Exploring Non-linear Dynamics: Constructing the Bifurcation Diagram of a Damped Driven Pendulum using Python

JOAN JANI
Department of Engineering Physics,
Polytechnic University of Tirana,
Rr. S. Delvina, 1001,
ALBANIA

Abstract: In this paper, we present a detailed analysis and construction of the bifurcation diagram for the damped-driven pendulum system. The bifurcation diagram in general presents the qualitative changes of the steady-state behavior for the pendulum. For this purpose, we implement the use of the Python programming language with the inclusion of scientific libraries. This nonlinear dynamical system is an example of a system that exhibits a chaotic regime, which is the sensitivity of its behavior to the initial conditions and the parameters of the system. We investigate the response of the system to a range of drive strengths γ applied. By changing the driving strength, the system reveals patterns of periodicity, quasi-periodic, and chaotic regimes. The critical values where it goes from regular motion to chaotic one are highlighted, offering further understanding of the mechanisms of the transitions. This study presents the use of the Python programming language for the modeling and visualization of non-dynamic systems and contributes to a deeper understanding of nonlinear oscillator dynamics.

Key-Words: - Bifurcation Diagram, Damped Driven Pendulum, Non-linear Dynamics, Python Programming, Numerical Methods, Chaotic Systems, Periodicity.

Received: March 19, 2024. Revised: August 11, 2024. Accepted: September 12, 2024. Published: October 24, 2024.

1 Introduction

In complex systems, the state of apparently stochastic and unpredictable behavior derived from a deterministic description of them is called chaos. This concept defines the limits of order and predictability, [1]. Its presence is ubiquitous, from turbulence in weather systems to the behavior of electronic circuits, [2]. This behavior generally comes from nonlinear systems under certain conditions; this means that not all nonlinear systems are chaotic, [3]. During a chaotic regime, the behavior of the system becomes unpredictable, giving the impression of stochastic and random operation, [4].

The bifurcation diagram is a powerful tool for understanding the evolution of a complex system by changing a parameter of them, [5]. The bifurcation diagram plots the possible steady state values of the system against a varying parameter, [6]. This diagram starts with a stable state, and as the parameters change, these states branch into multiple branches; this is an indication of different behaviors. The points where the graph is splitting are known as bifurcation points, [7].

The logistic map is an example of a bifurcation diagram in which the parameter of the growth rate increases. In this diagram, the population can first stabilize in a value, then oscillate between two values and eventually become chaotic, [8]. The stability and periodicity within the system reveal the intricate transition to chaos. This reveals the delicate interplay between order and disorder in natural phenomena, [9].

The use of advanced computational tools for modeling, data analysis, and visualization of the results in the fields of dynamical systems creates new opportunities for research, [10]. The widespread use of Python as a tool for studying dynamical systems has been well established during recent years. There are numerous examples presented in literature for the application of Python in studying dynamical systems, [11], [12], [13]. The potential use of chaotic systems for cryptography purposes and their use in secure communication systems in general has increased the interest in their study, [14].

In this paper, we investigate the dynamics of the non-linear dumped-driven pendulum and explore the construction of bifurcation diagrams using Python programming. Several analytical solutions exist in the literature for the damped pendulum, but when an external force is applied, the solution is typically only approximated numerically, [15], [16]. The proposed method is efficient and easy to implement in other chaotic systems. Another advantage is that our proposal is based on free source code, available in the public domain, and the program used and described in this paper.

2 Problem Formulation

We present a new approach to constructing the bifurcation diagram in a dynamic system. We have chosen Python as the programming language for this purpose. We will use it to model the system, simulate it, and present the results.

A nonlinear damped pendulum is an extension of the simple pendulum that incorporates both damping effects and nonlinearity in its restoring force. This system is governed by a differential equation that accounts for the pendulum's angle, damping, and nonlinear restoring forces, [6]. Such pendulums are found in various natural and engineered systems, and their study provides insight into the dynamics of oscillatory systems with energy dissipation and nonlinearities.

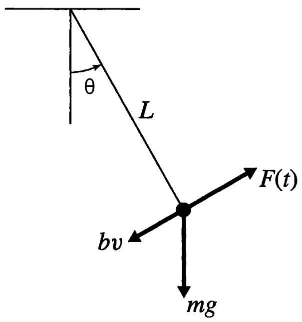

Fig. 1: A simple pendulum where three forces are applied, the weight, the resistive force and the driven force, [6]

The equation describing the motion of the pendulum can be derived from the equation of a rigid object under a net torque $I\ddot{\theta} = \Gamma$, where I denotes the moment of inertia of the object in our case $I = mL^2$, and Γ is the net torque about the pivot point. Figure (1) shows three forces applied on the object: the weight of the object mg applies a torque $-mgL\sin\theta$, the resistive force is proportional to the speed of the object and has a magnitude bv, hence exerts a torque $-Lbv = -bL^2\dot{\theta}$ and the driving force $F(t)$ applying a torque $LF(t)$. Thus, the equation of motion:

$$mL^2\ddot{\theta} = -mgL\sin\theta - bL^2\dot{\theta} + LF(t) \qquad (1)$$

The drive external force $F(t)$ is sinusoidal given by:

$$F(t) = A\cos(\omega_D t)$$

where A is the amplitude of the driving force (in Newtons) and ω_D is the drive frequency. Rearranging the equation (1) we have and substituting the expression for the external force we have:

$$\ddot{\theta} + \frac{g}{L}\sin\theta + \frac{b}{m}\dot{\theta} = \frac{A}{mL}\cos\omega_D t \qquad (2)$$

By substituting the coefficient $b/m = 2\beta$ where β is called the damping constant, $g/L = \omega_o^2$ where ω_o denotes the natural frequency of the pendulum and $\frac{A}{mL} = \gamma\omega_o^2$ where γ denotes the drive strength, we have the differential equation of the motion for a driven damped pendulum:

$$\ddot{\theta} + \omega_o^2\sin\theta + 2\beta\dot{\theta} = \gamma\omega_o^2\cos\omega t \qquad (3)$$

When the dimensionless parameter γ is less than 1, it indicates that the driving force is weaker than the pendulum's weight. In this case, the motion induced by the driving force is relatively small compared to the overall dynamics governed by gravity and damping, [6]. We investigate the motion of the pendulum by varying the parameter γ across a range of values.

3 Solution

We will use numerical methods to solve the equation presented in (3). Numerical methods have been made easier through their implementation in packages such as Scipy in the Python programming language, [17]. The results from the simulation will be presented using the matplotlib package, [18]. These tools are easy to use and do not require a lot of computing power. The parameters of the system are chosen $\omega_D = 2\pi$, $\omega_o = 1.5\omega$ and $\beta = \omega_o/4$.

The second-order differential equation (3) is rewritten as a system of two second-order equations as follows:

$$\begin{aligned} \frac{d\omega}{dt} &= -\omega_o^2\sin\theta - 2\beta\dot{\theta} + \gamma\omega_o^2\cos\omega_D t \\ \frac{d\theta}{dt} &= \omega \end{aligned} \qquad (4)$$

The system of differential equations can be solved through the function odeint, which is in the scipy package of the Python ecosystem. The function odeint is using lsoda from the FORTRAN library odepack. For solving the equations we use $\gamma = 0.9$, we use the initial conditions $\dot{\theta} = 0$ and several values for the initial angle listed bellow, $\theta_0 = [-2.5, -1.5, -0.5, 0.5, 0, 1.5, 2.5]$. The solutions for every initial angle value are plotted together and presented in Fig. 2.

The first we notice in Fig. 2 is that the system, after a transition state which depends on the initial conditions, ends up in a steady state which does not depend on the initial conditions. For a linear damped oscillator subjected to a sinusoidal driving force, there is a unique attractor that the system's motion will approach regardless of the initial conditions. Moreover, the motion of the attractor is sinusoidal and has a frequency that precisely corresponds to the drive frequency. As can be seen here, the period of

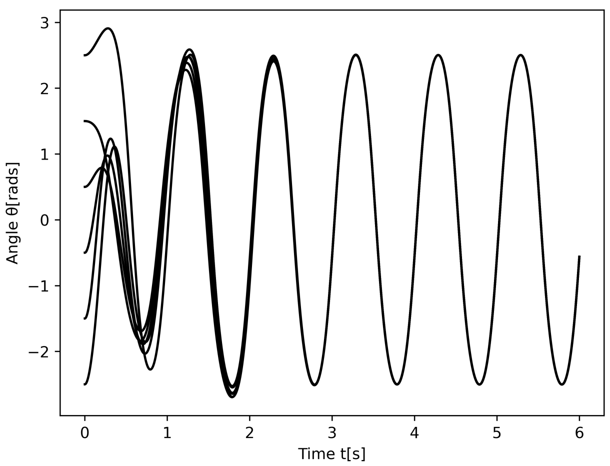

Fig. 2: The angle θ of the pendulum as a function of time for various initial conditions $\theta_0 = [-2.5, -1.5, -0.5, 0.5, 0, 1.5, 2.5]$ when $\gamma = 0.09$

the motion is $T = 1s$, which is equal to the period of the external force. So, in the linear regime, the system obeys the frequency of the external force. This behaviour is thoroughly analyzed in, [19]. This will continue to happen as the value for γ remains sufficiently small. For some value of drive strength γ, the attractor becomes unstable, and the system's behaviour becomes more complex.

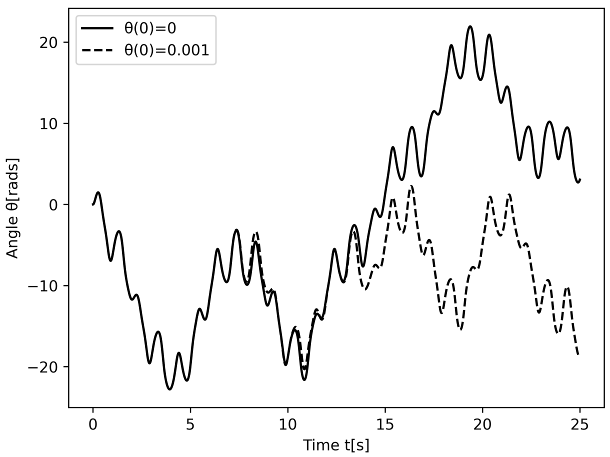

Fig. 3: The angle θ of the pendulum with time for two different initial conditions with $\theta = 0$ and $\theta = 0.001$, when drive strength $= 1.5$.

In Fig. 3, the angle θ is presented as a function of time when the value of $= 1.503$, two different initial conditions $\theta = 0$ and $\theta = 0.001$. As can be seen from the graph, at the beginning, the system's behaviour is the same for both initial conditions, but then their difference begins to increase exponentially. The Lyapunov exponent measures the rate of separation, [20]. For a nonlinear oscillator, different initial conditions can result in distinct attractors. Here, the behaviour is not periodic as in Fig. 2, and the slight difference in initial conditions leads the system to different trajectories.

One way to show the correlation between the parameter γ and the transition of the system to chaos is the bifurcation diagram, [6]. The construction of this diagram is presented in detail in the following paragraph.

4 Bifurcation

A bifurcation diagram presents the evolution of the motion of $\theta(t)$ as the driving strength γ changes. Our chosen values in this study are $1.03 \leq \gamma \leq 2.15$. A large number of values of γ evenly spaced, by an interval of $\Delta\gamma = 0.0001$ could be taken using the numpy function `arange` as follows: `numpy.arange(1.03,2.15,1e-5)`. By executing this command, we get a matrix of 112000 elements for the driving strength γ. Continuing our analysis, we are solving the equations describing the system's behaviour (4) for each value of γ presented in the matrix, with the same parameters and initial conditions given in the previous paragraph.

The time for simulation is from 0 to 500 seconds; the initial conditions are $\theta = -\pi/2$ and $\dot{\theta} = 0$. We choose the last 100 seconds to sample the angle θ that the oscillator is at. Sampling is done at the same frequency as the frequency of the external force $\omega_D = 2\pi$, so $T_s = 1s$. The calculated values of θ for each of the values of γ are stored in the matrix `biff`, and are presented in Fig. 4. The time needed to run the code is approximately 5 hours on a laptop. The code is publicly available and can be found here, [21].

When the system has periodic behaviour, sampling at times equal to its period will give us the same value. So, only one value will appear for a given value of γ in the bifurcation diagram. This occurs for small values of γ from $\gamma = 1.03$ to $\gamma = 1.06631$. After the value $\gamma = 1.06631$, the system displays two points for each value of γ. This is referred to in the literature as bifurcation, and the system has doubled its period. The period doubles again for $\gamma = 1.07951$, where we can get four different points for θ. The next bifurcation occurs when $\gamma = 1.08268$. In more detail, our plot shows the transition to chaos than the work done before, [22].

For constructing the bifurcation diagram, we do not require the pendulum's motion to be limited in space. We aim to study when the system goes into chaotic behaviour for a more extensive range of values $-\pi \leq \theta \leq \pi$ of the parameter g. We believe that changing the angle's values affects the emergence

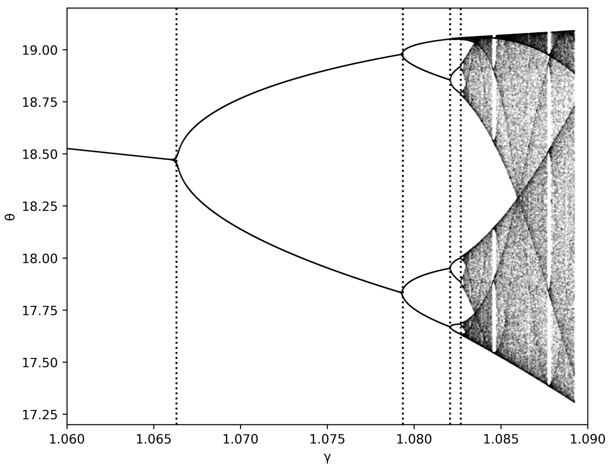

Fig. 4: Bifurcation diagram for $1.06 \leq \gamma \leq 1.09$. The period-doubling occurs for value γ equal to 1.06631, 1.07935, 1.08207 and 1.08268.

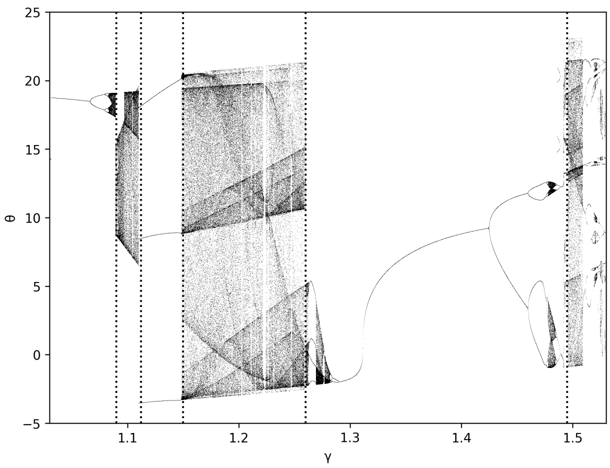

Fig. 5: Bifurcation diagram for $1.03 \leq \gamma \leq 1.53$.

of the system's behaviour and, in general, its chaotic operation.

In Fig. 5, the bifurcation diagram is present for a wide range of values of the parameter g where $1.03 \leq \gamma \leq 1.53$. For $1.26 \leq \gamma \leq 1.45$ we have a long stretch of period 1, followed by another period-doubling cascade and the final section which is primarily chaotic. What you notice is that in this diagram, the dynamics of the system are the same compared to diagram 12.18 form, [6].

In Fig. 6, after the chaotic section $1.495 \leq \gamma \leq 1.55$, the system exhibits periodic behaviour for the values $1.54 \leq \gamma \leq 1.68$.

The Fig. 7 presents the bifurcation diagram when drive strength $1.65 \leq \gamma \leq 2.15$. Here, the chaotic behaviour is present for values $1.68 \leq \gamma \leq 1.815$. The system exhibits periodic behaviour

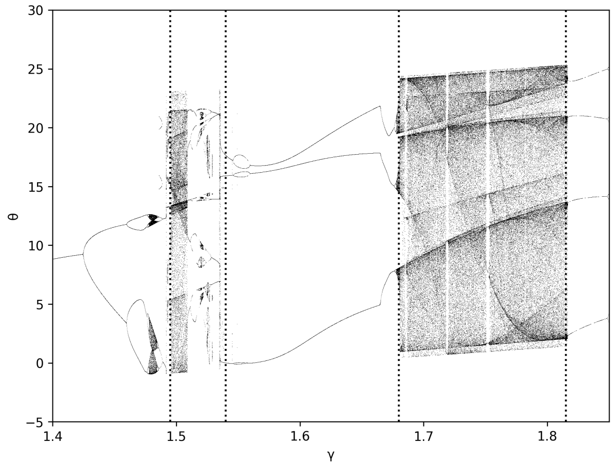

Fig. 6: Bifurcation diagram for $1.4 \leq \gamma \leq 1.85$.

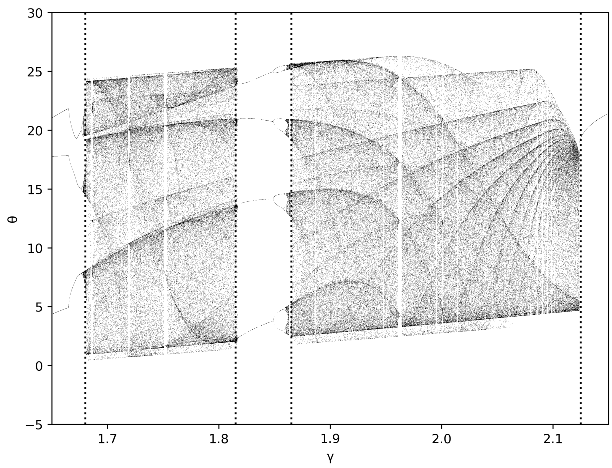

Fig. 7: Bifurcation diagram for $1.65 \leq \gamma \leq 2.15$.

between $1.815 \leq \gamma \leq 1.865$. The transition to chaos is done with a period-doubling cascade, as seen in Fig. 4. The chaotic regime is present for values $1.865 \leq \gamma \leq 2.125$. After that, the system became again periodic.

5 Conclusion

In this study, we have created the bifurcation diagram for the damped-driven pendulum, focusing on the transition from periodic to chaotic behavior. The bifurcation diagram was contracted over extended time periods. The results give a detailed bifurcation diagram, where the complex behavior of the system changes between periodic and chaotic regimes as the drive strength parameter, γ, is varied.

When the drive strength value γ is low, the system shows periodic behavior that is characterized by a single attractor that is not affected by the initial conditions of the system. As γ increases, a series of period-doubling bifurcations are observed, which are leading to the chaotic behavior. The transitions from periodic to chaotic regimes are presented in the bifurcation diagram, with the values of drive strength for which this occurs. For values of γ smaller than 1.06631, the system is periodic. The first period doubling occurs when $\gamma = 1.07935$ and the next at $\gamma = 1.0820$. For values of $\gamma > 1.08268$, the system enters into chaotic behavior.

The investigation of chaos generated by a damped-driven pendulum could be considered a case study with implications in various fields, like chaos-based cryptography for applications in secure communications. The investigation of conditions where chaos emerges in this system gives us potential for controlling chaotic systems through adjusting the parameters precisely.

The complex dynamical systems could be effectively analyzed with the Python programming language and libraries like SciPy for scientific computation and Matplotlib for visualization of results. Our approach facilitated the exploration of the pendulum's dynamics and underscored the usability of these methods and in other systems. The code is available publicly available, and it can be found on github, [21].

Overall, in this paper we have presented a methodology for studying the transitions of the system from periodic to chaotic behavior of a nonlinear system using the bifurcation diagram. Future work will extend this analysis to explore the effects of additional parameters, such as changing the damping parameter. Similar approaches could be applied in other nonlinear dynamic systems.

References:

[1] W. Szemplinska-Stupnicka, *Chaos, bifurcations and fractals around us: a brief introduction*, vol. 47. World Scientific, 2003.

[2] M. Hanias, I. Giannis, and G. Tombras, "Chaotic operation by a single transistor circuit in the reverse active region," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 20, no. 1, 2010.

[3] A. A. Tsonis, *Chaos: from theory to applications*. Springer Science & Business Media, 2012.

[4] A. C. Luo and Y. Guo, "Periodic motions to chaos in pendulum," *International Journal of Bifurcation and Chaos*, vol. 26, no. 09, p. 1650159, 2016.

[5] H. E. Nusse, J. A. Yorke, and E. J. Kostelich, *Bifurcation Diagrams*, pp. 229–268. New York, NY: Springer US, 1994.

[6] T. T. Taylor, *Mechanics: classical and quantum*. Elsevier, 2016.

[7] S. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*. CRC Press, 2018.

[8] S. C. Phatak and S. S. Rao, "Logistic map: A possible random-number generator," *Physical review E*, vol. 51, no. 4, p. 3670, 1995.

[9] E. N. Lorenz and K. Haman, "The essence of chaos," *Pure and Applied Geophysics*, vol. 147, no. 3, pp. 598–599, 1996.

[10] G. Yakubu, P. Olejnik, and J. Awrejcewicz, "On the modeling and simulation of variable-length pendulum systems: a review," *Archives of Computational Methods in Engineering*, vol. 29, no. 4, pp. 2397–2415, 2022.

[11] H. P. Langtangen and A. Logg, *Solving PDEs in python: the FEniCS tutorial I*. Springer Nature, 2017.

[12] S. Lynch, *Dynamical systems with applications using python*. Springer, 2018.

[13] J. Jani, "Simulation of chaotic operation of a damped driven pendulum using python," *WSEAS Transactions on Advances in Engineering Education*, vol. 20, pp. 1–6, 2023.

[14] N. Charalampidis, C. Volos, L. Moysis, H. E. Nistazakis, and I. Stouboulos, "A novel piecewise chaotic map for image encryption," in *2022 11th International Conference on Modern Circuits and Systems Technologies (MOCAST)*, pp. 1–4, IEEE, 2022.

[15] K. Johannessen, "An analytical solution to the equation of motion for the damped nonlinear pendulum," *European Journal of Physics*, vol. 35, p. 035014, mar 2014.

[16] S. A. E.-T. Wedad Albalawi, Alvaro H. Salas and A. A. A.-R. Youssef, "Approximate analytical and numerical solutions to the damped pendulum oscillator: Newton–raphson and moving boundary methods," *Journal of Taibah University for Science*, vol. 15, no. 1, pp. 479–485, 2021.

[17] P. Virtanen, R. Gommers, T. E. Oliphant, M. Haberland, T. Reddy, D. Cournapeau, E. Burovski, P. Peterson, W. Weckesser, J. Bright, *et al.*, "Scipy 1.0: fundamental algorithms for scientific computing in python,"

Nature methods, vol. 17, no. 3, pp. 261–272, 2020.

[18] J. D. Hunter, "Matplotlib: A 2d graphics environment," *Computing in science & engineering*, vol. 9, no. 3, pp. 90–95, 2007.

[19] K. N. Anagnostopoulos, *Computational Physics, Vol I: A Practical Introduction to Computational Physics and Scientific Computing*. Konstantinos Anagnostopoulos, 2014.

[20] J. Jani and P. Malkaj, "Numerical calculation of lyapunov exponents in various nonlinear chaotic systems," *International Journal of Scientific & Technology Research*, vol. 3, no. 7, pp. 87–90, 2014.

[21] J. Jani, "Bifurcation of the damped driven pendulum with python." `https://github.com/igiann/Bifurcation_DDP_Python.git`, 2024.

[22] P. Kurkarni and V. Borkar, "Oscillations in damped driven pendulum: A chaotic system," *International Journal of Scientific and Innovative Mathematical Research (IJSIMR)*, vol. 3, no. 10, pp. 14–27, 2015.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from formulating the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself
No funding was received for conducting this study.

Conflicts of Interest
The authors have no conflicts of interest to declare that are relevant to the content of this article.