On Period Annuli and Induced Chaos

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Abstract: - Nontrivial period annuli in the second order ordinary differential equation are continua of periodic trajectories that contain inside more than one critical point. They can appear in conservative equations, which are known to have no attractors. Nevertheless, according to some authors, their behavior may be done chaotic by adding a periodic external force. Is the period of the external force correlated with periods of solutions in period annuli? Is the chaotic behavior of a solution dependent on the initial value and, in turn, on a certain periodic annulus? These, and related questions are studied in the article.

Key-Words: - Differential equations, oscillation, period annuli, sensitive dependence, chaotic behavior, Lyapunov exponents.

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1 Introduction

The second order ordinary differential equations of the Newtonian form $x'' + g(x) = 0$ can have multiple period annuli. Period annuli are understood as continua of periodic solutions. The trivial example is a central region in a phase plane, associated with a critical point of the center type. The harmonic equation $x'' + \omega^2 x = 0$ is an evident example. But there may be regions filled with closed trajectories and surrounding more than one critical point. An example of this can be provided by the equation (1) with the phase plane as in Figure 1. Imagine that *g(x)* is an odd degree polynomial and the primitive *G(x)* has graph resembling the mountain range like in Figure 2. If there is a pair of mountains containing other (lower) mountains between them, then the period annuli appear. The periodic trajectories in these annuli have an interesting property. The periods of trajectories, locating close to the borders of a period annulus, are very long (tending to infinity) and the graph representing periods, has U-shape. A solution with a minimal period exists in any such period annulus. It is known that some of Duffing-type equations when excited by adding a periodic term on the right side of the equation can exhibit chaotic behavior. An autonomous Duffing type equation is the second order one, and it cannot be chaotic due to the Poincaré-Bendixson theory. However, there is no contradiction. The Duffing equation with an external force on the right side can be written in an equivalent form as a system of three ordinary differential equations. Such systems can be chaotic. The classical examples are the Lorenz and Rössler attractors. Examples and general information can be found in textbooks on differential equations, [1], [2], [3], [4], [5]. Do equations of the form $x'' + g(x) = 0$ with multiple period annuli possess the same property? We wish to gather information on this subject in this article.

For this, we recall the article by the authors with some examples. Consider first the equation:

$$x'' - (x+4)(x+2.5)(x+1.5)x(x-2)(x-3.6)(x-5) = 0 \quad (1)$$

This equation has period annuli described in Figure 1.

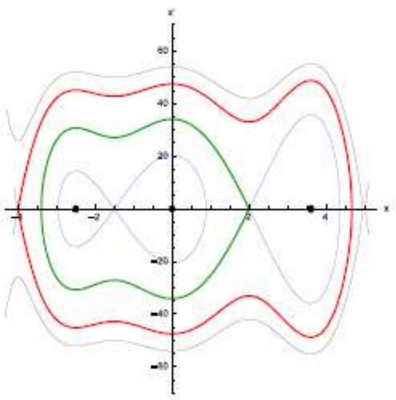

Fig. 1: Period annuli in equation (1). The borders are in green and red

This equation was investigated in the work, [6].
Introduce the notation:

$$g(x) = -(x+4)(x+2.5)(x+1.5)x(x-2)(x-3.6)(x-5) \quad (2)$$

and add a function $f(t)$ on the right, so the equation becomes:

$$x'' + g(x) = f(t) \quad (3)$$

The right side can be interpreted as an external force, which can be periodic, $f(t) = h\cos\omega t$.

Write this equation as the system:

$$\begin{cases} x' = y \\ y' = -g(x) + f(z) \\ z' = \omega \end{cases} \quad (4)$$

It is known that equations of the form (3) can exhibit chaotic behavior (Duffing equation, for instance). Our aim in this paper is to show, that the second order equations of the form (3) can be chaotic, provided that the shortened equation has period annuli.

$$x'' + g(x) = 0 \quad (5)$$

Remark, [7], [8]. A central region is the largest connected region covered with cycles surrounding the center. A period annulus is a connected region covered with concentric cycles. A period annulus associated with a central region is called a trivial period annulus. It contains a single critical point of the type center. Period annuli containing more than one critical point will be called nontrivial period annuli. The phase portrait depicted in Figure 1 contains exactly three trivial period annuli and two nontrivial period annuli.

We mention the following result, which allows easily constructing examples of equations with multiple period annuli, [6].

Assume that $g(x)$ is a polynomial with simple zeros. The primitive G(x)=$\int_0^t g(x)dx$ may have multiple maxima. It is easy to observe that the equivalent system $x'=y$, $y'=-g(x)$ has centers at the point $(m_i,0)$ and saddle points at $(M_j,0)$, where m_i and M_j are points of local minima and maxima respectively. In that case centers and saddles alternate. The following assertion is true.

Theorem. *Let M_1 and M_2 $(M_1<M_2)$ be non-neighboring points of maxima for the primitive $G(x)$. Suppose that $G(x)<\min\{G(M_1); G(M_2)\}$ for any $x\in(M_1, M_2)$.*

To illustrate this, the primitive $G(x)$ of the function $g(x)$ in (2) is depicted in Figure 2. There are four maxima. The first and the third maxima generate a nontrivial period annulus, which, in turn, is included in a greater one, generated by two side maxima.

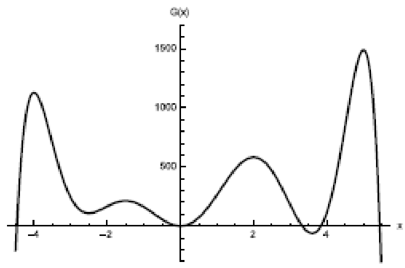

Fig. 2: Primitive G(x) of the function g(x), defined in (2). Each maximum point corresponds to a critical point of the type center

2 Chaotic Behavior

Chaotic behavior in dynamical systems is studied extensively. There are multiple definitions and criteria for chaos. One of the essential signs of chaotic behavior is the sensitive dependence of solutions on the initial data. To detect sensitive dependence, computational tools are widely used. Let us mention the mathematical apparatus, based on the Lyapunov exponents. The Lyapunov exponents reflect the evolution of the difference between two initially close solutions of a dynamical system. These differences for the three-dimensional systems, projected on the coordinate axis, result in three Lyapunov curves. If at least one of these curves is positive, this is a sign of sensitive dependence.

More on chaotic behavior in systems of ordinary differential equations and Lyapunov exponents can be learned from the book, [9] and the article [10], as well as from many other relevant sources. The chaotic behavior of trajectories may be observed near attractors. An extract from Table 1.1 in the book, [9], (Table S in a current text) shows the relation between types of attractors and the Lyapunov exponents.

Table S.

λ_1	λ_2	λ_3	Attractor	Dimension	Dynamic
–	–	–	Equilibrium	0	Static
0	–	–	Limit cycle	1	Periodic
+	0	–	Strange	2 or 3	Chaotic

Numbers λ_i , $i = 1{,}2$, indicate locations of the Lyapunov curves as positive (+), lying on the horizontal zero-axis, and negative (–). In the same book, [9], the equation:

$$x'' + \sin x = \sin t \tag{6}$$

was considered, which was called conservative meaning that it does not contain a damping term. This equation was shown to have chaotic behavior (chaotic sea with islands of periodicity). We will do the same for equations with period annuli. An extra question we wish to address is how the chaotic behavior depends on the initial conditions and the structure and number of period annuli.

3 Results

Example 1.

Consider equation:

$$x'' + g(x) = h \cos \omega t, \tag{7}$$

where $g(x)$ is given:

$$g(x) = -(x + 2.4)(x + 1.8)(x + 1)x(x - 0.8)(x - 1.7)(x - 2.4), \tag{8}$$

h is a parameter, ω is a coefficient, $h, k > 0$. The primitive G(x) is depicted in Figure 3 and the phase plane is shown in Figure 4. This equation (7) is equivalent to the system:

$$\begin{cases} x' = y \\ y' = -g(x) + h \cos z \\ z' = \omega. \end{cases} \tag{9}$$

We will examine system (9) and show that it can be chaotic.

Consider system (9), provided $h = 1$, $\omega = 3.2$, $\big(x(0), y(0), z(0)\big) = (1, 0, 0.4)$. Recall that in a scalar form, the equation (7) with zero right side has three trivial period annuli, and two nontrivial ones.

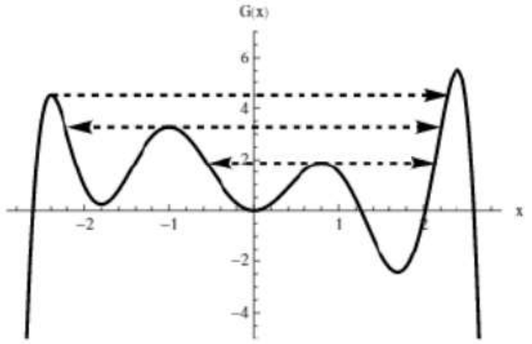

Fig. 3: The primitive G(x) of g(x) above

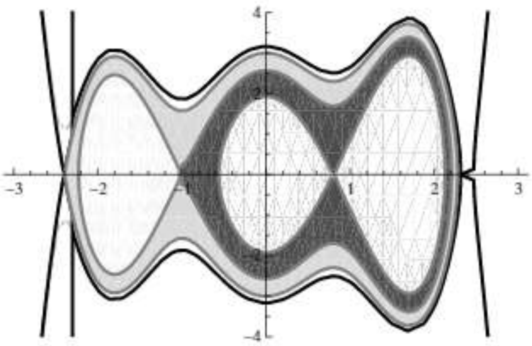

Fig. 4: Three nontrivial period annuli in the equation $x'' + g(x) = 0$, where g(x) is as in (8)

Figure 5, Figure 6 and Figure 7 respectively reflect the phase portrait, graphs of solutions and Lyapunov curves for the perturbed system (9).

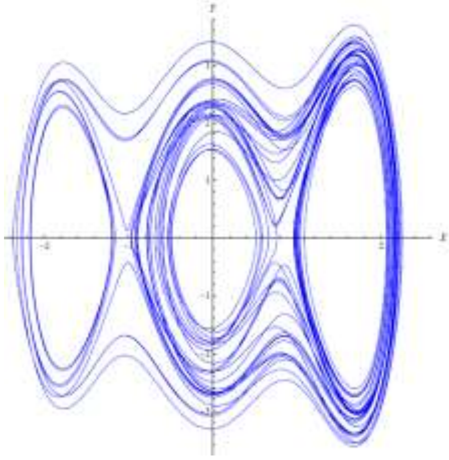

Fig. 5: Phase portrait for the system (9), h=1, ω =3.2, (x(0),y(0),z(0)) = (1, 0, 0.4)

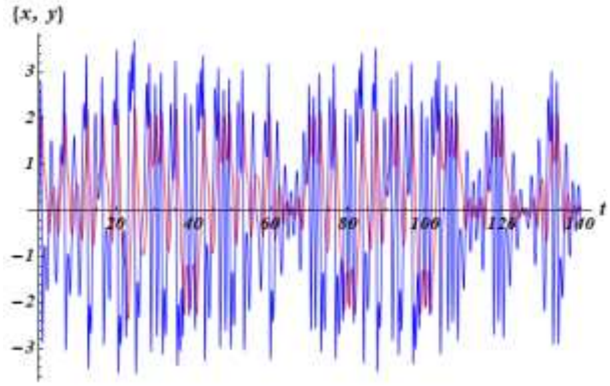

Fig. 6: Solutions of the system (9), h=1, ω =3.2, (x(0),y(0),z(0)) = (1, 0, 0.4)

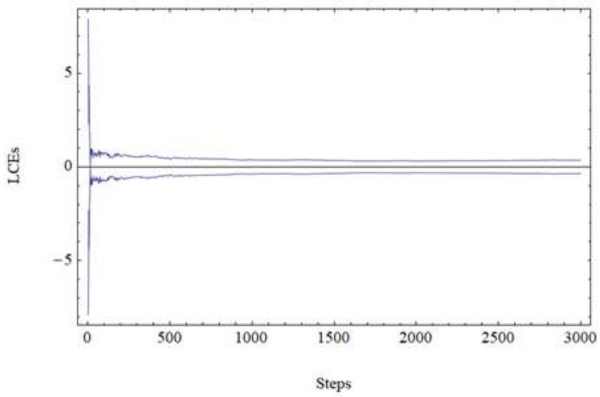

Fig. 7: Lyapunov curves for the system (9), h=1, ω=3.2, (x(0),y(0),z(0)) = (1, 0, 0.4). Lyapunov exponents (0.3552, 0, −0.3552)

These data are characteristic of chaotic behavior. The following table shows the dynamics of Lyapunov numbers under the change of the amplitude h.

Table 1. The Lyapunov numbers for the equation (7) with *ω=3.2, (x(0),y(0),z(0)) = (1, 0, 0.4)* and varying h

h	λ_1	λ_2	λ_3
0.2	0.281788	0	-0.281789
0.4	0.239252	0	-0.239252
0.6	0.376682	0	-0.376683
0.8	no data		
1.0	0.355192	0	-0.355192

For all h the sum of three Lyapunov exponents is zero. One of the exponents is positive, and that refers to chaotic behavior (Table 1).

Example 2.
One of the simple equations that can have a nontrivial period annulus, is equation (5), where *g(x)* is:

$$g(x) = (x+2)(x+1)x(x-1)(x-2). \qquad (10)$$

The vector field associated with (5) is depicted in Figure 8.

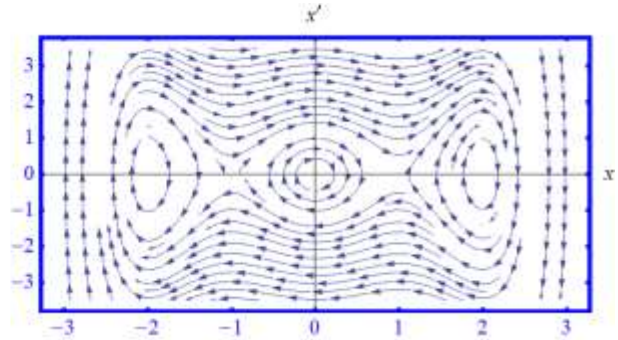

Fig. 8: The period annuli (three trivial ones, and one nontrivial spreading to infinity) in the equation $x'' + g(x) = 0$, where g(x) is as in (8)

Consider equation:

$$x'' + g(x) = h\cos\omega t \qquad (10)$$

and the equivalent system :

$$\begin{cases} x' = y \\ y' = -g(x) + h\cos z \\ z' = \omega. \end{cases} \qquad (11)$$

A series of computational experiments were performed. They yielded that for h=*3.9*, ω =*0.6* the trajectory that starts at x_0=0.62, y_0=0.9, z_0=0.4, exhibits irregular behavior. The Lyapunov exponents indicate chaos. This is shown in Figure 9. The Lyapunov curves and graphs of solutions to the system (9) are shown in Figure 10 and Figure 10a respectively.

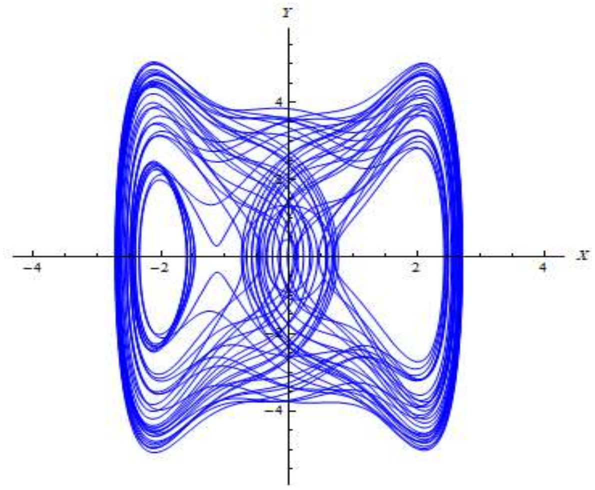

Fig. 9: Phase portrait for the system (11)

The Lyapunov exponents also indicate chaotic behavior.

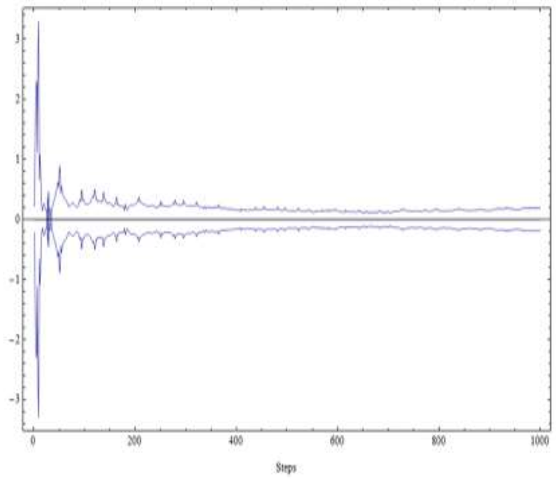

Fig. 10: Lyapunov exponents for the system (11)

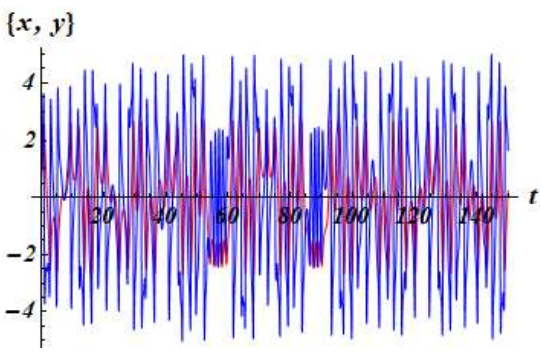

Fig. 10a: Solutions of the system (11)

The two, Table 2 and Table 3 show the dynamics of the Lyapunov exponents when the amplitude h and the coefficient ω change.

Table 2. The Lyapunov numbers for the equation (10) with *ω=0.6, (x(0),y(0),z(0)) = (0.62, 0.9, 0.4)* and varying *h*

h	λ_1	λ_2	λ_3
2.8	0.0271906	-0.027196	0
3.1	0.173493	-0.1735	0
3.5	0.196616	-0.196622	0
3.7	0.114328	-0.114336	0
3.9	0.188477	-0.188485	0

Table 3. The Lyapunov numbers for the equation (10) with *h=3.9, (x(0),y(0),z(0)) = (0.62, 0.9, 0.4)* and varying *ω*

ω	λ_1	λ_2	λ_3
0.1	0.0739052	-0.0739115	0
0.2	0.109937	-0.109944	0
0.4	0.145782	-0.145789	0
0.6	0.188477	-0.188485	0
2	0.302086	-0.302113	0

Both Table 2 and Table 3 for various h and ω contain a positive Lyapunov exponent.

Example 3.
Consider equation (5) and equation (12)

$$x'' + g(x) = h\cos\omega t, \tag{12}$$

where

$$g(x) = +(x+2.4)(x+1.8)(x+1)x(x-0.8)(x-1.7)(x-2.4). \tag{13}$$

The vector field for the unperturbed equation (12) is described by Figure 11.

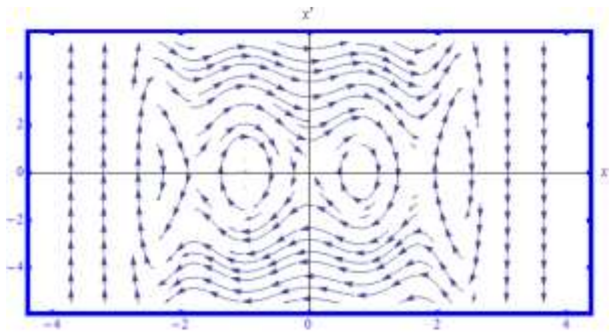

Fig. 11: The two trivial period annuli and one nontrivial in the equation $x'' + g(x) = 0$, where g(x) is as in (13)

The corresponding system :

$$\begin{cases} x' = y \\ y' = -g(x) + h\cos z \\ z' = \omega \end{cases} \tag{14}$$

differs from that in Example 1. Let the parameters be h=3.9, $\omega = 0.6$, $(x(0), y(0), z(0)) = (0.62, 0.9, 0.4)$. The vector field, the Lyapunov curves and the graphs of solutions for the system (14) are shown in Figure 12, Figure 13 and Figure 14 respectively.

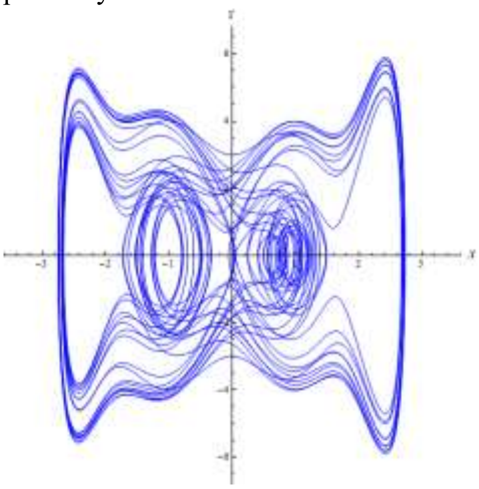

Fig. 12: Phase portrait for the system (14)

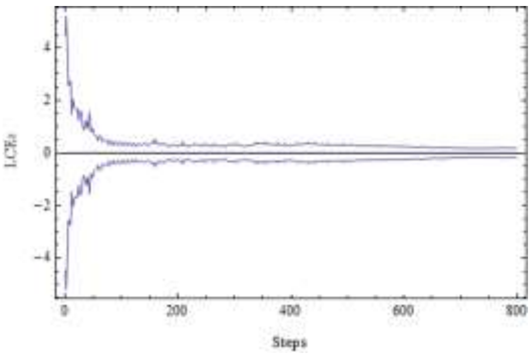

Fig. 13: Lyapunov curves for the system (14). The Lyapunov exponents are $(0.184021, -0.184494, 0.)$

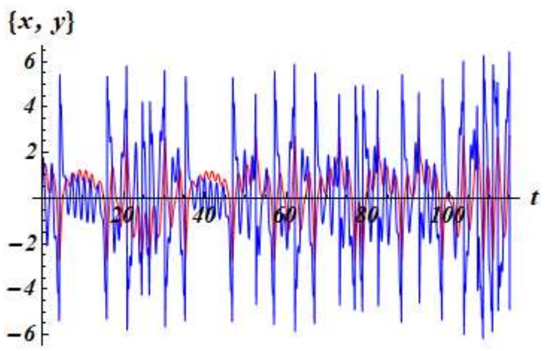

Fig. 14: Solutions of the system (14)

Example 4.

Finally, let us consider an example from the beginning of the article.

Consider equation (5) and equation (15)

$$x'' + g(x) = h\cos\omega t, \tag{15}$$

where

$$g(x) = -(x+4)(x+2.5)(x+1.5)x(x-2)(x-3.6)(x-5),$$

which can be written as a system of the

$$\begin{cases} x' = y \\ y' = -g(x) + h\cos z \\ z' = \omega \end{cases} \tag{16}$$

The amplitude of oscillations in period annuli is larger than in the previous examples. To affect the solutions in period annuli the amplitude h should be large also. Let h=176 , $\omega = 2.5$, $(x(0), y(0), z(0)) = (1.5, 5.0, 0.4)$. We provide below the phase portrait (Figure 15), the Lyapunov exponents (Figure 16) and the graphs of solutions (Figure 17) for the system (16). The solutions exhibit irregular behavior.

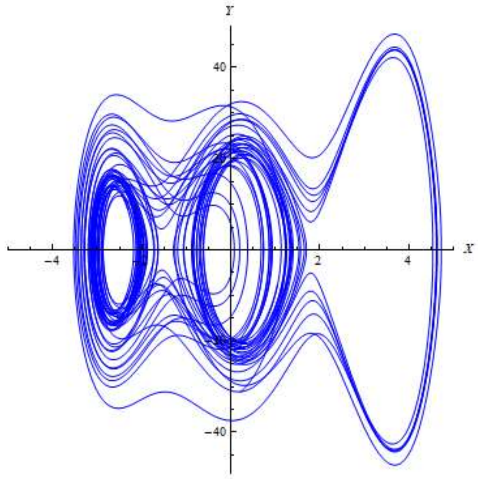

Fig. 15: Phase portrait for the system (16)

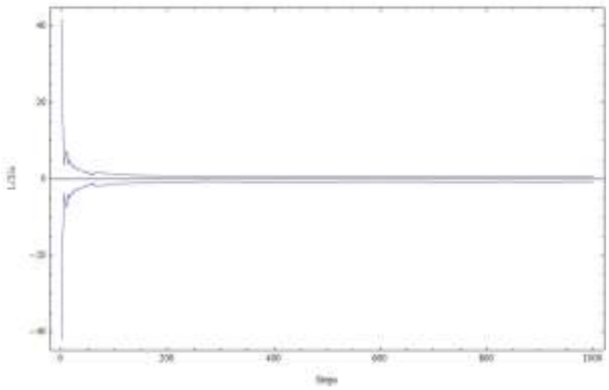

Fig. 16: Lyapunov curves for the system (16). The Lyapunov exponents are (0.667735,−0.678565, 0)

Fig. 17: Solutions of the system (16)

4 Conclusion

The second order ordinary differential equations with an external force can be chaotic. For this, they have to be nonlinear (the necessary condition). The nonlinearity should be suitable. The conditions for the nonlinearity can be found using computational experiments. We tried to detect chaotic behavior in periodically exited equations, which, without external forcing, have period annuli. We have examined four examples of equations of Newtonian form that possess one or several nontrivial period annuli. These equations were excited by a periodic external force with amplitude h and the period $2\pi/\omega$. As a rule, for appropriate values of h and ω the chaotic behavior was observed. In Example 4 the amplitudes of oscillations in period annuli are significantly greater than in the previously considered examples. Consequently, the parameter h in the excited equation is large to induce the chaotic behavior. For the criteria for chaotic behavior, the graph of the Lyapunov curves and Lyapunov exponents was chosen. The positivity of the Lyapunov curve was an indicator of chaos in the equation. The damping term is not needed for obtaining the chaotic behavior.

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