Quasi-periodic solutions for a three-dimensional system in gene regulatory network

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Abstract: This work introduces a three-dimensional system with quasi-periodic solutions for special values of parameters. The equations model the interactions between genes and their products. In gene regulatory networks, quasi-periodic solutions refer to a specific type of temporal behavior observed in the system. We show the dynamics of Lyapunov exponents. Visualizations are provided. It is important to note that the study of gene regulatory networks is a complex interdisciplinary field that combines biology, mathematics, and computer science.

Key-Words: gene regulatory network, Lyapunov exponents, quasi-periodic solution, nullclines, critical points

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1 Introduction

A genetic regulatory network (GRN in short) is a complex system that governs the interactions among genes and their products, including proteins, RNA molecules, and other regulatory elements, within a living organism, [1]. It plays an important role in controlling the development, growth, and function of cells and tissues, as well as coordinating various physiological processes throughout an organism's life, [2],[3]. Understanding genetic regulatory networks is of great interest in various fields, including developmental biology, systems biology, and biotechnology, [4], [5].

Computational models, such as Boolean networks and differential equations, are employed to simulate and predict the behavior of genetic regulatory networks under different conditions, [6]. These models help uncover the network's underlying principles and aid in the design of genetic engineering strategies to modulate specific biological processes for therapeutic or industrial purposes,[7].

The dynamical system of the form

$$\begin{cases} \dfrac{dx}{dt} = \\ \dfrac{1}{1+e^{-\mu_1(w_{11}x+w_{12}y+w_{13}z-\theta_1)}} - v_1 x, \\ \dfrac{dy}{dt} = \\ \dfrac{1}{1+e^{-\mu_2(w_{21}x+w_{22}y+w_{23}z-\theta_2)}} - v_2 y, \\ \dfrac{dz}{dt} = \\ \dfrac{1}{1+e^{-\mu_3(w_{31}x+w_{32}y+w_{33}z-\theta_3)}} - v_3 z, \end{cases} \tag{1}$$

is used to model genetic regulatory networks, where μ_i, θ_i and v_i are the parameters, w_{ij} are the coefficients of the so-called regulatory matrix

$$W = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}. \tag{2}$$

Such system was considered in the paper [8]. The parameters of the GRN have the following biological interpretations:

- v_i - degradation of the i-th gene expression product;
- w_{ij} - the connection weight or strength of control of gene j on gene i. Positive values of w_{ij} indicate activating influences while negative values define repressing influences;
- θ_i - influence of external input on gene i, which modulates the gene's sensitivity of response to activating or repressing influences.

Definition 1.1. *The j'th nullcline is the geometric shape for which* $\frac{dx_j}{dt} = 0$. *The critical points of the system are located where all of the nullclines intersect. In a two-dimensional linear system, the nullclines can be represented by two lines on a two-dimensional plot; in a general two-dimensional system they are arbitrary curves.*

The nullclines for the system are defined by the relations

$$\begin{cases} v_1 x = \dfrac{1}{1+e^{-\mu_1(w_{11}x+w_{12}y+w_{13}z-\theta_1)}}, \\ v_2 y = \dfrac{1}{1+e^{-\mu_2(w_{21}x+w_{22}y+w_{23}z-\theta_2)}}, \\ v_3 z = \dfrac{1}{1+e^{-\mu_3(w_{31}x+w_{32}y+w_{33}z-\theta_2)}}. \end{cases} \quad (3)$$

Critical points are solutions of the system (3).

Proposition 1.1. *System (1) has at least one equilibrium (critical point). All equilibria are located in the open box* $Q_3 := \{(x,y,z) : 0 < x < \frac{1}{v_1}, 0 < y < \frac{1}{v_2}, 0 < z < \frac{1}{v_3}\}$.

2 Critical points

The type of the critical point is determined by the position of the roots of the characteristic equation on the complex plane. Two main possibilities exist: either the three eigenvalues are real or two of them are complex conjugates. A critical point is stable if all eigenvalues have negative real parts; it is unstable if at least one eigenvalue has positive real part.

- **Node**. All eigenvalues are real and have the same sign. The node is stable (unstable) when the eigenvalues are negative (positive).
- **Saddle**. All eigenvalues are real and at least one of them is positive and at least one is negative. Saddles are always unstable.
- **Focus – Node**. It has one real eigenvalue and a pair of complex-conjugate eigenvalues, and all eigenvalues have real parts of the same sign. The critical point is stable (unstable) when the sign is negative (positive).
- **Saddle – Focus**. Negative real eigenvalue and complex eigenvalues with positive real part, and positive real eigenvalue and complex eigenvalues with negative real part. This type of critical point is unstable, [9].

3 Lyapunov exponents

Lyapunov exponents are a concept from chaos theory and dynamical systems that provide a quantitative measure of the sensitivity to initial conditions in a chaotic system, [10]. The concept is named after the Russian mathematician Aleksandr Lyapunov, who introduced it in the late 19th century.
Lyapunov exponents have applications

in various fields, including weather forecasting, fluid dynamics, biology, economics, and cryptography. They are particularly useful in studying chaotic systems, predicting long-term behavior, and understanding the stability of dynamical systems.
Calculating Lyapunov exponents can be a complex task, especially for high-dimensional systems, and often involves numerical methods and simulations.
The generally accepted convention is to write the Lyapunov exponents in descending order

$$\lambda_1 \geq \lambda_2 \geq \dots \lambda_n.$$

In 3D phase space, there exist four types of attractors: stable points, limit cycles, 2D tori and strange attractors, [11]. The following set of LEs characterizes possible dynamical situations to be met:

- $(LE_1, LE_2, LE_3) = (-,-,-)$ - stable fixed point;
- $(LE_1, LE_2, LE_3) = (0,-,-)$ - stable limit cycle;
- $(LE_1, LE_2, LE_3) = (0,0,-)$ - stable 2D tori;
- $(LE_1, LE_2, LE_3) = (+,0,-)$ - strange attractor.

Any dissipative dynamical system will have at least one negative exponent, the sum of all of exponents is negative, [12],[14],[15]. In general, a system has a set of Lyapunov exponents, each characterizing the average stretching or shrinking of phase space in a particular direction, [13].

3.0.1 Properties of Lyapunov exponents

1. The number of Lyapunov exponents is equal to the number of phase space dimensions, or the order of the system of differential equations. They are arranged in descending order [15].
2. The largest Lyapunov exponent of a stable system does not exceed zero [16].
3. A chaotic system has at least one positive Lyapunov exponent, and the more positive the largest Lyapunov exponent, the more unpredictable the system is [16].
4. To have a dissipative dynamical system, the values of all Lyapunov exponents should sum to a negative number [15].
5. A hyperchaotic system is defined as a chaotic system with at least two positive Lyapunov exponents. Combined with one null exponent and one negative exponent, the minimal dimension for a hyperchaotic system is four [15].

Proposition 3.1. *The largest Lyapunov exponent of a stable system does not exceed zero, [16].*

Proposition 3.2. *Only dissipative dynamical systems have attractors, [9].*

4 Quasi-periodic solutions

Quasi-periodicity refers to a pattern or behavior that exhibits some level of periodicity but is not perfectly periodic. In other words, it shows repetitive characteristics, but the repetition is not strictly regular. This concept is encountered in various fields, including physics, mathematics, signal processing, and even in everyday life, [17]. They can describe the behavior of certain celestial objects in astronomy, vibrations in mechanical systems, or oscillations in chemical reactions. Quasi-periodicity is a valuable concept that helps us understand and model complex systems that exhibit both regular and irregular behaviors.
Quasi-periodicity describes in dynamical systems solutions, which neither exhibit a

finite period length nor are chaotic, [18]. Quasi-periodic behavior can be challenging to visualize and analyze. Advanced mathematical tools and numerical simulations are often used to study and understand these solutions.

Definition 4.1. *Quasi-periodic solutions are characterized by a discrete frequency spectrum, which does not consists of integer multiples of one single base frequency. The spectrum consists of linear combinations of at least two independent base frequencies,[18].*

Two zero Lapunov values in 3D systems indicates quasi-periodic dynamic, [19].

4.1 Example

Consider the system (1) and the regulatory matrix

$$W = \begin{pmatrix} 0.025 & 0.5 & 1.14 \\ -0.9 & 0.2 & 5.5881 \\ 0.08 & -1 & 2 \end{pmatrix} \quad (4)$$

$v_1 = 0.2505, v_2 = 0.1407, v_3 = 0.4305; \mu_1 = 4.6, \mu_2 = 4.2, \mu_3 = 7; \theta_1 = 0.565, \theta_2 = 0.8355, \theta_3 = 0.49$. Let initial conditions be $(0; 0.5; 0.1)$. There is the critical point $P = (3.94; 1.30; 0.67)$. Characteristic values for critical point P are $\lambda_1 = -0.23, \lambda_{2,3} = 1.21 \pm 1.87i$. The type of critical point is a saddle-focus. The nullclines of the system (1) with the regulatory matrix (4) are considered in Figure 1.
The trajectory of the system (1) with the regulatory matrix (4) is considered in Figure 2.
Solutions $(x(t), y(t), z(t))$ of the system (1) with the regulatory matrix (4) are shown in Figure 3.
The dynamics of Lyapunov exponents are shown in Figure 4. Lyapunov exponents are $\lambda_1 = 0.00; \lambda_2 = 0.00; \lambda_3 = -0.05$;

The presence of two zero values indicates the quasi-periodicality of the dynamics.

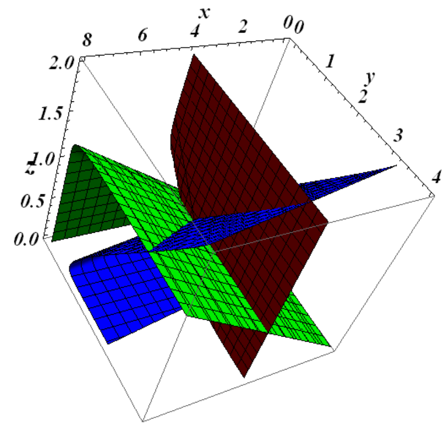

Figure 1: The nullclines of the system (1) with the regulatory matrix (4).

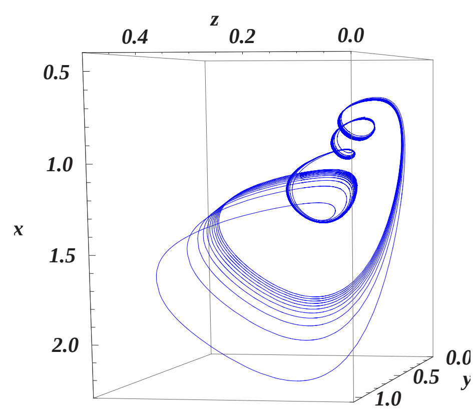

Figure 2: The trajectory of the system (1) with the regulatory matrix (4).

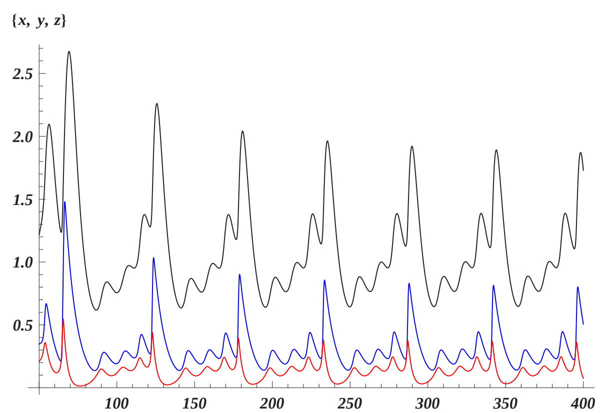

Figure 3: Solutions $(x(t), y(t), z(t))$ of the system (1) with the regulatory matrix (4)

Figure 3 demonstrate a discrete frequency spectrum, which does not consists of integer multiples of one single base frequency.

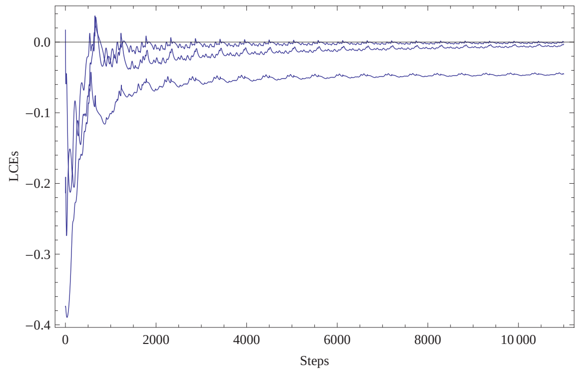

Figure 4: The dynamics of Lyapunov exponents.

Figure 2 demonstrate complicated trajectory motion. Despite trajectories complex motion, the quasi-periodic motion is predictable. Trajectories starting close to each other stay close to each other, and the long-term prognosis is guaranteed.

5 Conclusions

The three-dimensional GRN system is considered. The dynamics of Lyapunov exponents are shown. The nullclines of the system (1) with the regulatory matrix (4) are shown. Also the trajectory of the system (1) with the regulatory matrix (4) is shown. Getting quasy-periodicity in 3D GRN system is very important.
The next step of devolopment of GRN system after quasy-periodicity is chaotic behavior. Chaotic behavior in 3D cases in GRN occur extremely rare, [20].

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