On a Single Server Vacation Queue with Two Types of Service and Two Types of Vacation

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Abstract: - We study a single server queueing system that receives singly arriving customers according to a Poisson process. The server offers one of the two types of heterogeneous services. Before the beginning of a service, , the customer can choose an exponential service with probability p_1 or a deterministic service with probability $p_{2,}$, where $p_1 + p_2 = 1$. Immediately after a service is completed, the server has a choice of taking a vacation with probability δ, or, with probability $1-\delta$, the server may continue staying in the system. We further assume that if the server opts to take a vacation, then with probability α_1, he may take a vacation of an exponential duration with mean vacation time $1/\upsilon\,(\upsilon > 0)$ or with probability α_2 he may want to take a deterministic vacation with constant duration d>0, where $\alpha_1+\alpha_2=1$. After a vacation is complete, the server instantly starts providing service if there is at least one customer in the system or the server remains idle in the system till a new customer arrives for service. We find a steady state solution in terms of the generating function of the queue length as well as the steady state probabilities for all different states of the system.

Key-Words: - Single server, Poisson arrivals; exponential service, deterministic service; exponential vacation, deterministic vacation; generating function; queue length, States of the system, steady state.

Received: June 14, 2023. Revised: November 25, 2023. Accepted: December 29, 2023. Published: February 15, 2024.

1 Introduction

In the majority of queueing systems, the server provides the same kind of service to customers and the service time follows the same distribution, [1], [2], [3]. In addition, as it happens in the majority of the vacation queueing systems, the server's vacation follows the same distribution, [4], [5], [6], [7], [8]. In the last couple of decades, vacation queues have been studied extensively. The work done by all these, and many other authors deals with service interruptions either due to random system failures or due to optional server vacations with many different vacation policies. Queueing systems with deterministic service or deterministic vacations have been studied by many authors including, [9], in which the author deals with a queueing system which allows the server to opt for either an exponential vacation or for a deterministic vacation. In the present paper, we extend the idea in [9] and study a queueing system in which the customer has a choice of either taking a service with exponential duration or a deterministic duration in addition to the server having the choice of taking a vacation of an exponential length or a deterministic vacation or no vacation after each service. Symbolically, we denote our system as $M/\binom{M}{D}/\binom{M}{D}/1$ queueing system. We find steady state generating functions of queue lengths of all different states of the system and derive results corresponding to various interesting special cases including the earlier known results of the systems M/M/D/1, M/D/M/1, M/M1 and M/D/1.

2 Model Description

- Customers arrive at the system one by one according to a Poisson process with mean arrival rate $\lambda > 0$
- Before his service starts, a customer can opt for an exponential service with mean service time $\frac{1}{\mu}$ $(\mu > 0)$ with probability p_1 or a deterministic service of constant duration 'k' with probability p_2, where $p_1 + p_2 = 1$.
- As soon as a service of a customer is complete, the server may decide to go on a vacation with probability δ, or may not take a vacation with probability $1 - \delta$.. Next, we assume that if the server decides to take a vacation, then with probability α_1, he may take a vacation of random

length which follows an exponential distribution with mean vacation time $\frac{1}{\vartheta}$ ($\vartheta > 0$) or with probability α_2, he may take a deterministic vacation with constant duration ' d '. Where $\alpha_1 + \alpha_2 = 1$.

- As soon as his vacation is over, the server immediately takes up a customer at the head of the queue for service, if a customer is waiting in the queue. However, if on returning the server finds the queue empty, then he still joins the system and remains idle until a new customer arrives in the system.
- All stochastic processes involved in the system are independent of each other.

3 Definitions and Equations

We assume that $W_n^{(1)}$ is the steady state probability that there are $n(\geq 0)$ customers in the queue excluding one customer in service and the server is providing exponential service, $W_n^{(2)}$ is the steady state probability that there are $n(\geq 0)$ customers in the queue and the server is providing a deterministic service, $V_n^{(1)}$ is the steady state probability that there are $n(\geq 0)$ customers in the queue and the server is on exponential vacation, $V_n^{(2)}$ is the steady state probability that there are $n(\geq 0)$ customers in the queue and the server is on deterministic vacation. Further, let $P_n = W_n^{(1)} + W_n^{(2)} + V_n^{(1)} + V_n^{(2)}$ be the steady state probability that there are $n(\geq 0)$ customers in the queue irrespective of whether the server is providing any type of service or is on any type of vacation.

Next, we define Q to be the steady-state probability that there is no customer in the system and the server is idle.

We further assume that a_r is the probability of r arrivals during the period of deterministic service time k and therefore,

$$a_r = \frac{\exp(-\lambda k)\,(\lambda k)^r}{r!}, \quad r = 0, 1, 2, 3, \ \ldots \tag{1}$$

Next, we assume that b_r is the probability of r arrivals during the period of deterministic vacation d and therefore,

$$b_r = \frac{\exp(-\lambda d)(\lambda d)^r}{r!}, \quad r = 0, 1, 2, 3, \ \ldots \tag{2}$$

Now, we define the following Probability Generating Functions (PGFs):

$$W^{(1)}(z) = \sum_{n=0}^{\infty} W_n^{(1)} z^n \tag{3}$$

$$W^{(2)}(z) = \sum_{n=0}^{\infty} W_n^{(2)} z^n \tag{4}$$

$$V^{(1)}(z) = \sum_{n=0}^{\infty} V_n^{(1)} z^n \tag{5}$$

$$V^{(2)}(z) = \sum_{n=0}^{\infty} V_n^{(2)} z^n \tag{6}$$

$$A(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{\exp(-(-\lambda k)(\lambda k)^n}{n!} = \exp\big(-\lambda k(1-z)\big),\ |z| \leq 1, \tag{7}$$

$$B(z) = \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} \frac{\exp(-\lambda d)(\lambda d)^n}{n!} z^n = \exp\big(-\lambda d(1-z)\big),\ |z| \leq 1. \tag{8}$$

4 Equations Governing the System

We use probability reasoning to obtain the following steady-state equations:

$$(\lambda+\mu)W_n^{(1)} = \lambda W_{n-1}^{(1)} + \mu(1-\delta)p_1 W_{n+1}^{(1)} + \vartheta p_1 V_{n+1}^{(1)} + p_1(V_0^{(2)} b_{n+1} + V_1^{(2)} b_n + \ldots + V_{n+1}^{(2)} b_0) + (1-\delta)p_1(W_0^{(2)} a_{n+1} + W_1^{(2)} a_n + \ldots + W_{n+1}^{(2)} a_0),\ n \geq 1, \tag{9}$$

$$(\lambda+\mu)W_0^{(1)} = \mu(1-\delta)p_1 W_1^{(1)} + \vartheta p_1 V_1^{(1)} + p_1(V_0^{(2)} b_1 + V_1^{(2)} b_0) + (1-\delta)p_1(W_0^{(2)} a_1 + W_1^{(2)} a_0) + \lambda p_1 Q \tag{10}$$

$$W_n^{(2)} = \mu(1-\delta)p_2 W_{n+1}^{(1)} + \vartheta p_2 V_{n+1}^{(1)} + p_2(V_0^{(2)} b_{n+1} + V_1^{(2)} b_n + \ldots + V_{n+1}^{(2)} b_0) + (1-\delta)p_2(W_0^{(2)} a_{n+1} + W_1^{(2)} a_n + \ldots + W_{n+1}^{(2)} a_0),,\ n \geq 1 \tag{11}$$

$$W_0^{(2)} = \mu(1-\delta)p_2 W_1^{(1)} + \vartheta p_2 V_1^{(1)} + p_2(V_0^{(2)} b_1 + V_1^{(2)} b_0) + (1-\delta)p_2(W_0^{(2)} a_1 + W_1^{(2)} a_0) + \lambda p_2 Q, \tag{12}$$

$$(\lambda+\vartheta)V_n^{(1)}=\lambda V_{n-1}^{(1)}+\ \mu\delta\propto_1 W_n^{(1)}+ +\delta\propto_1(W_0^{(2)}a_n+ W_1^{(2)}a_{n-1}+\ .\ .\ .+W_n^{(2)}a_0),\ n\geq 1 \qquad (13)$$

$$(\lambda+\vartheta)V_0^{(1)}=\ \mu\delta\propto_1 W_0^{(1)}+\delta\propto_1 W_0^{(2)}a_0 \qquad (14)$$

$$V_n^{(2)}=\mu\delta\propto_2 W_n^{(1)}+\delta\propto_2(W_0^{(2)}a_n+ W_1^{(2)}a_{n-1}+\ .\ .\ .+W_n^{(2)}a_0),\ n\geq 0, \qquad (15)$$

$$\lambda\mathrm{Q}=\mu(1-\delta)W_0^{(1)}+\ (1-\delta)W_0^{(2)}a_0 +\vartheta V_0^{(1)}+V_0^{(2)}b_0. \qquad (16)$$

5 Steady State Solution

Use of the standard generating function approach, equations (9), (10) give.

$$[(\lambda+\mu)z-\lambda z^2-\mu(1-\delta)p_1]W^{(1)}(z) =\ \vartheta p_1V^{(1)}(z)+p_1V^{(2)}(z)B(z) +(1-\delta)p_1W^{(2)}(z)A(z)+p_1\lambda(1-z)Q. \qquad (17)$$

Similar operations on (11) and (12) yield

$$zW^{(2)}(z)=\mu(1-\delta)p_2W^{(1)}(z)+ p_2V^{(2)}(z)B(z) +(1-\delta)p_2W^{(2)}(z)A(z) +\vartheta p_2V^{(1)}(z)+p_2\lambda(1-z)Q \qquad (18)$$

Next, from equations (13) and (14) we obtain.

$$(\lambda-\lambda z+\vartheta)V^{(1)}(z)=\mu\delta\propto_1 W^{(1)}(z) +\delta\propto_1 W^{(2)}(z)A(z) \qquad (19)$$

And from (15), we get.

$$V^{(2)}(z)=\mu\delta\propto_2 W^{(1)}(z) +\delta\propto_2 W^{(2)}(z)A(z). \qquad (20)$$

Next, we make use of equations (19) and (20) into equations (17) and (18), simplify and get.

$$[(\lambda+\mu)z-\lambda z^2-\mu(1-\delta)p_1]W^{(1)}(z)$$
$$=\vartheta p_1\left\{\frac{\mu\delta\propto_1 W^{(1)}(z)\ +\ \delta\propto_1 W^{(2)}(z)A(z)}{(\lambda-\lambda z+\vartheta)}\right\} +p_1B(z)\left\{\begin{matrix}\mu\delta\propto_2 W^{(1)}(z)\\ +\delta\propto_2 W^{(2)}(z)A(z)\end{matrix}\right\} +(1-\delta)p_1W^{(2)}(z)A(z) +p_1\lambda(1-z)Q \qquad (21)$$

$$zW^{(2)}(z)=\mu(1-\delta)p_2W^{(1)}(z) +p_2B(z)\left\{\begin{matrix}\mu\delta\propto_2 W^{(1)}(z)\\ +\delta\propto_2 W^{(2)}(z)A(z)\end{matrix}\right\} +(1-\delta)p_2W^{(2)}(z)A(z) +p_2\left\{\frac{\mu\delta\propto_1 W^{(1)}(z)\ +\delta\propto_1 W^{(2)}(z)A(z)}{(\lambda-\lambda z+\vartheta)}\right\} +p_2\lambda(1-z)Q \qquad (22)$$

Now, we re-write equations (21) and (22) as:

$$\begin{bmatrix}(\lambda+\mu)z-\lambda z^2-\mu(1-\delta)p_1\\ -\frac{\mu\delta\propto_1\vartheta p_1}{(\lambda-\lambda z+\vartheta)}+\ \mu\delta\propto_2 p_1B(z)\end{bmatrix}W^{(1)}(z) -\begin{bmatrix}\frac{\delta\propto_1\vartheta p_1}{(\lambda-\lambda z+\vartheta)}A(z)+\delta\propto_2 A(z)p_1B(z)\\ +(1-\delta)p_1A(z)\end{bmatrix}W^{(2)}(z) =p_1\lambda(1-z)Q \qquad (23)$$

$$-\begin{bmatrix}\mu(1-\delta)p_2+p_2B(z)\mu\delta\propto_2\\ +\dfrac{\vartheta p_2\mu\delta\propto_1}{(\lambda-\lambda z+\vartheta)}\end{bmatrix}W^{(1)}(z) +\begin{bmatrix}z-(1-\delta)p_2A(z)-p_2B(z)\delta\alpha_2A(z)\\ -\dfrac{\vartheta p_2\delta\propto_1 A(z)}{(\lambda-\lambda z+\vartheta)}\end{bmatrix}W^{(2)}(z) =p_2\lambda(1-z)Q \qquad (24)$$

Re-writing equations (23) and (24) in matrix form as:

$$\begin{bmatrix}G_{11} & -G_{12}\\ -G_{21} & G_{22}\end{bmatrix}\ \begin{bmatrix}W^{(1)}(z)\\ W^{(2)}(z)\end{bmatrix}=\begin{bmatrix}p_1(\lambda(z-1)Q\\ p_2(\lambda(z-1)Q\end{bmatrix}, \qquad (25)$$

Where:

$$G_{11}(z)=(\lambda+\mu)z-\lambda z^2 -\mu(1-\delta)p_1-\left(\frac{\vartheta p_1\mu\delta\propto_1}{\lambda-\lambda z+\vartheta}\right) -p_1B(z)\,\mu\,\delta\propto_2$$

$$G_{12}(z)=\ \left(\frac{\vartheta p_1\delta\propto_1 A(z)}{\lambda-\lambda z+\vartheta}\right) +p_1B(z)\delta\propto_2 A(z) +(1-\delta)p_1A(z),$$

$$G_{21}(z)=\mu(1-\delta)p_2 +p_2B(z)\,\mu\delta\propto_2+\left(\frac{\vartheta p_2\mu\delta\propto_1}{\lambda-\lambda z+\vartheta}\right).$$

$$G_{22}(z) = z - p_2 B(z)\delta \propto_2 A(z) -(1-\delta)p_2 A(z) - \left(\frac{\vartheta p_2 \delta \propto_1}{\lambda - \lambda z + \vartheta}\right) A(z),$$

Solving (25) simultaneously for $W^{(1)}(z)$ and $W^{(2)}(z)$, we obtain:

$$W^{(1)}(z) = \frac{\begin{bmatrix} p_1(\lambda(z-1)Q & -G_{12}(z) \\ p_2(\lambda(z-1)Q & G_{22}(z) \end{bmatrix}}{\begin{bmatrix} G_{11}(z) & -G_{12}(z) \\ -G_{21}(z) & G_{22}(z) \end{bmatrix}} = \frac{N_1(z)\,Q}{D(z)}, \quad (26)$$

$$W^{(2)}(z) = \frac{\begin{bmatrix} G_{11}(z) & p_1\lambda(z-1)Q \\ -G_{21}(z) & p_2(\lambda(z-1)Q \end{bmatrix}}{\begin{bmatrix} G_{11}(z) & -G_{12}(z) \\ -G_{21}(z) & G_{22}(z) \end{bmatrix}} = \frac{N_2(z)\,Q}{D(z)}. \quad (27)$$

Next, we use (26) and (27) into (19), (20), simplify and obtain:

$$V^{(1)}(z) = \frac{\delta \propto_1 \left\{\frac{[\mu N_1(z) + A(z) N_2(z)]}{\vartheta}\right\} Q}{[\lambda - \lambda z + \vartheta]\, D(z)}, \quad (28)$$

$$V^{(2)}(z) = \frac{\delta \propto_2 [\mu N_1(z) + A(z) N_2(z)]\, Q}{D(z)}, \quad (29)$$

Where:

$N_1(z) = \lambda(z-1)[p_1 G_{22}(z) + p_2 G_{12}(z)]$,

$N_2(z) = \lambda(z-1)[p_1 G_{21}(z) + p_2 G_{11}(z)]$,

$D(z) = G_{11}(z) G_{22}(z) - G_{12}(z) G_{21}(z)$.

Now, at z=1, the generating functions found above yield:

$$W^{(1)}(1) = \lim_{z\to 1} W^{(1)}(z) = \frac{p_1 \lambda Q}{p_1\left[\mu - \lambda - \lambda\mu\delta\left\{\frac{\alpha_1}{v} + \alpha_2 d\right\}\right] + \mu p_2\left[1 - \frac{\lambda\delta\alpha_1}{\vartheta} - \lambda k - \alpha_2 \delta\lambda d\right]} \quad (30)$$

$$W^{(2)}(1) = \lim_{z\to 1} W^{(2)}(z) = \frac{\mu p_2 \lambda k Q}{p_1\left[\mu - \lambda - \lambda\mu\delta\left\{\frac{\alpha_1}{v} + \alpha_2 d\right\}\right] + \mu p_2\left[1 - \frac{\lambda\delta\alpha_1}{\vartheta} - \lambda k - \alpha_2 \delta\lambda d\right]} \quad (31)$$

$$V^{(1)}(1) = \lim_{z\to 1} V^{(1)}(z) \quad (32)$$

$$V^{(2)}(1) = \lim_{z\to 1} V^{(2)}(z) = \frac{\mu\, \delta \propto_2 \lambda Q}{p_1\left[\mu - \lambda - \lambda\mu\delta\left\{\frac{\alpha_1}{v} + \alpha_2 d\right\}\right] + \mu p_2\left[1 - \frac{\lambda\delta\alpha_1}{\vartheta} - \lambda k - \alpha_2 \delta\lambda d\right]} \quad (33)$$

It may be noted that Q is the only unknown which remains to be determined. For this purpose, we will use the following normalizing condition:

$$W^{(1)}(1) + W^{(1)}(1) + V^{(1)}(1) + V^{(2)}(1) + Q = 1. \quad (34)$$

On simplifying, (5.18) yields:

$$Q = \frac{p_1\left[\mu - \lambda - \lambda\mu\delta\left\{\frac{\alpha_1}{v} + \alpha_2 d\right\}\right] + \mu p_2\left[1 - \frac{\lambda\delta\alpha_1}{\vartheta} - \lambda k - \alpha_2 \delta\lambda d\right]}{p_1\lambda + \mu p_2 \lambda k + \left(\frac{\mu\,\delta \propto_1 \lambda}{\vartheta}\right) + \mu\,\delta \propto_2 \lambda + p_1\left[\mu - \lambda - \lambda\mu\delta\left\{\frac{\alpha_1}{v} + \alpha_2 d\right\}\right] + \mu p_2\left[1 - \frac{\lambda\delta\alpha_1}{\vartheta} - \lambda k - \alpha_2 \delta\lambda d\right]} \quad (35)$$

If we use the value of Q from (35) into the results (30) to (33) we would be able to determine all the above steady state probabilities in explicit form. Also, using the value of Q from (35) into the main results (26) to (29), all the steady state probability generating functions of the queue length are found in explicit form.

6 Some Special Cases

6.1 $M/\binom{M}{D}/M/1$ Queue

We substitute $\propto_2 = 0$, $\propto_1 = 1$ in the system state probabilities found above in (30) to (33) and (35). Therefore, we obtain:

$$W^{(1)}(1) = \frac{p_1 \lambda Q}{p_1\left[\mu - \lambda - \lambda\mu\delta\left\{\frac{\alpha_1}{v} + \alpha_2 d\right\}\right] + \mu p_2\left[1 - \frac{\lambda\delta\alpha_1}{\vartheta} - \lambda k - \alpha_2 \delta\lambda d\right]} \quad (36)$$

$$W^{(2)}(1) = \frac{\mu p_2 \lambda k Q}{p_1\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right] + \mu p_2\left[1-\frac{\lambda\delta\alpha_1}{\vartheta}-\lambda k - \alpha_2\delta\lambda d\right]} \tag{37}$$

$$V^{(1)}(1) = \frac{\left(\frac{\mu\,\delta\,\alpha_1\lambda}{\vartheta}\right)Q}{p_1\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right] + \mu p_2\left[1-\frac{\lambda\delta\alpha_1}{\vartheta}-\lambda k - \alpha_2\delta\lambda d\right]} \tag{38}$$

$$V^{(2)}(1) = \frac{\mu\,\delta\alpha_2\lambda Q}{p_1\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right] + \mu p_2\left[1-\frac{\lambda\delta\alpha_1}{\vartheta}-\lambda k - \alpha_2\delta\lambda d\right]} \tag{39}$$

$$Q = \frac{p_1\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right] + \mu p_2\left[1-\frac{\lambda\delta\alpha_1}{\vartheta}-\lambda k - \alpha_2\delta\lambda d\right]}{p_1\lambda+\mu p_2\lambda k+\left(\frac{\mu\,\delta\,\lambda}{\vartheta}\right) + p_1\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right] + \mu p_2\left[1-\frac{\lambda\delta\alpha_1}{\vartheta}-\lambda k - \alpha_2\delta\lambda d\right]} \tag{40}$$

6.2 $M/\binom{M}{D}/D/1$ Queue

Substituting $\alpha_1 = 0$ and $\alpha_2 = 1$ in the system state probabilities, we obtain:

$$W^{(1)}(1) = \frac{p_1\lambda Q}{p_1[\mu-\lambda-\lambda\mu\delta\{d\}] + \mu p_2\left[1-\lambda k - \delta\lambda d\right]} \tag{41}$$

$$W^{(2)}(1) = \frac{\mu p_2 \lambda k Q}{p_1[\mu-\lambda]+\mu p_2(1-\lambda k)} \tag{42}$$

$$V^{(1)}(1) = \frac{\left(\frac{\mu\,\delta\,\alpha_1\lambda}{\vartheta}\right)Q}{p_1\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right] + \mu p_2\left[1-\frac{\lambda\delta\alpha_1}{\vartheta}-\lambda k - \alpha_2\delta\lambda d\right]} \tag{43}$$

$$V^{(2)}(1) = \frac{\mu\,\delta\alpha_2\lambda Q}{p_1\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right] + \mu p_2\left[1-\frac{\lambda\delta\alpha_1}{\vartheta}-\lambda k - \alpha_2\delta\lambda d\right]} \tag{44}$$

$$Q = \frac{p_1\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right] + \mu p_2\left[1-\frac{\lambda\delta\alpha_1}{\vartheta}-\lambda k - \alpha_2\delta\lambda d\right]}{p_1\lambda+\mu p_2\lambda k+\left(\frac{\mu\,\delta\,\alpha_1\lambda}{\vartheta}\right)+\mu\,\delta\alpha_2\lambda + p_1\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right] + \mu p_2\left[1-\frac{\lambda\delta\alpha_1}{\vartheta}-\lambda k - \alpha_2\delta\lambda d\right]} \tag{45}$$

6.3 $M/\binom{M}{D}/1$ Queue

Putting $\delta = 0$ in the system state probabilities, we get:

$$W^{(1)}(1) = \frac{p_1\lambda Q}{p_1(\mu-\lambda)+\mu p_2(1-\lambda k)} \tag{46}$$

$$W^{(2)}(1) = \frac{\mu p_2 \lambda k Q}{p_1(\mu-\lambda)+\mu p_2(1-\lambda k)} \tag{47}$$

$$V^{(1)}(1) = 0 \tag{48}$$

$$V^{(2)}(1) = 0 \tag{49}$$

$$Q = \frac{p_1(\mu-\lambda)+\mu p_2(1-\lambda k)}{p_1\lambda+\mu p_2\lambda k + p_1(\mu-\lambda)+\mu p_2(1-\lambda k} \tag{50}$$

6.4 $M/M/\binom{M}{D}/1$ Queue: Only Exponential Service with Both Types of Vacation

Letting $p_2 = 0$ and $p_1 = 1$ in the system state probabilities found above, we get:

$$W^{(1)}(1) = \frac{\lambda Q}{\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right]} \tag{51}$$

$$W^{(2)}(1) = 0 \tag{52}$$

$$V^{(1)}(1) = \frac{\left(\frac{\mu\,\delta\,\alpha_1\lambda}{\vartheta}\right)Q}{\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right]} \tag{53}$$

$$V^{(2)}(1) = \frac{\mu\,\delta\alpha_2\lambda Q}{\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right]} \tag{54}$$

$$Q = \frac{\left[\begin{array}{c}\mu-\lambda \\ -\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\end{array}\right]}{\lambda+\left(\frac{\mu\,\delta\,\alpha_1\lambda}{\vartheta}\right)+\mu\,\delta\alpha_2\lambda+\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right]} \tag{55}$$

6.5 $M/D/\binom{M}{D}/1$ Queue: Only Deterministic Service with Both Types of Vacation

We substitute $p_1 = 0$ and $p_2 = 1$ in the main results obtained above and obtain.

$$W^{(1)}(1) = 0 \tag{56}$$

$$W^{(2)}(1) = \frac{\mu\lambda Q}{\mu\left[\begin{array}{c}1-\frac{\lambda\delta\alpha_1}{\vartheta}-\lambda k \\ -\alpha_2\delta\lambda d\end{array}\right]} \tag{57}$$

$$V^{(1)}(1) = \frac{\left(\frac{\mu\,\delta\,\alpha_1\lambda}{\vartheta}\right)Q}{\mu\left[\begin{array}{c}1-\frac{\lambda\delta\alpha_1}{\vartheta}-\lambda k \\ -\alpha_2\delta\lambda d\end{array}\right]} \tag{58}$$

$$V^{(2)}(1) = \frac{\mu\,\delta\alpha_2\lambda Q}{\mu\left[\begin{array}{c}1-\frac{\lambda\delta\alpha_1}{\vartheta}-\lambda k \\ -\alpha_2\delta\lambda d\end{array}\right]} \tag{59}$$

$$Q = \frac{p_1\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right]+\mu p_2\left[\begin{array}{c}1-\frac{\lambda\delta\alpha_1}{\vartheta}-\lambda k \\ -\alpha_2\delta\lambda d\end{array}\right]}{p_1\lambda+\mu p_2\lambda k+\left(\frac{\mu\,\delta\,\alpha_1\lambda}{\vartheta}\right)+\mu\,\delta\alpha_2\lambda+p_1\left[\mu-\lambda-\lambda\mu\delta\left\{\frac{\alpha_1}{v}+\alpha_2 d\right\}\right]+\mu p_2\left[\begin{array}{c}1-\frac{\lambda\delta\alpha_1}{\vartheta}-\lambda k \\ -\alpha_2\delta\lambda d\end{array}\right]} \tag{60}$$

6.6 $M/M/1$ Queue: Only Exponential Service with No Server Vacation

We substitute $\delta = 0$, $p_2 = 0$ and $p_1 = 1$ to obtain.

$$W^{(1)}(1) = \frac{\lambda Q}{[\mu-\lambda]} \tag{61}$$

$$W^{(2)}(1) = 0 \tag{62}$$

$$V^{(1)}(1) = 0 \tag{63}$$

$$V^{(2)}(1) = 0 \tag{64}$$

$$Q = 1 - \frac{\lambda}{\mu} \tag{65}$$

6.7 $M/D/1$ Queue: Only Deterministic Service with No Vacation

Putting $\delta = 0$, $p_1 = 0$ and $p_2 = 1$ in the above results we get:

$$W^{(1)}(1) = 0 \tag{66}$$

$$W^{(2)}(1) = \frac{\lambda k Q}{1-\lambda k} \tag{67}$$

$$V^{(1)}(1) = 0 \tag{68}$$

$$V^{(2)}(1) = 0 \tag{69}$$

$$Q = 1 - \lambda k \tag{70}$$

The results derived in special cases 6 and 7 are known results in queueing theory literature.

Acknowledgment:
The author is sincerely thankful to all reviewers for their valuable comments and suggestions in revising the paper in the final form.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflicts of interest to declare.