectory optimization. In subsection 4.2, as an illustration of advanced control systems, we consider a robotic arm that needs to follow a complex path with high precision. For vibration analysis and control, we study and illustrate higher derivatives of position in the context of a car suspension system in subsection 4.3. Subsection 4.4 is devoted to seismology, where we provide an analysis of ground motion behavior during earthquakes. We address the question of how higher-order derivatives of position provide valuable insights into ground motion during an earthquake and discuss the importance of higher-order derivatives in seismology. In Section 5, we provide additional practical applications. Our conclusions are presented in Section 6.

2 Basic Concepts and Higher-Order Derivatives

In physics, derivatives of the position with respect to time have specific names and interpretations.

Let's present the well-known ones from the first to the tenth derivative:

The position $x(t)$ gives the location of an object in space at any given time.

- **Velocity** or the speed of an object: Is the magnitude of the change of its position x over time.

$$v(t) = \frac{dx(t)}{dt}. \quad (1)$$

Therefore, first time derivative of position gives the object's velocity, indicating both how fast and in which direction the object is moving.

- **Acceleration:** Is the second time derivative of position, representing the rate at which the velocity changes.

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt^2}. \quad (2)$$

It explains how the object's velocity changes over time, which is essential for understanding the forces acting upon it.

- **Jerk:** Is the third derivative of position with respect to time. It is the rate of change of acceleration. The jerk is described by the following equation:

$$j(t) = \frac{da(t)}{dt} = \frac{d^2v(t)}{dt^2} = \frac{d^3x(t)}{dt^3}. \quad (3)$$

Its physical dimension is $[j] = LT^{-3}$. Known as jerk because of the abrupt shift in acceleration that can cause a person to be jerked backward or forward.

Exactly, the jerk shows how the acceleration is changing with time. It plays a crucial role in systems where sudden changes in acceleration can affect the behavior or structural integrity of an object.

Applying a high jerk, which means a rapid change in acceleration within a brief period, subjects the object to a sudden and significant force. Here the forces involved can be very large, even if applied for a short time. This can lead to the object breaking or tearing because the sudden force can exceed the material's ability to withstand it. For example, if you pull a piece of paper slowly, the force you apply is spread out over a longer period. The paper's fibers can gradually adjust to the force, and it likely won't tear because the force at any given moment is not sufficient to break the fibers. But, if you grab the paper in the middle and jerk it quickly, you're applying a high jerk. The sudden force is concentrated in a very short time, which overwhelms the paper's fibers, causing them to break almost instantly. This is why the paper tears more easily with a quick jerk compared to a slow pull. This example provides a clear illustration of how jerk appears.

- **Snap** (**Jounce** or **hyper-jerk**): Rate of change of jerk. It is the fourth derivative of position with respect to time.

$$s(t) = \frac{dj(t)}{dt} = \frac{d^4x(t)}{dt^4}. \quad (4)$$

The physical dimension of snap is $[s] = LT^{-4}$. The snap measures the rate of change of jerk. This can be useful for fine-tuning the control mechanisms in precise engineering applications to ensure smooth transitions and avoid mechanical stress.

- **Crackle**: Rate of change of snap. The fifth derivative of position with respect to time.

$$c(t) = \frac{d^5x(t)}{dt^5}. \quad (5)$$

- **Pop**: The sixth derivative of position with respect to time. Rate of change of crackle.

$$p(t) = \frac{d^6x(t)}{dt^6}. \quad (6)$$

Crackle and pop are less commonly used but can be essential in extremely high-precision applications where even minor fluctuations need to be accounted for.

2.1 Sixth and Beyond: Higher-Order Derivatives

For the seventh and higher-order derivatives, there are no widely accepted names in classical mechanics, but sometimes physicists or engineers use whimsical names or terms in specialized contexts.

- **Lock** l: The sixth derivative of position with respect to time. Rate of change of pop.

$$l(t) = \frac{d^7x(t)}{dt^7}. \quad (7)$$

This can be useful in highly dynamic systems where changes in the rate of change of crackle need to be monitored, though such situations are rare in typical applications.

- **Drop** d: The sixth derivative of position with respect to time. Rate of change of lock.

$$d(t) = \frac{d^8x(t)}{dt^8}. \quad (8)$$

This could potentially be relevant in systems requiring extremely precise control over higher-order dynamics, such as in advanced robotics or certain types of control theory.

- **Shot** s': The sixth derivative of position with respect to time. Rate of change of drop.

$$s'(t) = \frac{d^9 x(t)}{dt^9}. \quad (9)$$

This is even more specialized and might be relevant in theoretical contexts or in simulations where extremely fine control of motion is necessary.

- **Put** p': The sixth derivative of position with respect to time. Rate of change of shot.

$$p'(t) = \frac{d^{10} x(t)}{dt^{10}}. \quad (10)$$

The practical applications of such a high-order derivative are highly specialized and rare, possibly useful in detailed simulations of complex systems or in fields like advanced materials science or aerospace engineering.

Generally, physical dimensions of higher order derivatives of position are defined to be quantities with

$$[Q] = LT^{-q}, \quad (11)$$

for any integer number q greater or equal than zero. These derivatives provide a hierarchical framework to describe motion with increasing levels of precision regarding the behavior of a moving object. In general, the n-th derivative of the position can be referred to as the n-th order rate of change of the position. However, beyond the fifth or sixth derivative, these terms are not typically used in practical physics or engineering due to the diminishing physical significance of higher-order derivatives in most applications. If a specific higher-order derivative needs to be referred to, it is often described by its order, e.g., "seventh-order derivative of position" or "eighth-order derivative of position," rather than a specific name.

2.2 Applicable Formulas of Higher-Order Derivatives

The yank Y is the derivative of the force $F(t)$ with respect to time, it is as well the mass m multiplied by the jerk:

$$Y(t) = \frac{dF(t)}{dt} = mj(t). \quad (12)$$

The tug T is the derivative of the yank with respect to time, or the mass multiplied by the snap, also it equals to the third derivative of momentum P with respect to time

$$T(t) = \frac{dY(t)}{dt} = \frac{d^2 F(t)}{dt^2} = \frac{d^3 P(t)}{dt^3} = ms(t). \quad (13)$$

With $P = mv$, and $F = ma$. Here m is considered constant, [31].

There is also the snatch S, which is the fourth derivative of momentum with respect to time, or it is the mass multiplied by the crackle:

$$S(t) = \frac{d^4 P(t)}{dt^4} = mc(t). \quad (14)$$

We have the shake S', which is the fifth derivative of momentum or it is the mass multiplied by the pop:

$$S'(t) = \frac{d^5 P(t)}{dt^5} = mp(t). \quad (15)$$

3 Series Expansion of Position Model

The Taylor series expansion of the position $x(t)$ around t=0, known as the Maclaurin series, incorporates all the higher-order derivatives of the position with respect to time. This series can describe the motion of an object with high precision. Exploring a model based on higher-order derivatives, potentially extending to an infinite derivative order, is intriguing and may have potential applications in various advanced fields of physics and engineering.

The following equation is used for higher-order derivatives of the position (series expansion of the position in terms of its time derivatives around t=0):

$$x(t) = x(0) + x^{(1)}(0)t + x^{(2)}(0)\frac{t^2}{2} + x^{(3)}(0)\frac{t^3}{3!} + x^{(4)}(0)\frac{t^4}{4!} + \cdots = \sum_{n=0}^{\infty} x^{(n)}(0)\frac{t^n}{n!}, \quad (16)$$

Here $x^{(n)}(0)$ represents the n-th time derivative of the position around t=0. There is an infinite derivative series. However, this series can be extended to infinite order, provided the series converges. Note that in this series expansion, each term represents a time derivative of the position evaluated at t=0.

Now, we move to show how study the convergence of the Taylor series expansion $\sum_{n=0}^{\infty} x^{(n)}(0)\frac{t^n}{n!}$.

We need to consider the following aspects:

1. Radius of Convergence: The radius of convergence of a Taylor series can be determined using the ratio test or the root test as follows:

- **Ratio Test**: The ratio test involves examining the limit of the absolute value of the ratio of consecutive terms: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$, for our case:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{(n+1)}(0)}{(n+1)x^{(n)}(0)}\right| t. \quad (17)$$

Here the general term $a_n = \frac{x^{(n)}(0)}{n!} t^n$. If this limit exists and is less than 1, the series converges.

- **Root Test**: The root test involves examining the limit $\lim_{n\to\infty} \sqrt[n]{|a_n|}$, and for our series:

$$\sqrt[n]{|a_n|} = \left| \frac{x^{(n)}(0)}{n!} \right|^{\frac{1}{n}} t. \tag{18}$$

The series converges if this limit is less than 1.

2. Practical Steps to Determine Convergence:

- Behavior of derivatives: Analyze the growth of the derivatives $x^{(n)}(0)$. If they grow factorially or more slowly, the series is more likely to converge for a larger range of t.

- Analytic Functions: If $x(t)$ is an analytic function (i.e., it can be represented by a convergent power series around t=0), then the radius of convergence R is non-zero, and the series converges for $t < R$.

- Special Cases:

(i) For many common functions (e.g., exponential, sine, cosine), the Taylor series converges for all t.
(ii) For more complex functions, specific analysis of the growth rate of $x^{(n)}(0)$ is required.

To determine the convergence of the series $\sum_{n=0}^{\infty} x^{(n)}(0) \frac{t^n}{n!}$, we analyze the behavior of the derivatives $x^{(n)}(0)$. We use the ratio test or root test to find the radius of convergence and ensure the series converges within this radius. For many functions encountered in practice, particularly those that are analytic, this approach will confirm the convergence of their Taylor series expansions.

This series expansion model is a powerful tool in describing the motion of objects with high precision by incorporating the higher-order derivatives. This model has potential to enhance the precision and smoothness of control in a variety of physical systems in fields such as robotics, aerospace, and theoretical physics, although there are significant challenges about convergence, computational complexity, and practical application. The potential advantages of this model make it an extremely valuable tool.

4 Practical Applications

In everyday physics, pop, crackle, and other higher-order time derivatives are rarely discussed. But, they are very relevant in specialized and advanced fields of physics and engineering like control systems, robotics, and applied mathematics. In the following, we provide some examples in which these higher-order time derivatives play a particularly important role.

4.1 Spacecraft Trajectory Planning

In space missions, precise control of a spacecraft's trajectory is vital. Engineers need to ensure smooth transitions between the different phases of mission, requiring careful management of higher-order time derivatives of position to avoid sudden changes in motion that could strain the spacecraft's structure or cause instability. Furthermore, snap, crackle, and pop should be considered when planning orbital maneuvers or landings in order to ensure smooth and continuous trajectory adjustments, thereby reducing stress on the spacecraft and optimizing fuel efficiency.

For illustrating higher-order time derivatives of position in the context of spacecraft trajectory optimization, we can depict their vectors within three-dimensional (3D) space. In the following, we model the spacecraft position, velocity, acceleration, and other derivatives vectors employing sinusoidal functions where the connection between these vectors is demonstrated.

We use the following helical trajectory parameters: The Amplitude A=1 represents the radius of the helix; the growth rate B=1, which determines the linear progression along the z-axis. The angular frequency $\omega = 1$ that controls how many turns the helix makes over the specified period.

Also, we use the following position components:

$$\begin{cases} x(t) = A\cos(\omega t), \\ y(t) = A\sin(\omega t), \\ z(t) = Bt. \end{cases} \tag{19}$$

Now, by using a helical path, a common representation in orbital mechanics for a stable and periodic motion, we plot a 3D spacecraft trajectory in Figure 1. Note that the visualization helps in understanding the trajectory in 3D space, exactly, it is important for effective planning and analysis in spacecraft trajectory optimization.

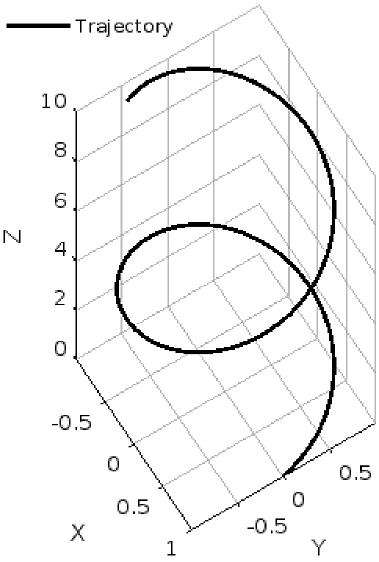

Fig. 1: Visualization of a helical 3D trajectory for a spacecraft

Knowing that for simulating different trajectory shapes and behaviors in Figure 1, the parameters A, B, and ω can be adjusted.

Subsequently, we include velocity, acceleration, jerk, crackle and pop vectors along the 3D helical trajectory to have the following Figure 2:

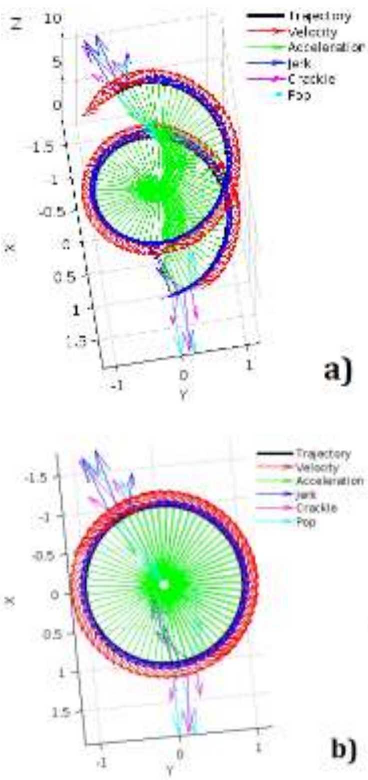

Fig. 2: Plot of the helical trajectory, along with the velocity, acceleration, jerk, crackle and pop vectors in sub-fig a). The projection of this helical trajectory in xy-plane, along with these vectors is displayed in sub-fig b)

However, for further illustration, we include the velocity, force, and yank vectors in the 3D helical trajectory of the spacecraft, as shown in Figure 3.

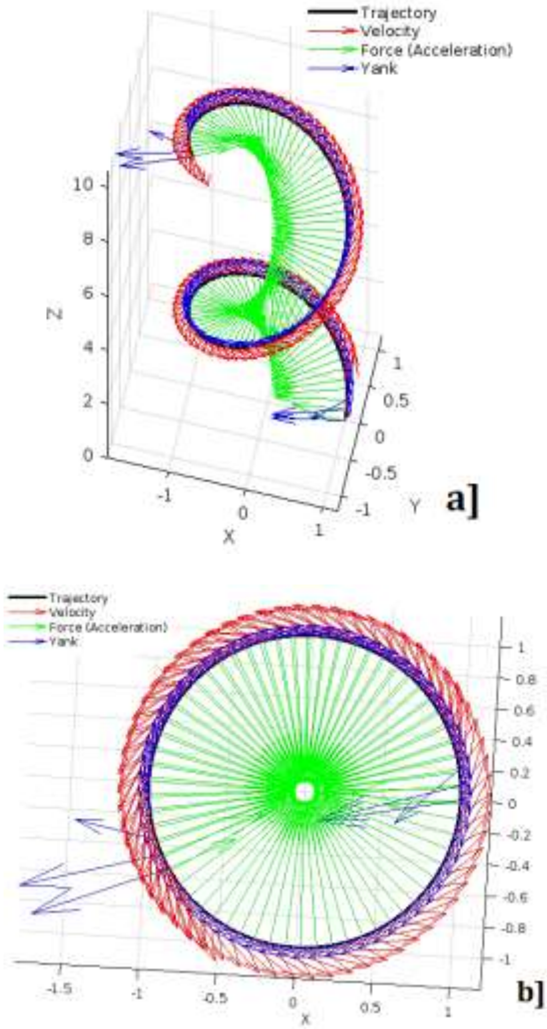

Fig. 3: The helical trajectory, accompanied by velocity, force, and yank vectors. The 3D plot is shown in sub-fig a], with its projection onto xy-plane shown in sub-fig b]

Knowing that the interval variable can be adjusted to modify the density of the plotted vectors as necessary in Figure 2 and Figure 3.

These figures provide a good detailed view of the spacecraft's trajectory and the dynamic forces acting upon it, which are essential for trajectory planning and analysis.

Now, we move to depict the evolution of the jerk, crackle, and pop components over time as 2D plots of the higher-order time derivatives of position, for the helical trajectory example.

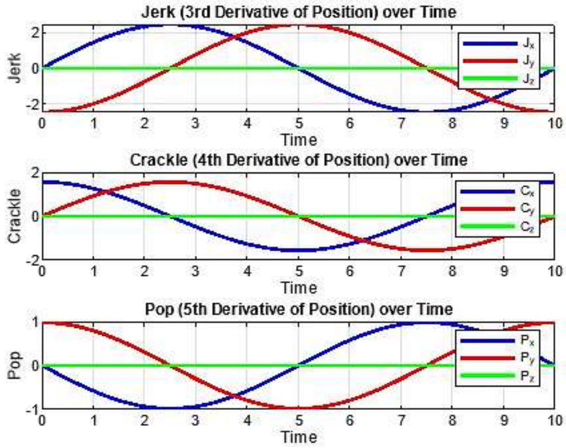

Fig. 4: A figure consisting of three subplots displaying the evolution of jerk, crackle, and pop components over time

A comprehensive view of higher-order derivatives of the spacecraft's trajectory is given in Figure 4. Again, we confirm that higher-order derivatives are essential for optimizing spacecraft path, as they offer valuable insights into the motion, ensuring smoother and more efficient trajectories.

4.2 Control Systems: Robotics and Mechanical Systems

In robotic systems, particularly those requiring precise movements such as robotic arms or surgical robots, higher-order derivatives are used to achieve smooth and controlled motion. This is because in motion, abrupt changes, which are characterized by high jerk, snap, crackle, or pop, can cause wear or damage to mechanical components and by minimizing these sudden changes, robotic systems can operate with greater precision, resulting in more reliable and efficient performance. So, advanced control algorithms often involve the higher-order time derivatives of position to optimize the robot's path ensuring smooth and efficient movement.

We consider a robotic arm that must follow a complex path with high precision as an example of an advanced control system. Traditional control systems typically use velocity, acceleration, and sometimes jerk for path planning, but the infinite derivative model given in equation (16) provide a control input $u(t)$ that takes into account all higher-order derivatives in order to ensure the arm moves with the highest level of smoothness and precision.

$$u(t) = \sum_{n=0}^{\infty} K_n \frac{d^n x(t)}{dt^n}, \qquad (20)$$

where K_n are control coefficients that could be optimized. Then, the implementation steps are as follows:

1- We clearly define the infinite series and the conditions for its convergence.
2- We develop and create algorithms to compute the higher-order derivatives efficiently.
3- We apply the model to simulated systems to test its performance and stability.
4- We use optimization techniques to find the best coefficients for the specific application.

Let's generate similar visualizations for a robotic arm (or surgical robot), where the position, velocity, acceleration, jerk, snap, crackle, and pop vectors are derived from sinusoidal functions to demonstrate the relationships between these vectors. So, for the circular trajectory parameters R=5 (radius of the circle), and $\omega = \frac{2\pi}{10}$ (angular frequency of the circular motion). The position components (circular trajectory) are as follows:

$$\begin{cases} x(t) = R\cos(\omega t), \\ y(t) = R\sin(\omega t), \\ \text{Linear movement in z direction.} \end{cases} \qquad (21)$$

We used a circular trajectory in the xy-plane with a linear motion in the z-direction. In Figure 5, we simulate a simple robotic arm with a fixed base and an end effector that moves along a predefined trajectory (e.g., a circular path).

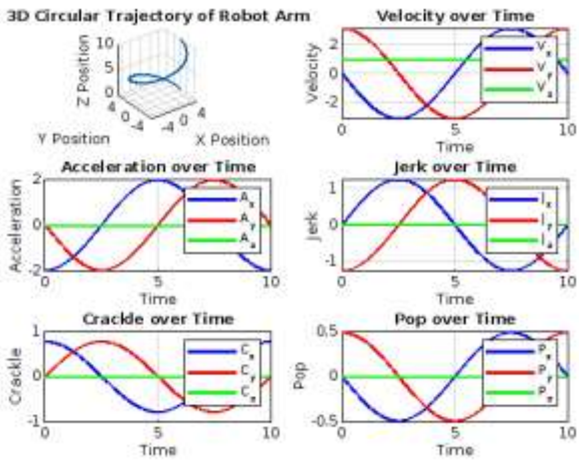

Fig. 5: Movement of a robotic arm in 3D space and its various derivatives (velocity, acceleration, jerk, crackle, and pop) over time

Next, we illustrate the vectors at various points (position, velocity, acceleration and jerk) along the trajectory of a robotic arm with joints in 3D.

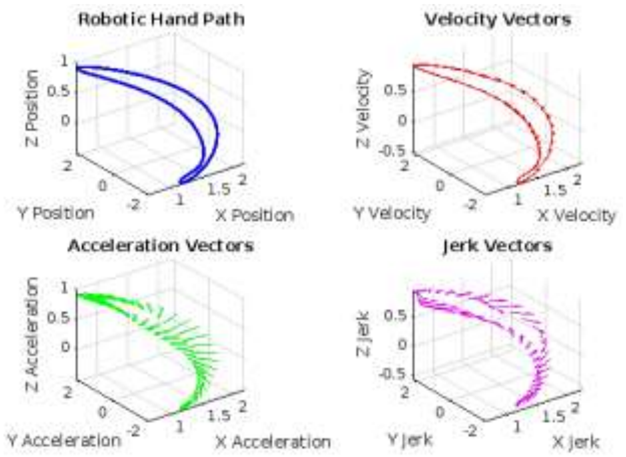

Fig. 6: visualization of velocity, acceleration, and jerk vectors on a hand-like robotic arm with joints

Here, in Figure 6, we use the following calculations:

For joint angles, we have:

$$\theta_1 = \sin(t)\,;\ \theta_2 = \cos(t)\,;\ \theta_3 = \sin(2t). \quad (22)$$

For, arm segment lengths, we use:

$$L_1 = 1;\ L_1 = 0.8;\ L_1 = 0.6. \quad (23)$$

Then, the positions are:

$$x = L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3), \quad (24)$$

$$y = L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2) + L_3\sin(\theta_1+\theta_2+\theta_3), \quad (25)$$

$$z = \frac{1}{2}(\theta_1+\theta_2+\theta_3). \quad (26)$$

This comprehensive visualization helps in understanding the dynamics and control of the robotic arm.

Besides, we can find applications of higher time derivatives of position in automotive control. In advanced driver assistance systems and autonomous vehicles, controlling jerk is essential for passenger comfort and safety. Sudden changes in acceleration can lead to discomfort and potential loss of control. For control systems, especially in mechanical systems, it is essential to analyze the system's response to control inputs, thus we consider the time derivatives of position.

4.3 Vibration Analysis and Control

In engineering, especially in structures subject to dynamic loads such as bridges, buildings, or vehicle suspensions, understanding and controlling vibrations is crucial. Higher-order time derivatives of position can describe the detailed response of a system to external forces like wind, earthquakes, and traffic. For example, in designing a car suspension system, considering snap, crackle, and pop can help engineers predict and mitigate complex vibrational behaviors, improving ride comfort and vehicle stability.

In engineering, particularly for structures and systems subject to dynamic loads, these higher-order derivatives are crucial for understanding and controlling vibrations. By analyzing these vectors, engineers can design systems that mitigate complex vibrational behaviors, thereby improving stability, longevity, and performance. For example, in a car suspension system, minimizing high snap, crackle, and pop values can lead to a smoother ride and reduced mechanical wear. To illustrate higher time derivatives of position vectors in the context of a car suspension system, we show how these derivatives manifest in response to a bump in the diagram provided in Figure 7.

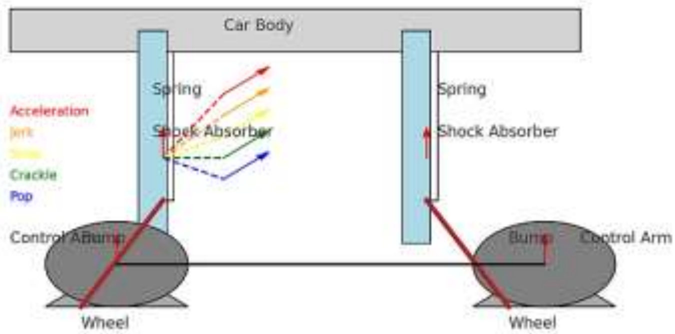

Fig. 7: A diagram showing the suspension system responding to a bump, with annotations indicating the higher derivatives of position

In the context of vehicle suspension, encountering a bump leads to a series of dynamic responses related to acceleration and other higher-order derivatives. Thus, higher-order derivatives are necessary for analyzing and predicting the detailed conduct of the suspension, basically contributing to better ride comfort and vehicle stability. So, to illustrate how these higher-order derivatives vary over time as the suspension reacts to a bump, we depict acceleration, jerk, snap, crackle, and pop as functions of time in Figure 8. Knowing that basic functions are used to represent these higher-order derivatives and demonstrate their conduct. In a real-world scenario, these used functions are taken from the well-known physical equations that describe the dynamics of the suspension system.

The position function $x(t)$ is defined as a damped sinusoid:

$$x(t) = Ae^{-\beta t}\sin(\omega t), \quad (27)$$

where the amplitude A=1, the damping coefficient $\beta = 0.1$, and the angular frequency $\omega = 2\pi$.

In Figure 8, we show how the derivative of position evolves as the system responds to a bump.

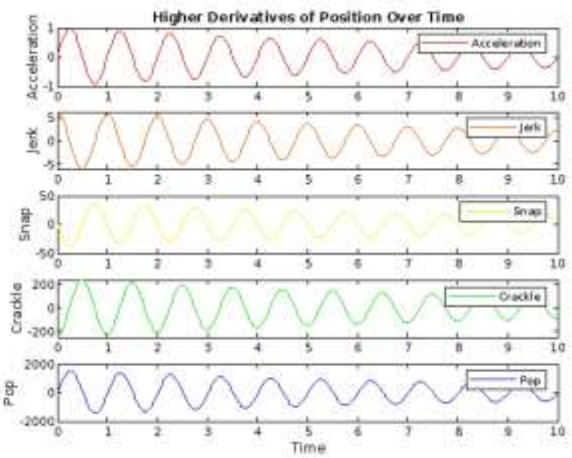

Fig. 8: The evolution of acceleration, jerk, snap, crackle, and pop over time in the context of a car suspension system responding to a bump

4.4 Seismology

In earthquake studies and investigations, the ground motion can be described using higher-order derivatives to understand the intensity and the impact of seismic waves. While acceleration is commonly emphasized, high-order derivatives such as jerk, snap, crackle, and pop can provide more detailed insights into the characteristics of ground motion and its effects on structures. Thus, this is very relevant for understanding and predicting seismic activity.

In seismology, analyzing ground motion through higher-order time derivatives of ground displacement provides a more comprehensive understanding of the characteristics of earthquakes and their potential effects on both structures and landscapes.

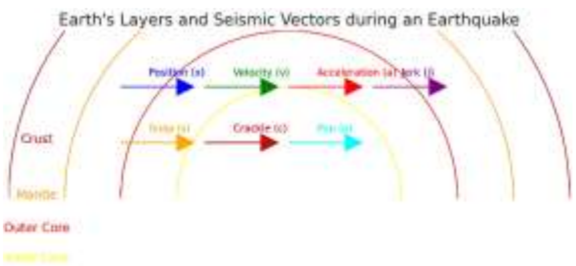

Fig. 9: A basic cross-sectional illustration of the Earth that shows the crust, mantle, and core along with vector representations for position, velocity, jerk, and additional higher-order derivatives

The crust is the outermost layer, the Mantle is the layer beneath the crust, the Outer Core is the liquid layer beneath the mantle, and the inner Core is the solid innermost layer.

Note that in Figure 9, a diagram illustrating the Earth's layers during an earthquake, the higher-order derivatives (velocity, acceleration, etc.) are represented as vectors. However, these vectors indicate how the motion of the ground changes at different rates, providing a detailed understanding of seismic wave propagation and its effects. We also give a simulation of ground motion over time, calculating the higher-order derivatives (position, velocity, acceleration, jerk, snap, crackle, and pop). We plot these derivatives to visualize how they change over time during an earthquake in Figure 10. This provides a clear visual representation of how higher-order derivatives change over time during an earthquake, offering detailed insights into the ground motion characteristics and potential impacts on structures.

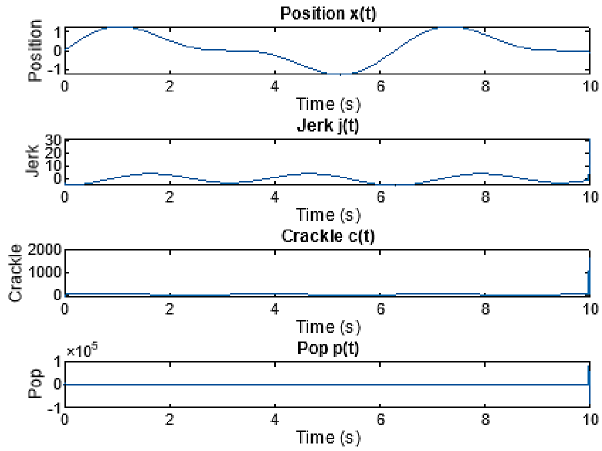

Fig. 10: A representation of how higher-order derivatives change over time during an earthquake

Here, we simulated ground motion (position) as a combination of sine and cosine functions:

$$x(t) = \sin(t) + \frac{1}{2}\sin(2t). \qquad (28)$$

Now, let's address the following question: *How do higher-order derivatives of position provide valuable insights into ground motion during an earthquake, and what is the importance of higher-order derivatives of the position vector in seismology?*

We answer this question through the following explanations:

1. Understanding seismic wave propagation:

- Position or displacement: This shows the actual movement of the ground, which is crucial for determining the extent of the earthquake.
- Velocity: Helps in understanding how fast the seismic waves are traveling. Higher velocities can indicate more intense shaking.
- Acceleration: Critical for assessing the forces that buildings and infrastructure will experience. High accelerations can cause severe damage.
- Jerk: Indicates rapid changes in acceleration, which can be particularly damaging as structures may not have time to adapt to these changes.
- Snap, crackle, and pop offer even more precise insights into ground motion. These derivatives help assess the smoothness and continuity of seismic waves. Sudden shifts in these derivatives signal areas of potential structural weakness or the onset of stronger shaking.

2. Impact on structural design:
   - Most building codes emphasize acceleration as it directly relates to the constraints that a structure needs to endure.
   - Jerk and higher-order derivatives impose additional stress on structure. Buildings and materials can fail not only due to the magnitude of constraints but also because of the speed at which these constraints change. For instance, sudden shifts in acceleration can create a high dynamic load on structural components, potentially causing failure points.
   - Snap, crackle, and pop help in identifying resonant frequencies or periods during which the structure may undergo amplified vibrations, potentially increasing damage.

3. Seismic hazard assessment: Analyzing higher-order derivatives in detail leads to the development of more accurate seismic hazard models. Also, understanding time variations in ground motion, improves predictions of earthquake impact locations and intensities across various regions.

4. Early warning systems: Integrating higher-order derivatives into early warning systems leads to more accurate alerts, which facilitate better preparation and response.

5. Post-earthquake analysis: Examine recorded data for higher-order derivatives following an earthquake, providing valuable insights into the behavior of ground motion and the specific factors that contribute to the most damage. This analysis helps to refine building codes and improves the design of future structures.

6. Improving building codes: Information obtained from higher-order derivatives leads to revising and updating the building codes, ensuring new constructions are better equipped to withstand complex seismic forces. Thus, using jerk and snap derivatives, building codes can be updated to incorporate rapid acceleration changes and their impact on structural integrity.

7. Practical example: If we consider a scenario where the data of ground motion during an earthquake is collected, then by analyzing this data, we have:

- High acceleration, indicates that strong shaking forces can cause considerable damage to buildings.
- High jerk, this implies that the structure will be subjected to quick and jarring movements that can lead to damage and crack the structural components.
- High snap, this implies a significant rate of change in jerk or rapid jarring abrupt movements, increasing the risk of structural failure.

Last but not least, engineers by understanding these factors, can design strong buildings that withstand high acceleration, as well as flexible structures that absorb and dissipate energy from rapid force changes. Materials can also be developed to withstand these dynamic loads, and structural elements can be reinforced and well-equipped by damping systems.

Indeed high-order derivatives analysis provides valuable insights into ground motion during earthquakes, which leads to improve structural design, along with, enhanced safety measures, and more accurate seismic risk assessments. This in turn helps to better prepare structures that withstand seismic events, reduce damages, and protect lives.

5 Additional Practical Applications

Finally, yet importantly, higher-order time derivatives of position actually have many other applications. Here are a few additional important examples that we will briefly touch on without going into more detail as before.

5.1 Animation and Special Effects

In the world of animation and computer graphics, particularly when simulating realistic movements, higher-order time derivatives are essential for achieving smooth, natural-looking motions. Animators and simulation software use these derivatives to ensure fluid transitions between frames, in order to eliminate abrupt, jerky motions.

In addition, higher-order derivatives are applied in predictive modeling, such as weather forecasting, leading to the capture of subtle dynamics that simpler models may ignore, enhancing forecast accuracy. For instance, *Numerical Weather Prediction* (NWP), and *General Circulation Models* (GCMs) are likely to include higher-order time derivatives of position through physical processes governed by differential equations, particularly in fluid dynamics and atmospheric modeling.

5.2 Theories in Theoretical Physics

- Non-local field theories: Such theories frequently use infinite derivatives to present interactions extending beyond a single point in space-time, offering insights into phenomena that are challenging for local theories to explain.

- String field theory, here non-local actions with infinite derivatives naturally arise, with the aim of describing the fundamental nature of particles and forces within a unified framework.

- p-Adic string theory: A form of string theory where the action includes infinite derivatives, resulting in non-local equations of motion.

- Higher-order gravity, here some approaches incorporate higher-order derivatives of the metric tensor to develop a quantum theory of gravity. Models based on infinite derivative gravity aim to address challenges like renormalizability and the singularity problem in general relativity.

- Infinite derivative gravity, here a non-local extension of general relativity that incorporates an infinite series of higher-order curvature terms presumably addresses singularities and provides a pathway toward a consistent quantum theory of gravity.

Finally, the aforementioned examples explain how important the higher-order time derivatives of position are for achieving precise, smooth, and well-controlled motion across diverse fields in physics and engineering.

6 Conclusion

In this work, we studied the higher-order time derivatives of position up to the tenth order and presented a mathematical model that accurately describes the motion of objects and smoother control. It is confirmed that higher-order time derivatives of position provide important information about the dynamics of object motion, which allows for more precise control, improved performance, and increased safety across a variety of applications in physics and engineering. In most practical scenarios, time derivatives beyond jerk or fourth-order snap are rarely utilized due to their limited physical impact and the challenges associated with measurement control. But in highly specialized fields like advanced robotics, aerospace engineering, or theoretical physics, these higher-order time derivatives may be essential for accurately modeling and controlling complex, high-precision dynamic systems.

Although higher-order time derivatives of position can be defined and sometimes named, their practical usefulness declines significantly beyond the sixth time derivative, making them largely theoretical rather than practical in most real-world contexts.

Our results can be used as a basis for further investigations and explorations across other scenarios. For example, combining higher derivatives with fractional calculus allows for more sophisticated modeling and control strategies, enabling more precise and flexible approaches to complex problems in engineering and physics, as well as providing insights into systems where the memory effect is important. Such as in spacecraft trajectory planning, fractional calculus can be used to model the effects of perturbations and non-idealities in the system, while higher-order derivatives help design smooth and precise trajectories. This represents a promising avenue for future research.

Declaration of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work the author used Chatgpt in order to modify some phrases in the research work to enhance academic quality. After using this tool/service, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.

References:

[1] Flash, T., Hogan, N. The coordination of arm movements: an experimentally confirmed mathematical model. *Journal of neuroscience*, 5(7), 1985, 1688-1703. https://doi.org/10.1523/JNEUROSCI.05-07-01688.1985.

[2] Laczko, J., Scheidt, R. A., Simo, L. S., Piovesan, D. Inter-joint coordination deficits revealed in the decomposition of endpoint jerk during goal-directed arm movement after stroke. *IEEE Transactions on neural systems and rehabilitation engineering*, 25(7), 2017, 798-810. https://doi.org/10.1109/TNSRE.2017.2652393

[3] De Lucena, D. S., Stoller, O., Rowe, J. B., Chan, V., Reinkensmeyer, D. J. Wearable sensing for rehabilitation after stroke: bimanual jerk asymmetry encodes unique information about the variability of upper extremity recovery. *2017 International conference on rehabilitation robotics*, London, UK, 2017, 1603-1608. https://doi.org/10.1109/ICORR.2017.8009477

[4] Zhang, L., Diraneyya, M. M., Ryu, J., Haas, C. T., Abdel-Rahman, E. M. Jerk as an indicator of physical exertion and fatigue. *Automation in construction,* 104, 2019, 120-128. https://doi.org/10.1016/j.autcon.2019.04.016.

[5] Siczek, K.; Stefanski, A. Analysis of dynamics and friction resistance in the cam-tappet system. *J. Theor. Appl. Mech.* 57, 2019, 273–286. https://doi.org/10.15632/jtam-pl.57.1.273.

[6] Sajjadi, S. S., Baleanu, D., Jajarmi, A., Pirouz, H. M. A new adaptive synchronization and hyperchaos control of a biological snap oscillator. *Chaos Solitons Fractals.* (2020), 138, 109919. https://doi.org/10.1016/j.chaos.2020.109919.

[7] Thompson, P.M. Snap crackle and pop. *In Proceedings of the AIAA Southern California Aerospace Systems and Technology Conference,* Hawthorne, CA, USA, 2011.

[8] Hayati, H., Eager, D., Pendrill, A. M., Alberg, H. Jerk within the context of science and engineering - a systematic review. *Vibration,* 3(4), 2020, 371-409. https://doi.org/10.3390/vibration3040025.

[9] Baran, E. A., Uzunovic, T., and Sabanovic, A. Performance improvement of bilateral control systems using derivative of force. *Robotica,* 36(11), 2018, 1627-1640. https://doi.org/10.1017/S0263574718000607.

[10] Lin, D. C., McGowan, C. P., Blum, K. P., Ting, L. H. Yank: the time derivative of force is an important biomechanical variable in sensorimotor systems. *Journal of experimental biology,* 222(18), 2019, jeb180414. https://doi.org/10.1242/jeb.180414.

[11] Lo Bianco, C.G. Evaluation of generalized force derivatives by means of a recursive Newton–Euler approach. *IEEE Transaction on robotics,* 25(4), 2009, 954-959. https://doi.org/10.1109/TRO.2009.2024787.

[12] Yang, K., Yang, W., Wang, C. Inverse dynamic analysis and position error evaluation of the heavy-duty industrial robot with elastic joints: an efficient approach based on Lie group. *Nonlinear Dynamics,* 93, 2018, 487-504. https://doi.org/10.1007/s11071-018-4205-2.

[13] Sommer Iii, H. J. Kinematic jerk and jounce for multibody dynamics with joint constraints. *Mechanism and Machine Theory,* 196, 2024, 105613. https://doi.org/10.1016/j.mechmachtheory.2024.105613.

[14] Visser, M. Jerk, snap and the cosmological equation of state. *Classical and Quantum Gravity,* 21(11), 2004, 2603. https://doi.org/10.1088/0264-9381/21/11/006.

[15] Novak, K. E., Miller, L. E., Houk, J. C. Kinematic properties of rapid hand movements in a knob turning task. *Experimental brain research,* 132, 2000, 419-433. https://doi.org/10.1007/s002210000366.

[16] Edelman, S., Flash, T. A model of handwriting. *Biological cybernetics,* 57(1-2), 1987, 25-36. https://doi.org/10.1007/BF00318713.

[17] Saho, K., Sugano, K., Kita, M., Uemura, K., Matsumoto, M. Classification of health literacy and cognitive impairments using higher-order kinematic parameters of the sit-to-stand movement from a monostatic Doppler radar. *IEEE Sensors journal,* 21(8), 2021, 10183-10192. https://doi.org/10.1109/JSEN.2021.3060050

[18] Résal, H. *Treatise on Pure Cinematics (Traité de Cinématique Pure)*, Mallet-Bachelier: Paris, France, 1862.

[19] Özen, K.E.; Dündar, F.S.; Tosun, M. An alternative approach to jerk in motion along a space curve with applications. *J. Theor. Appl. Mech.* 57, 2019, 435–444. https://doi.org/10.15632/jtam-pl/104595.

[20] Özen, K.E.; Güner, M.; Tosun, M. A note on the acceleration and jerk in motion along a space curve. *An. St. Univ. Ovidius Constanta.* 28, 2020, 151–164. https://doi.org/10.2478/auom-2020-0011.

[21] Güner, M. On the jerk in motion along a space curve. *Math. Methods Appl. Sci.* 44, 2021, 7407–7415. https://doi.org/10.1002/mma.6094.

[22] Tosun, M.; Hızarcıoglu, M. On the jerk in motion along a regular surface curve. *TBILISI-MATHEMATICS,* 3, 2020, 67–75.

[23] Hamouda E, Cesarano C, Askar S, Elsharkawy A. Resolutions of the Jerk and Snap Vectors for a Quasi Curve in Euclidean 3-Space. *Mathematics.* 9(23), 2021, 3128. https://doi.org/10.3390/math9233128.

[24] Eager, D.; Pendrill, A.M.; Reistad, N. Beyond velocity and acceleration: Jerk, snap and higher derivatives. *Eur. J. Phys.* 37, 2016, 065008. https://doi.org/10.1088/0143-0807/37/6/065008.

[25] Nemes, A.; Mester, G. Energy Efficient Feasible Autonomous Multi-Rotor Unmanned Aerial Vehicles Trajectories. *In Proceedings of the 4th International Scientific Conference on Advances in Mechanical, Engineering,* Debrecen, Hungary, 13–15, 2016, 369–376.

[26] Coats, T.W.; Haupt, K.D.; Murphy, H.P.; Ganey, N.C.; Riley, M.R. A Guide for Measuring, Analyzing, and Evaluating Accelerations Recorded During Seakeeping Trials of High-Speed Craft; Report; *Naval Surface Warfare Center Carderock Division:* Norfolk, VA, USA, 2016.

[27] Bae, I.; Moon, J.; and Seo, J. Toward a Comfortable Driving Experience for a Self-Driving Shuttle Bus. *Electronics.* 2019, 8, 943. https://doi.org/10.3390/electronics8090943.

[28] Lalanne, C. *Mechanical Vibration and Shock Analysis, Fatigue Damage,* John Wiley & Sons: Hoboken, NJ, USA, 2010.

[29] Smith, J.D. *Vibration Measurement and Analysis,* Butterworth-Heinemann: Oxford, UK, 2013.

[30] Jia, J. *Essentials of Applied Dynamic Analysis*, Springer: Berlin/Heidelberg, Germany, 2014.

[31] Haouam, I. On the Monotonous Evolutions: Model and Applications. *WSEAS Transactions on Systems*, 22, 2023, 255-263. https://doi.org/10.37394/23202.2023.22.27.

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