Local Splines and the Least Squares Method

I. G. BUROVA
The Department of Computational Mathematics,
St. Petersburg State University,
7-8 Universitetskaya Embankment, St.Petersburg,
RUSSIA

Abstract: - The least squares method is widely used in processing quantitative results of natural science experiments, technical data, astronomical and geodetic observations and measurements. This paper proposes the construction of a modified least squares method based on the use of basis splines of a non-zero level. This modification allows us to obtain a continuously differentiable (required number of times) solution to the problem. The resulting solution is convenient to use to further solve other related problems. The construction of a continuously differentiable solution and a twice continuously differentiable solution is considered in more detail. These solutions are constructed based on the use of basis Hermitian splines of the fourth and sixth orders of approximation. The numerical results are presented for processing inaccurately specified experimental data, as well as for smoothing curves.

Key-Words: - polynomial splines, cubic splines, linear splines, least squares method, Hermitian splines, Lagrangian splines, fourth order approximation, second order approximation.

Received: January 25, 2024. Revised: March 2, 2024. Accepted: May 3, 2024. Published: June 17, 2024.

1 Introduction

As is known, if a large number of interpolation nodes is unsuccessfully selected, the approximation error can be catastrophically large. Improving the quality of approximation can be achieved in different ways. One of these methods is the least squares method.

The specifics of applying the least squares method are described in many books, [1]. In a lot of research papers the least squares method is used to solve various practical problems, for example, channel estimation in wireless systems, [2]. As is noted in this paper, “accurate channel estimation is the key factor to improving the receiver quality”.

Based on the least squares method, various algorithms may be constructed. For example, in paper [3], a least mean square based algorithm is proposed to estimate the phase noise.

In paper [4], it was shown that through the partial least square equation modeling, the big data analytics capability affects the organizational agility and firm performance positively. In paper [5], the robust filtering problem is introduced based on the minimization of the mean-square value of the filtering errors for the system states. Paper [6], discovers a link between an insurance company's initial capital and the likelihood of ruin. The least squares regression method is utilized to calculate the minimum initial capital, and the simulation approach would be used to determine the ruin probability.

Many authors use splines and least squares method in their research, [7]. The B-spline is often used when we solve such problems. Note, that papers [8], [9] use B-splines.

The least squares minimization method is involved in study [10] to calculate control points to minimize the approximation error.

The least squares method and B-splines were used in [11] to solve a nonlinear problem in fluid dynamics using the Navier–Stokes equations as a mathematical model.

The purpose of the paper [12] is to show the results of a study focused on the occurrence of damage heterogeneous materials, especially on the issue of modelling crack formation and propagation.

In paper [13], the authors estimated the spatial pattern and temporal trend of the resilience of subtropical evergreen forests in China.

Paper [14], presents an original approach to generate a 2D high detail riverbed based on a drone photogrammetric survey.

Monograph [15], discusses the use of quadratic splines for constructing a mean-square approximation.

In this paper**,** we use the local splines of the Lagrangian splines and the Hermitian splines to construct a mean-square approximation.

1.1 Some Information

In various practical cases we are often given grid (tabular) functions $f(x_j), j = 0,1, \dots n,$ at the equidistant set of nodes $\{x_j\}$. So, for example, the grid functions are obtained during experimental studies. The grid functions are often specified when designing structural elements. So, at the grid nodes $x_j, j = 0,1, \dots, n,$ the values of the function $f(x_j)$ are known. But sometimes in addition to the values of the function $f(x_j)$, the values of the first derivative of this function at the nodes, are known. And it is also possible that in addition to the values of the function $f(x_j)$, the values of the first and the second derivatives of this function at the nodes, are known. It is required to carry out a continuous or continuously differentiable function passing through the given points.

Very often, the input data has measurement errors. In this case, it is required to draw a continuous or continuously differentiable function passing close to the given points.

Different smoothing methods are known. Smoothing methods are used if the input data has measurement errors. Typically, two main methods are used to solve this problem: the least squares method (best mean square approximation method) and the best integral approximation method.

In this case, the algebraic polynomials

$$\boldsymbol{\tilde{f}(x, a) = \sum_{j=0}^{m} a_j x^j}, \qquad \boldsymbol{m \le n},$$

where a_j are coefficients, $a = (a_0, \dots, a_n)$, or the generalized polynomials

$$\tilde{f}(x, a) = \sum_{j=0}^{m} a_j \varphi_j(x)$$

are often used. We can take $\varphi_j(x)$ as the algebraic polynomials. In the case $a_j = f(x_j)$, $j = 0,1, \dots, n,$ we have the Lagrange interpolation. When constructing smoothing approximations, we use one of the following conditions**:**

$$\sqrt{\frac{1}{n+1} \sum_{i=0}^{n} \left(\tilde{f}(x_i, a) - f(x_i)\right)^2} \to \min_a, \quad (1)$$

or

$$\int_{x_j}^{x_{j+1}} \tilde{f}(x, a) dx - I_j^{j+1} = 0,$$

where

$$I_j^{j+1} = \int_{x_j}^{x_{j+1}} f(x) dx.$$

The least squares method is a variational method. Variational methods are based on the fact that the problem of solving an equation is reduced to the problem of finding the minimum of a certain function.

The minimum in (1) is achieved if

$$\delta = \sum_{i=0}^{n} \left(\tilde{f}(x_i, a) - f(x_i)\right)^2 \to \min_a$$

The extremum conditions are as follows

$$\frac{\partial \delta}{\partial a_j} = 0, j = 0,1, \dots, m.$$

As a result, we obtain a system of equations. Using the scalar product

$$\left(\varphi_k, \varphi_j\right) = \sum_{i=0}^{n} \varphi_k(x_i)\, \varphi_j(x_i),$$

we write the system in the form

$$(\varphi_k, \varphi_0) a_0 + (\varphi_k, \varphi_1) a_1 + \cdots + (\varphi_k, \varphi_m) a_m = (f, \varphi_k),\ k = 0,1, \dots, m,$$

where

$$(f, \varphi_k) = \sum_{i=0}^{n} f(x_i)\, \varphi_k(x_i)\,.$$

The determinant of the system is symmetric. If the basis functions are linearly independent, then the determinant of the system is not equal to zero. Therefore, the solution to the system exists and is unique. This determinant is called the Gram determinant.

In this paper, we consider the use of local spline approximations to construct smoothing curves. This problem arises, for example, if the input data is given with errors or we want to construct a smooth line using a given line. In the case of using local splines of a non-zero level, we can also obtain the restoration and smoothing of the derivatives. The results of using continuously differentiable splines of the first level, as well as twice continuously differentiable splines of the second level are used. Section 3 presents the results of the numerical experiments.

1.2 Lagrangian Splines and Hermitian Splines

First, we discuss splines of a zero level (Lagrangian splines) and a non-zero level (Hermitian splines).
Let an ordered grid of nodes be constructed on the interval $[a, b]$

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Local spline approximations on each grid interval $[x_k, x_{k+1}]$ are constructed as the sum of the products of the values of the function and its derivatives at the grid nodes and the basis splines of a non-zero level. By level we mean the maximum value of the derivatives of the functions included in the approximation. Splines of the non-zero level can have the basis splines with narrow and wide support, [16]. The supports of the basis splines with narrow support occupy two grid intervals. In paper [16], spline approximations $\tilde{u}(x)$ with a narrow support of the following form were considered

$$\tilde{u}(x) = \sum_{j=k,k+1} \sum_{\alpha=0}^{s} u_j^{(\alpha)} w_{j,\alpha}(x), x \in [x_k, x_{k+1}]$$

We determine the basis splines from the relations
$\tilde{u}(x) = u(x), u = x^k,\ k = 0,1,2, \dots m,$
$x \in [x_k, x_{k+1}]$.
In this case, the approximation is said to be exact on polynomials of degree at most m. The approximation theorem is proved, [17]:

$$|\, \tilde{u}(x) - u(x)| \le K h^{m+1},$$

here K is a constant, and $m + 1$ is called the order of approximation. A distinctive feature of these splines is that their construction does not require solving systems of linear algebraic equations. Therefore, the construction of approximations can be carried out in real time on an infinite grid of nodes and on a finite grid of nodes on the interval $[a, b]$.

$$\tilde{u}(x) = u(x), u = x^k,\ k = 0,1,2, \dots m,\quad x \in [x_k, x_{k+1}].$$

The basis splines $w_{j,\alpha}(x)$ turn out to be α times continuously differentiable local functions and satisfy the conditions $w_{j,\alpha}^{(\beta)}(x_k) = \delta_{jk}\delta_{\alpha\beta}$. Here δ_{jk}, $\delta_{\alpha\beta}$ the Kronecker symbols. Thus, the piecewise given function $\widetilde{U}(x)$ coinciding on each grid interval with the function $\tilde{u}(x)$, turns out to be a Hermite interpolation, interpolating the function u at the grid nodes. As is known, if a large number of interpolation nodes is unsuccessfully selected, the approximation error can be catastrophically large. Improving the quality of approximation can be achieved in different ways.

2 Smoothing Methods Construction

Suppose, the values of the function f is given at the nodes $\{x_j\}$. In the simplest case, the grid of nodes $\{x_j\}$ is given with step $h = (b - a)/n$.

$$a = x_0 < \cdots < x_{j-1} < x_j < x_{j+1} < \cdots < x_n = b.$$

Suppose, experimental data $f(x_j)$ has measurement errors ε_j:

$$f(x_j) = u(x_j) + \varepsilon_j.$$

We construct the smoothing curve in two stages. First, we connect the points with a piecewise line using the basis splines of the second order of approximation

$$w_j(x) = \frac{x - x_{j+1}}{x_j - x_{j+1}},$$
$$w_{j+1}(x) = \frac{x - x_j}{x_{j+1} - x_j}.$$

On each interval $[x_j, x_{j+1}]$ we construct $\tilde{u}^{<j>}(x)$:

$$\tilde{u}^{<j>}(x) = f(x_j) w_j(x) + f(x_{j+1}) w_{j+1}(x).$$

If the function $u \in C^2[x_{k-1}, x_{k+1}]$, then the estimate is valid

$$|\, \tilde{u}(x) - u(x)| \le \frac{h^2}{8} \| \ u^{(2)} \ \|_{[x_{k-1}, x_{k+1}]}.$$

Here $h = x_{k+1} - x_k$.

Now we get the piecewise function $U(x)$, $x \in [x_0, x_n]$:

$$U(x) = \begin{cases} \tilde{u}^{<0>}(x), & x \in [x_0, x_1] \\ & \dots \\ \tilde{u}^{<n-1>}(x), & x \in [x_{n-1}, x_n] \end{cases}$$

The error of approximation with such splines in the general case is given in the author's paper, [3].

Example 1. Let $n = 50$. The values of the function u at nodes x_j are given as shown in Figure 1.

When the points using second-order approximation splines are connected, we get the continuous piecewise function (Figure 2)

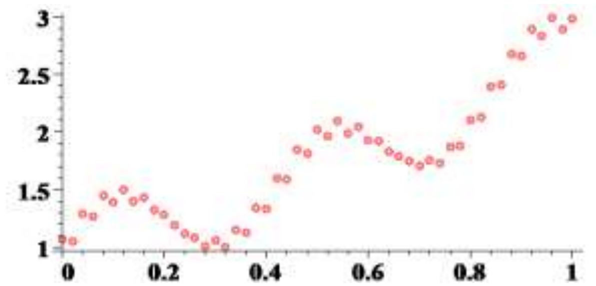

Fig. 1: The values of function f

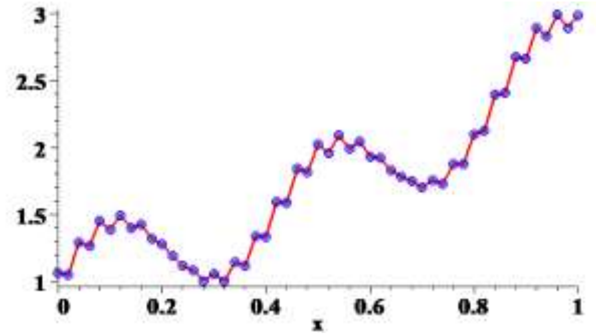

Fig. 2: The continuous piecewise function after connecting the points using the second-order approximation splines

2.1 Construction of Basis Splines of the Fourth Order of Approximation of the First Level

Let a sparse grid of nodes $X_k, k = 0,1, \dots N,\ N < n$, also be given as follows

$$\dots < X_{k-1} < x_{j-1} < x_j < x_{j+1} = X_k < x_{j+2} < \cdots$$

Let us assume that at each grid node X_k the values of the function $u(x)$ and its first derivative are known.

In this case, we construct an approximation of function $u(x)$, $x \in [X_k, X_{k+1}]$, in the form

$$\tilde{u}(x) = \sum_{j=k,k+1} u(x_j)\, \omega_{j,0}(x) + u'(x_j)\omega_{j,1}(x).$$

We assume that supp $\omega_{j,0}$=supp $\omega_{j,1} = [X_{k-1}, X_{k+1}]$. Let the basis functions be determined from the conditions

$$u \equiv \tilde{u}, u = 1, x, x^2, x^3.$$

From these conditions on the interval $[X_k, X_{k+1}]$ we obtain a system of equations for determining the basis functions

$$\omega_{j,0}(x) + \omega_{j,1}(x) = 1\,,$$
$$x_j\omega_{j,0}(x) + x_{j+1}\omega_{j+1,0}(x) + \omega_{j,1}(x) + \omega_{j+1,1}(x) = x\,,$$
$$x_j^2\ \omega_{j,0}(x) + x_{j+1}^2\omega_{j+1,0}(x) + 2\,x_j\ \omega_{j,1}(x) + 2\,x_{j+1}\ \omega_{j+1,1}(x) = x^2\,,$$
$$x_j^3\ \omega_{j,0}(x) + x_{j+1}^3\omega_{j+1,0}(x) + 3\,x_j^2\ \omega_{j,1}(x) + 3\,x_{j+1}^2\ \omega_{j+1,1}(x) = x^3\,.$$

The basis splines of the first level have the form

$$\omega_{j,0}(x) = \frac{(x - x_{j+1})^2}{(x_{j+1} - x_j)^2} + \frac{2(x - x_j)(x - x_{j+1})^2}{(x_{j+1} - x_j)^3},$$
$$\omega_{j+1,0}(x) = \frac{(x - x_j)^2}{(x_{j+1} - x_j)^2} + \frac{2(x_{j+1} - x)(x - x_j)^2}{(x_{j+1} - x_j)^3},$$
$$\omega_{j,1}(x) = \frac{(x - x_j)(x - x_{j+1})^2}{(x_{j+1} - x_j)^2},$$
$$\omega_{j+1,1}(x) = \frac{(x - x_{j+1})(x - x_j)^2}{(x_{j+1} - x_j)^2}.$$

Now we get formulas for the approximation function u,

$$\tilde{u}(x) = u(x_j)\omega_{j,0}(x) + u(x_{j+1})\omega_{j+1,0}(x) + u'(x_j)\omega_{j,1}(x) + u'(x_{j+1})\omega_{j+1,1}(x),$$
$$x \in [X_j, X_{j+1}].$$

If the function $u \in C^4[X_{k-1}, X_{k+1}]$, then the estimate is valid

$$|\,\tilde{u}(x) - u(x)| \leq Kh^4\ \|\ u^{(4)}\ \|_{[X_{k-1},X_{k+1}]}\,.$$

Here $h = X_{k+1} - X_k, K = 1/384$.

3 Construction of the Solution

To construct a continuously differentiable smoothing curve, we use splines of the fourth order of approximation of the first level.

To construct a twice continuously differentiable smoothing curve, we use splines of the sixth order of approximation of the second level.

3.1 Construction of the Twice Continuous Smoothing Functions

On each interval $[X_k, X_{k+1}]$, we consider the expression $\tilde{U}$ with the basis splines of the fourth order of approximation and the first level in the form

$$\tilde{U}(x) = \sum_{j=k,k+1} c_{j,0}\ \omega_{j,0}(x) + c_{j,1}\omega_{j,1}(x).$$

Next, we use the scalar production of the form

$$(\zeta,\varsigma)=\int_a^b \zeta(x)\varsigma(x)dx.$$

We find the coefficients $c_{j,i}$ by solving the system of equations

$$MC=F,$$

where M is the Gram matrix

$$M=\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}.$$

Let $m_{i,j}^k$ be an element of the matrix $M_k, k=1,2,3,4$, then

$$m_{i,j}^1=\int_a^b \omega_{i,0}(x)\omega_{j,0}(x)dx,$$
$$m_{i,j}^2=\int_a^b \omega_{i,1}(x)\omega_{j,0}(x)dx,$$
$$m_{i,j}^3=\int_a^b \omega_{i,0}(x)\omega_{j,1}(x)dx,$$
$$m_{i,j}^4=\int_a^b \omega_{i,1}(x)\omega_{j,1}(x)dx.$$

Let's construct the vector F:

$$F=(F_1,F_2)^T$$

Let f_i^k be an element of the vector F_k. Then

$$f_i^1=\int_a^b u(x)\omega_{i,0}(x)dx,$$
$$f_i^2=\int_a^b u(x)\omega_{i,1}(x)dx.$$

Let the grid nodes be equally spaced, that is, $h=x_{j+1}-x_j$. Then it is easy to calculate the elements of the matrix M. Matrix M consists of 4 block matrices: M_1, M_2, M_3, M_4.

The non-zero elements of the matrix M_1 are as follows**:**

$$m_{1,1}^1=\frac{13}{35}h,$$
$$m_{n,n}^1=\frac{13}{35}h,$$
$$m_{j,j}^1=\frac{26}{35}h,$$
$$m_{j,j+1}^1=m_{j,j-1}^1=\frac{9}{70}h.$$

The remaining elements in this block are zeros. The non-zero elements of the matrix M_2 are as follows**:**

$$m_{1,1}^2=\frac{11}{210}h^2$$
$$m_{n,n}^2=\frac{-11}{210}h^2$$
$$m_{j,j}^2=0,$$
$$m_{j,j+1}^2=m_{j,j-1}^1=\frac{-13}{420}h^2 .$$

The remaining elements in this block are zeros.

The M_3 matrix is as follows $M_3=M_2^2$.

The non-zero elements of the matrix M_4 are as follows**:**

$$m_{1,1}^4=\frac{1}{105}h^3,$$
$$m_{n,n}^4=\frac{1}{105}h^3,$$
$$m_{j,j}^4=\frac{2}{105}h^3,$$
$$m_{j,j+1}^4=m_{j,j-1}^1=\frac{-1}{140}h^3 .$$

The remaining elements are zero. Thus, all blocks of the M matrix have a tridiagonal structure with identical elements on the diagonals (except for the corner elements).

When $N=5$**,** we get the values of the function at the nodes (blue points) in Figure 3 and Figure 4. In Figure 3 we also see the initial data (red points).

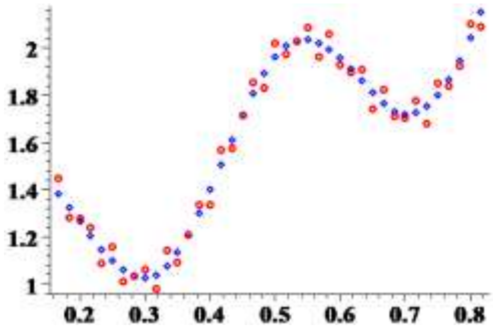

Fig. 3: The values of the function at the nodes (blue points) and the initial data (red points)

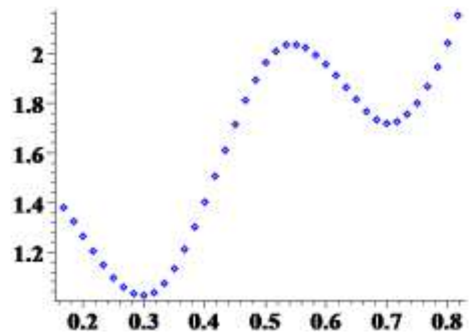

Fig. 4: The values of the function at the nodes (blue points)

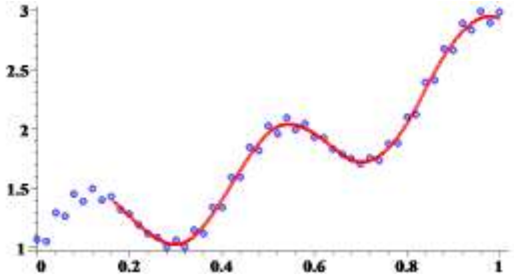

Fig. 5: The values of function u (red line) and the initial data (blue points)

When the found points with splines of the fourth order of approximation of the first level are connected, we obtain a continuously differentiable function, shown in Figure 5 with the red line.

If the original function is given by some expression, then this same method allows us to easily construct a continuously differentiable smoothing curve.

Example 2. Let the function $y(x)$ be given by the expression

$$y(x) = \exp(x) + (\tfrac{1}{3})\sin(15x) + (\tfrac{1}{15})\cos(150x)$$

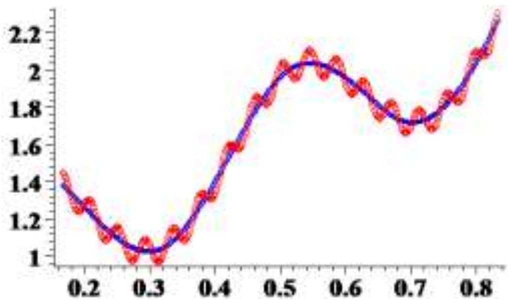

Fig. 6: The values of the result smooth function (blue line) and the initial line (red line)

The initial line and the result smooth function are shown in Figure 6.

3.2 Construction of the Twice Continuous Smoothing Functions

Let the function be such that $u \in C^6[a,b]$ and $\{x_j\}$ be the set of nodes on $[a,b]$, such that $x_0 = a$, $x_n = a$.

Suppose that the support of the basis spline $\omega_{j,i}$ occupies two grid intervals supp $\omega_{j,i} = [x_{j-1}, x_{j+1}]$, $i = 0, 1, 2$.

For $x \in [x_j, x_{j+1}]$, consider the approximation

$$\tilde{u}(x) = u(x_j)\omega_{j,0}(x) + u(x_{j+1})\omega_{j+1,0}(x) + u'(x_j)\omega_{j,1}(x) + u'(x_{j+1})\omega_{j+1,1}(x) + u''(x_j)\omega_{j,2}(x) + u''(x_{j+1})\omega_{j+1,2}(x),$$
$$x \in [x_j, x_{j+1}].$$

The plots of basis splines $\omega_{j,i}$ are presented in Figure 7, Figure 8 and Figure 9.

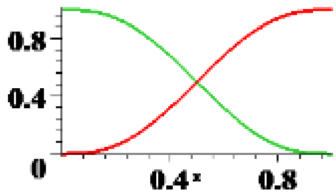

Fig. 7: The plot of the basis splines $\omega_{0,0}$ (green line) and $\omega_{1,0}$ (red line)

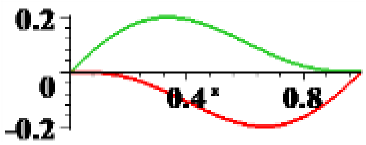

Fig. 8: The plot of the basis splines $\omega_{0,1}$ (green line) and $\omega_{1,1}$ (red line)

Fig. 9: The plot of the basis splines $\omega_{0,2}$ (green line) and $\omega_{1,2}$ (red line)

Next, our task is to determine the coefficients $C_{j,i}$ in the expression

$$\tilde{u}(x) = C_{j,0}\omega_{j,0}(x) + C_{j+1,0}\omega_{j+1,0}(x) + C_{j,1}\omega_{j,1}(x) + C_{j+1,1}\omega_{j+1,1}(x) + C_{j,2}\omega_{j,2}(x) + C_{j+1,2}\omega_{j+1,2}(x),$$
$$x \in [x_j, x_{j+1}].$$

To calculate the coefficients, we create a system of linear algebraic equations

$$MC = F\,.$$

Matrix M has a block form. It consists of 9 blocks M_i. Each block M_i has a strip structure. We calculate the elements of $M_i$using scalar products.

Example 2. Let function $y(x)$ be given by the expression

$$y(x) = \exp(x) + \left(\frac{1}{3}\right)\sin(15\,x) + \left(\frac{1}{15}\right)\cos(250\,x).$$

Let $N = 5$. The initial line and the result smooth function are shown in Figure 10.

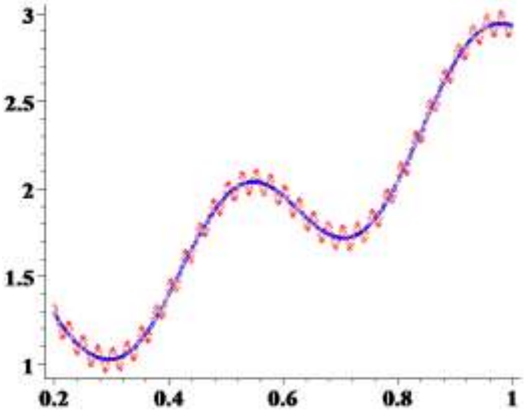

Fig. 10: The values of the result smooth function (blue line) and the initial line (red line)

4 Conclusion

In this paper, we use the local splines of the Lagrangian splines and the Hermitian splines to construct a modification of the construction of a mean-square approximation. The basis splines of a non-zero level make it possible to construct m-times continuously differentiable smoothing lines. The details of the construction of the three times continuous smoothing functions with the Hermitian splines and the approximations with the non-polynomial splines will be considered in the future.

Acknowledgement:
The authors are gratefully indebted to St. Petersburg University for their financial support in the preparation of this paper (ID 118437477), as well as to a resource center for providing the Maple package.

References:
[1] R.W.Hamming. *Numerical Methods for Scientists and Engineers,*New York, San Francisco, Toronto, London, 1962.
[2] Nisha Mary Joseph, Puttamadappa C. Highly Accurate Technique for CO-OFDM Channel Estimation Technique Using Extreme Learning Machine (ELM), *WSEAS Transactions on Electronics*, Vol. 14, 2023, pp.7-23, https://doi.org/10.37394/232017.2023.14.2.
[3] M. Bharti, Performance enhancement of coherent optical OFDM system using LMS algorithm, *J. Telecommun. Inf. Technol.*, Vol. 1, No 2020, 2020, pp. 1–5.
[4] Muhamad Luqman Khalil, Norzalita Abd Aziz, Ahmad Azmi M. Ariffin, Abdul Hafaz Ngah, Big Data Analytics Capability and Firm Performance in the Hotel Industry: The Mediating Role of Organizational Agility, *WSEAS Transactions on Business and Economics,* Vol. 20, 2023, pp. 440-453, https://doi.org/10.37394/23207.2023.20.40.
[5] Seiichi Nakamori, Robust Recursive Least-Squares Wiener Filter for Linear Continuous Time Uncertain Stochastic System, *WSEAS Transactions on Signal Processing*, Vol. 19, 2023, pp.108-117, https://doi.org/10.37394/232014.2023.19.12.
[6] Soontorn Boonta, Kanoktip Kotsamran, Comparison of the Minimum Initial Capital of Investment Discrete Time Surplus Process in Case Separated Claims, *WSEAS Transactions on Business and Economics,* Vol. 20, 2023, pp. 1260-1267, https://doi.org/10.37394/23207.2023.20.112.
[7] Betul Kan Kilinc, Semra Malkoc, A. Savas Koparal, Berna Yazici, Using multivariate adaptive regression splines to estimate pollution in soil, *International Journal of Advanced and Applied Sciences*, Vol.4, No2, 2017, pp 10-16.
[8] Juan Zapata and Ramon Ruiz, On Speckle Noise Reduction in Medical Ultrasound Images, *Recent Advances in Signals and Systems*, 2009, pp.126-131.
[9] Tetsuzo Kuragano and Kazuhiro Kasono, B-spline Curve Generation and Modification based on Specified Radius of Curvature, *WSEAS Transactions on Information Science and Applications*, Issue 11, Vol. 5, November 2008, pp.1607-1617.
[10] Nur Azliana Azlin Ahmad Munir; Normi Abdul Hadi; Mohd Agos Salim Nasir; Suhaila Abd Halim, Modelling fruit outline using cubic trigonometric spline**,** *AIP Conf. Proc.* 2905, 030006, 2024. https://doi.org/10.1063/5.0172468.
[11] Maria Luminita Scutaru, Sohaib Guendaoui, Ouadie Koubaiti, Lahcen El Ouadefli, Abdeslam El Akkad, Ahmed Elkhalfi and Sorin Vlase, Flow of Newtonian Incompressible Fluids in Square Media: Isogeometric vs. Standard Finite Element Method, *Mathematics*, 2023, Vol. 11, No 17, 3702, pp.1-14, https://doi.org/10.3390/math11173702.
[12] Vladislav Kozák, Jiří Vala, Modelling of crack formation and growth using FEM for selected structural materials at static loading, WSEAS *Transactions on Applied and Theoretical Mechanics*, Vol. 18, 2023,

pp.243-254, https://doi.org/10.37394/232011.2023.18.23.

[13] Jinghua Chen, Shaoqiang Wang, Hao Shi, Bin Chen, Junbang Wang, Chen Zheng and Kai Zhu, Radiation and temperature dominate the spatiotemporal variability in resilience of subtropical evergreen forests in China, *Front. For. Glob. Change, 17 April 2023, Sec. Forest Disturbance*, Vol. 6, 2023, pp.1-12, https://doi.org/10.3389/ffgc.2023.1166481.

[14] L. Stucchi, D. Fugazza, A. Sharifi, G. Traversa, G. Diolaiuti, D. Bocchiola, An algorithm to generate 2D bathymetry of an Alpine river for habitat suitability assessment, *Science of the Total Environment*, Vol. 918, 2024, 170703, pp.1-14.

[15] J.M.Ortega, W.G.Poole, *An introduction to numerical methods for differential equations*, London, 1981.

[16] S. G. Mikhlin, Variational-grid approximation. *Notes of scientific seminars of the LOMI Academy of Sciences of the USSR*. 48, 1974, pp. 32–188. (In Russian)

[17] I.G. Burova, On left integro-differential splines and Cauchy problem, *International Journal of Mathematical Models and Methods in Applied Sciences*, Vol.9, 2015, pp. 683-690.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

I.G. Burova was responsible for ideas, conducting a research and investigation process, specifically performing the experiments.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

The authors are gratefully indebted to St. Petersburg University for their financial support in the preparation of this paper (ID 118437477), as well as to a resource center for providing the package Maple.

Conflict of Interest

The authors have no conflicts of interest to declare.