On the finite presentation of operads

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Abstract: Operads were introduced to describe compositional structures arising in algebraic topology. Recently, some researches were interested in using operads in applied mathematics, to model composition of structures in logic, databases, and dynamical systems. In, we focus on finite presentation of an operad and its associated algebra. More precisely, we prove the general result stating that if an operad $\mathcal{O}$ has a finite presentation, then the associate $\mathcal{O}$-algebra has also a corresponding one. Some application in physics, especially in wiring diagrams will be discussed.

Key-Words: operad, operad algebra, wiring diagrams, finite presentation, compositionality.

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1 Introduction

Operad theory is a field of algebra (more precisely abstract algebra) used to describe algebraic structure that models some algebraic properties such as commutativity, associativity, Lie brackets, ...

An operad (or colored operad) is a structure that consists of a bunch of elements that are viewed as abstract operations, each one having a multiple inputs, where inputs are finite ordered list (possibly zero ones) of elements called colors in a fixed non-empty finite class $\mathcal{S}$, these operations are equipped with a specification of how to compose them in one output (element in $\mathcal{S}$), and subject to associativity and unity axioms, [1], these operations are represented by trees which can be grafted onto each other to represent the composition. Just like a monoid can be viewed as a single object category, likewise an operad can be seen as a single object multicategorie, hence multicategories are also named as operads, or colored operads, and in case of ambiguity, it can be identified the class $\mathcal{S}$, then it will be called an $\mathcal{S}$-colored operad.

An operad-algebras (algebra over on operad) is a generalization of the notion of a module over a ring. One can define an operad-algebra as a concrete realization of the abstract operations of the operad, in other words, it can be defined as a set combined with concrete operations on this set whereby their behaviour is analogous to the abstract operations in the operad : an object combined with operations as defined by the operad, subject to the composition condition as defined by the operad, [1]. Operad-algebra is to its associate operad as group representation is to its group. They form a category analog of that of universal algebras. Operads were firstly introduced in algebraic topology in the early of 1970s by [2], notably to model iterated loop spaces, and the original definition of operad is due to J.Peter in his book "The Geometry of Iterated Loop Spaces", and he was the first one to coin the term operad. Operads are basic in homotopy theory, and they have many applications in many branches of mathematics, [3], [4], such as string topology, category theory, combinatories of trees, algebraic deformation theory, homotopical algebras and vertex operator algebras. Furthermore, operads are essential in mathematical physics, computer science, biology, and others. For more details on operads, we refer the interested reader to, [5], [6].

Here, we focus on two specific operads, that of wiring diagrams (resp. undirected wiring diagrams) denoted through this paper by $\mathcal{WD}$ (resp. $\mathcal{UWD}$). The original definition of wiring diagram is given by Rupel and Spivak in [7] who observed that the set of wiring diagrams form an operad called the operad of wiring diagrams. Wiring diagrams are a simplified representation of electrical systems or circuits (graphical language) composed of such operations, each one having a multiple inputs and multiple outputs, each element of which is allowed to carry a such value, and describes how these operations are connected between them to form a larger one operation more complicated. Contrary to a wiring diagram, an undirected wiring diagram is version of wiring diagrams that each operation can be seen as a finite set, which each element is allowed to carry a value. Mathematically speaking, let $\mathcal{S}$ be a class, a $\mathcal{S}$-wiring diagram (we can drop $\mathcal{S}$, and call it a wiring diagram if $\mathcal{S}$ is clear from the context) in [1] is given by $\psi = (\underline{X}, Y, DN, v, s)$ where Y is the output box of ψ and $\underline{X} = (X_1, ..., X_n)$ is a $Box_{\mathcal{S}}$-profile with X_i the i-th input box of ψ, (DN, v) is an $\mathcal{S}$-finite set, and s is the supplier assignment for ψ. Similarly a $\mathcal{S}$-undirected wiring diagram (we can drop $\mathcal{S}$, and call it an undirected wiring diagram if $\mathcal{S}$ is clear from the context) in [1] is given by $\psi = (\underline{X}, Y, C, f, g)$ where Y is the output box of ψ and $\underline{X} = (X_1, ..., X_n)$ is a $Fin_{\mathcal{S}}$-profile with X_i the i-th input box of ψ, $C \in Fin_{\mathcal{S}}$ the set of cables of ψ, and f, g are maps in the cospan diagram. More details about the explicit definition of wiring diagram and undirected wiring diagram can be found in [1].

In [1], D. Yau established that for each class $\mathcal{S}$, the collection of $\mathcal{S}$-wiring diagrams is a $Box_{\mathcal{S}}$-colored operad denoted $\mathcal{WD}$ that has 8 generating wiring diagrams that they generate the operad $\mathcal{WD}$ and 28 elementary generating relations that they generate together with the associativity and the unity axioms defined by the operad $\mathcal{WD}$ all the relations in $\mathcal{WD}$, and he also established that for each class $\mathcal{S}$, the collection of $\mathcal{S}$-undirected wiring diagrams is a $Fin_{\mathcal{S}}$-colored operad denoted $\mathcal{UWD}$ that has 6 generating undirected wiring diagrams that they generate the operad $\mathcal{UWD}$ and 17 elementary generating relations that they generate together with the associativity and the unity axioms defined by the operad $\mathcal{UWD}$ all the relations in $\mathcal{UWD}$. Then every wiring diagram (respectively undirected wiring diagram) has a presentation in terms of finitely many generating wiring diagrams (respectively undirected wiring diagrams) as a finite iterated operadic composition, diagrams in both $\mathcal{WD}$ and $\mathcal{UWD}$ can be built as an operadic composition of generating diagrams in different ways, so they can have many different presentations expressed as a finite iterated operadic composition. Then the concept of a simplex was crucial to develop the necessary language to check if two any presentations of the same diagram are equivalent, meaning connected by a finite sequence of elementary equivalences. According to [1] the operad $\mathcal{WD}$ (resp $\mathcal{UWD}$) has a finite presentation if and only if $\mathcal{WD}$ (resp $\mathcal{UWD}$) satisfies the two following assertions : the first is that every wiring diagram (resp. undirected wiring diagram) can be generated by the generating wiring diagrams (resp. undirected wiring diagram) as a finite iterated operadic composition, and the second is that if a wiring diagram (resp. undirected wiring diagram) can be expressed as an operadic composition of the generating wiring diagrams (resp. undirected wiring diagram) in two different ways, then it can be found a finite sequence of elementary equivalences from the first operadic composition to the other one, that's what it call a finite presentation theorem for $\mathcal{WD}$ (resp. $\mathcal{UWD}$).

In [1], D. Yau used these finite presentations to describe the $\mathcal{WD}$-algebra and $\mathcal{UWD}$-algebra in terms of finitely many generating structure maps and generating axioms corresponding to the generating wiring diagrams and elementary relations in their associted operads. His proof was based on an equivalence between two different definitions of $\mathcal{O}$-algebra (where $\mathcal{O} = \mathcal{WD}$ or $\mathcal{O} = \mathcal{UWD}$). In fact, the $\mathcal{WD}$-algebra (resp. the $\mathcal{UWD}$-algebra) has 8 generating structure maps corresponding to the 8 generating wiring diagrams, and 28 generating axioms corresponding to the 28 elementary relations in $\mathcal{WD}$ (resp. 6 generating structure maps corresponding to the 6 generating undirected wiring diagrams, and 17 generating axioms corresponding to the 17 elementary relations in $\mathcal{UWD}$). Finally D.Yau proved that the operad algebra $\mathcal{WD}$-algebra has a finite presentation corresponding to the one of $\mathcal{WD}$.

A natural extension, is to investigate the analogous of D. Yau's claim for any $\mathcal{O}$ operad, then our approach is based on the operad $\mathcal{WD}$ (resp. $\mathcal{UWD}$) which initially invented in Spivak, [7], [8]. This paper deals with an operadic approach to formulating and proving a more general result that consider both of these finite presentation theorems as special cases, our main result stating that if an operad $\mathcal{O}$ has a finite presentation, then the associate $\mathcal{O}$-algebra has also a corresponding one, a finite presentation of $\mathcal{O}$ (resp. $\mathcal{O}$-algebra) means that the operad $\mathcal{O}$ (resp. $\mathcal{O}$-algebra) has a finite generating set and any two equivalent simplices in $\mathcal{O}$ (resp. $\mathcal{O}$-algebra) are connected by a finite sequence of elementary equivalences in $\mathcal{O}$ (resp. $\mathcal{O}$-algebra). Our results state the following :

Theorem 1.1. *If the operad $\mathcal{O}$ has a finite generating set $\mathcal{T}$, then its associated $\mathcal{O}$-algebra, $\mathcal{A}$, has a corresponding finite generating set $\mathcal{T}_\mu$.*

That will be sufficient to prove the first part of our main theorem, to establish the second part of our theorem, we will define the concept of a simplex and an elementary equivalence, we will see that every elementary equivalence is induced by an elementray relation or an operad associativity unity equivariance axiom, then we develop the necessary language that allows us to define what an elementary relation means, and prove the following result :

Theorem 1.2. *Let $\mathcal{T}$ be a finite generating set for $\mathcal{O}$, and take ζ and ξ two simplices in $\mathcal{O}$. If $|\zeta| = |\xi|$ is an elementary relation in $\mathcal{O}$, then its corresponding elementary relation in $\mathcal{A}$ is $\mu_{|\zeta|} = \mu_{|\xi|}$.*

Afterwards, we will show that the operad associativity unity equivariance axiom in $\mathcal{O}$ has a corresponding associativity unity equivariance axiom in $\mathcal{O}$-algebra, and this together with the above results permits us to announce the next result :

Theorem 1.3. *If $\mathcal{W}$ the set of all elementary equivalences in $\mathcal{T}$ is a strong generating set of $\mathcal{O}$ in $\mathcal{T}$, then its corresponding set $\mathcal{W}_\mu$ of all elementary equivalences in $\mathcal{T}_\mu$ is a strong generating set of $\mathcal{A}$ in $\mathcal{T}_\mu$.*

This last theorem yields the second part of our main theorem. and that will be sufficient to prove the following theorem :

Theorem 1.4. *If $\mathcal{O}$ has a finite presentation, then its associated $\mathcal{O}$-algebra, $\mathcal{A}$ has a corresponding finite presentation one.*

The rest of the paper is broken down as follows : in section 2 we will summarize the necessary background to prove our results, that will be proved in section 3. In section 4, we apply see how our result fits in the cases of both directed and undirected wiring diagrams operads.

2 Materials

Let $\mathcal{S}$ be a class, $(n,m) \in \mathbb{N}^2$, and $Prof(\mathcal{S})$ the class of finite ordered sequences of elements in $\mathcal{S}$. Elements $\underline{a} = (a_1, \ldots, a_n)$ of $Prof(\mathcal{S})$ of length n are also called $\mathcal{S}$-profiles.

Definition 2.1. *A $\mathcal{S}$-colored operad $(\mathcal{O}, 1, \circ)$ is defined as follows:*
To any $b \in \mathcal{S}$ and any two $\mathcal{S}$-profiles, $\underline{a} = (a_1, \ldots, a_n)$ and $\underline{c} = (c_1, \ldots, c_m)$, $(\mathcal{O}, 1, \circ)$ is equipped with

- *a class $\mathcal{O}\begin{pmatrix} b \\ \underline{a} \end{pmatrix} = \mathcal{O}\begin{pmatrix} b \\ a_1, \ldots, a_n \end{pmatrix}$ which elements are called multimaps;*
- *a bijection*

$$\mathcal{O}\begin{pmatrix} b \\ \underline{a} \end{pmatrix} \xrightarrow{\sigma} \mathcal{O}\begin{pmatrix} b \\ \underline{a}\sigma \end{pmatrix}$$

  where $\sigma \in \mathfrak{S}_n$, and $\underline{a}\sigma = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$

- *a specific element called the b-colored unit $1_b \in \mathcal{O}\begin{pmatrix} b \\ b \end{pmatrix}$*
- *a map called the operadic composition $\circ_i$ is defined by*

$$\mathcal{O}\begin{pmatrix} b \\ \underline{a} \end{pmatrix} \times \mathcal{O}\begin{pmatrix} a_i \\ \underline{c} \end{pmatrix} \xrightarrow{\circ_i} \mathcal{O}\begin{pmatrix} b \\ \underline{a} \circ_i \underline{c} \end{pmatrix}$$

  where $\underline{a} \circ_i \underline{c} = (a_1, \ldots, a_{i-1}, c_1, \ldots, c_m, a_{i+1}, \ldots, a_n)$.

This is enough to satisfy the associativity, the unity, and the equivariance axioms.

The best general refernce here is, [1].

Definition 2.2. *Let $\mathcal{C}$ be a collection of multimaps of $\mathcal{O}$, and ψ a multimap in $\mathcal{O}$, we say that ψ admits a presentation in $\mathcal{C}$ if ψ can be expressed as an iterated operadic composition of multimaps in $\mathcal{C}$.*

For example, consider the collection $\mathcal{C} = \{\psi_1, \psi_2, \psi_3\}$, where ψ_1, ψ_2, ψ_3 are multimaps in $\mathcal{O}$.
Suppose $\psi \in \mathcal{O}$ such that $\psi = (\psi_1 \circ_3 \psi_2) \circ_4 \psi_3$, then ψ has a presentation in $\mathcal{C}$.
Note that the iterated operadic composition $\psi_1 \circ_3 (\psi_2 \circ_4 \psi_3)$ is not necessarily a presentation of ψ in $\mathcal{C}$, since the equality $\psi = (\psi_1 \circ_3 \psi_2) \circ_4 \psi_3 = \psi_1 \circ_3 (\psi_2 \circ_4 \psi_3)$ is not assured, for this reason, we need to define the concept of a *simplex* later.

Definition 2.3. *Let $\mathcal{O}$ be an operad.*

1. *A set (or collection) $\mathcal{T}$ of multimaps of $\mathcal{O}$ is called a generating set for $\mathcal{O}$ if every multimap in $\mathcal{O}$ has a such presentation in $\mathcal{T}$, meaning that for all ψ in $\mathcal{O}$ there exist $\psi_1, \ldots, \psi_r$ in $\mathcal{T}$ such that ψ can be decomposed as an iterated operadic composition (possibly infinite) of $\psi_1, \ldots, \psi_r$, for some $r \geq 1$.*
2. *A set (or collection) $\mathcal{T}$ of multimaps of $\mathcal{O}$ is called a finite generating set for $\mathcal{O}$, when $\mathcal{T}$ is a finite set (or collection) of mutimaps of $\mathcal{O}$, and every multimap in $\mathcal{O}$ has a such presentation in $\mathcal{T}$, meaning that for all ψ in $\mathcal{O}$ there exist $\psi_1, \ldots, \psi_r$ in $\mathcal{T}$ such that ψ can be decomposed as a finite iterated operadic composition of $\psi_1, \ldots, \psi_r$, for some $r \geq 1$.*

The elements of $\mathcal{T}$ are called generating multimaps.

Given an operad $\mathcal{O}$, an algebra over an operad, or just $\mathcal{O}$-algebra for simplicity, roughly, is a left module over $\mathcal{O}$ with multiplications parametrized by $\mathcal{O}$. Formally meaning

Definition 2.4. *Let $\mathcal{O}$ be an operad. An operad-algebra (shortly $\mathcal{O}$-operad), is a pair $(\mathcal{A}, \mu)$ equipped with a class , denoted $\mathcal{A}_b$, for any $b \in \mathcal{S}$, called the b-colored entry of $\mathcal{A}$, and the multiplication μ is defined to be*

$$\mathcal{A}_{\underline{a}} = \prod_{i=1}^{n} \mathcal{A}_{a_i} \xrightarrow{\mu_\psi} \mathcal{A}_b,$$

called the structure map, where $b \in \mathcal{S}, \underline{a} = (a_1, \ldots, a_n) \in Prof(\mathcal{S})$ and $\psi \in \mathcal{O}\begin{pmatrix} b \\ \underline{a} \end{pmatrix}$, such that the associativity, unity, and equivariance axioms are respected.

Each structure map has one entry of $\mathcal{O}$. The best general refernce here is, [1].

Simplices:

Let $\mathcal{O}$ be an operad with a finite generating set $\mathcal{T}$, let $n \in \mathbb{N}^*$, we define inductively a n-simplex ψ and its composition $|\psi|$ in $\mathcal{O}$ as follows :

- A 1-simplex is a generating multimap, and its composition $|\psi|$ is defined to be ψ itself, i.e $|\psi| = \psi$;
- A n-simplex, for $n \geq 2$ is defined to be a tuple $\psi = (\varphi, i, \phi)$ consists of :

$$\begin{cases} i \in \mathbb{N}^* \\ p - \text{simplex } \varphi \text{ with } p \geq 1 \\ q - \text{simplex } \phi \text{ with } q \geq 1 \end{cases}$$

  such that $p + q = n$, and the operadic composition $|\psi| = |\varphi| \circ_i |\phi|$.
  Here, the k-simplices for all $1 \leq k \leq n-1$ and their compositions are supposed to be well defined in $\mathcal{O}$.

For simplicity of notation, we also denote a n-simplex by $\psi = (\psi_1, \ldots, \psi_n)$, where the generating multimaps $\psi_1, \ldots, \psi_n$ are ordered in which they appear in the composition and we write $|\psi| = \psi_1 \circ_{i_1} \cdots \circ_{i_{n-1}} \psi_n$. Unless otherwise stated a simplex in $\mathcal{O}$ is a m-simplex in $\mathcal{O}$ for some $m \geq 1$. Moreover we say that a simplex ψ is a *presentation* of $|\psi|$, and the set of all presentations of ψ in $\mathcal{O}$, denoted by

$$\overline{\psi} = \{\Psi \text{ } simplex \text{ } in \text{ } \mathcal{O} \text{ } / |\Psi| = \psi\}.$$

For example, we can consider for some $i, j, k, l \in \mathbb{N}$ the case of a 5-simplex in $\mathcal{O}$ which is an iterated operadic composition in the operad $\mathcal{O}$ of the form:

$(((\psi_1 \circ_i \psi_2) \circ_j \psi_3) \circ_k \psi_4) \circ_l \psi_5$, shortly $((((\psi_1, i, \psi_2), j, \psi_3), k, \psi_4), l, \psi_5)$
$((\psi_1 \circ_i (\psi_2 \circ_j \psi_3)) \circ_k \psi_4) \circ_l \psi_5$, shortly $(((\psi_1, i, (\psi_2, j, \psi_3)), k, \psi_4), l, \psi_5)$
$(\psi_1 \circ_i ((\psi_2 \circ_j \psi_3) \circ_k \psi_4)) \circ_l \psi_5$, shortly $((\psi_1, i, ((\psi_2, j, \psi_3), k, \psi_4)), l, \psi_5)$
$(\psi_1 \circ_i (\psi_2 \circ_j (\psi_3 \circ_k \psi_4))) \circ_l \psi_5$, shortly $((((\psi_1, i, \psi_2), j, \psi_3), k, \psi_4), l, \psi_5)$
$((\psi_1 \circ_i \psi_2) \circ_j (\psi_3 \circ_k \psi_4)) \circ_l \psi_5$, shortly $(((\psi_1, i, \psi_2), j, (\psi_3, k, \psi_4)), l, \psi_5)$
$\psi_1 \circ_i (\psi_2 \circ_j (\psi_3 \circ_k (\psi_4 \circ_l \psi_5)))$, shortly $(\psi_1, i, (\psi_2, j, (\psi_3, k, (\psi_4, l, \psi_5))))$
$\psi_1 \circ_i (\psi_2 \circ_j ((\psi_3 \circ_k \psi_4) \circ_l \psi_5))$, shortly $(\psi_1, i, (\psi_2, j, ((\psi_3, k, \psi_4), l, \psi_5)))$
$\psi_1 \circ_i ((\psi_2 \circ_j (\psi_3 \circ_k \psi_4)) \circ_l \psi_5)$, shortly $(\psi_1, i, ((\psi_2, j, (\psi_3, k, \psi_4)), l, \psi_5))$
$\psi_1 \circ_i (((\psi_2 \circ_j \psi_3) \circ_k \psi_4) \circ_l \psi_5)$, shortly $(\psi_1, i, (((\psi_2, j, \psi_3), k, \psi_4), l, \psi_5))$
$\psi_1 \circ_i ((\psi_2 \circ_j \psi_3) \circ_k (\psi_4 \circ_l \psi_5))$, shortly $(\psi_1, i, ((\psi_2, j, \psi_3), k, (\psi_4, l, \psi_5)))$
$(\psi_1 \circ_i \psi_2) \circ_j ((\psi_3 \circ_k \psi_4) \circ_l \psi_5)$, shortly $((\psi_1, i, \psi_2), j, ((\psi_3, k, \psi_4), l, \psi_5))$
$(\psi_1 \circ_i \psi_2) \circ_j (\psi_3 \circ_k (\psi_4 \circ_l \psi_5))$, shortly $((\psi_1, i, \psi_2), j, (\psi_3, k, (\psi_4, l, \psi_5)))$
$((\psi_1 \circ_i \psi_2) \circ_j \psi_3) \circ_k (\psi_4 \circ_l \psi_5)$, shortly $(((\psi_1, i, \psi_2), j, \psi_3), k, (\psi_4, l, \psi_5))$
$(\psi_1 \circ_i (\psi_2 \circ_j \psi_3)) \circ_k (\psi_4 \circ_l \psi_5)$, shortly $((\psi_1, i, (\psi_2, j, \psi_3)), k, (\psi_4, l, \psi_5))$.

In our example, for $\psi = (((\psi_1 \circ_i \psi_2) \circ_j \psi_3) \circ_k \psi_4) \circ_l \psi_5$, there exists a 5-simplex $\Psi = ((((\psi_1, i, \psi_2), j, \psi_3), k, \psi_4), l, \psi_5)$ (shortly $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5\ )$) whose composition is $\psi = |\Psi| = (((\psi_1 \circ_i \psi_2) \circ_j \psi_3) \circ_k \psi_4) \circ_l \psi_5$, then the 5-simplex Ψ is a presentation of $\psi = |\Psi|$, but neither $((((\psi_1, j, \psi_2), i, \psi_3), k, \psi_4), l, \psi_5)$ nor $(((\psi_1, i, (\psi_2, j, \psi_3)), k, \psi_4), l, \psi_5)$ is not necessarily a presentation of ψ.

Definition 2.5. *Two simplices ψ and ϕ are said to be equivalent in $\mathcal{O}$, whenever their compositions $|\psi|$ and $|\phi|$ are equal in $\mathcal{O}$.*

A sub-simplex $\widehat{\psi}$ of ψ is defined to be itself if ψ is a 1-simplex, since otherwise $\psi = (\varphi, i, \phi)$ is defined as above, hence a sub-simplex of ψ is defined to be either a sub-simplex of φ, or of ϕ, or ψ itself, and we write $\widehat{\psi} \subseteq \psi$.

Once again in our example, for $\psi = (\psi_1 \circ_i (\psi_2 \circ_j \psi_3)) \circ_k (\psi_4 \circ_l \psi_5)$, both $\psi_1 \circ_i (\psi_2 \circ_j \psi_3) = (\psi_1, i, (\psi_2, j, \psi_3))$, $\psi_2 \circ_j \psi_3 = (\psi_2, j, \psi_3)$, or $\psi_4 \circ_l \psi_5 = (\psi_4, l, \psi_5)$ are sub-simplices of ψ, however, both $\psi_1 \circ_i \psi_2 = (\psi_1, i, \psi_2)$ and $\psi_3 \circ_k (\psi_4 \circ_l \psi_5) = (\psi_3, k, (\psi_4, l, \psi_5))$ are not.

In what follows, we will see how two such simplices that are presentations of the same multimap can be related, more precisely we develop the precise concept that will allow us to be able to replace (or substitue) a sub-simplex of a simplex by another one.

Definition 2.6. *Given a n-simplex ψ in $\mathcal{O}$ with $\psi = (\psi_1, ..., \psi_n)$ and $k < n$. A relaxed k-moves of ψ is the given of some $1 \leq k', k" \leq k$; $1 \leq s \leq n-k'$ and a $(n-k'+k")$-simplex ϕ such that :*

1. $\psi_j = \phi_j$ *for all* $1 \leq j \leq s-1$,
2. $\psi_j = \phi_{j-k'+k"}$ *for all* $s+k' \leq j \leq n$,
3. $|\zeta| = |\xi|$, *for* $\zeta = (\psi_s, ..., \psi_{s+k'-1})$ *a sub-simplex of* ψ, *and* $\xi = (\phi_s, ..., \phi_{s+k"-1})$ *a sub-simplex of* ϕ *such that*

Remarks.

- Firstly it is worth to point out the compatibility with the operadic composition $\circ_i$ of $\psi_{s+k'} = \phi_{s+k"}$,
- a relaxed k-moves of ψ, is finally a l-simplex ϕ, where $l = n-k'+k"$ such that $n-k+1 \leq l \leq n+k-1$. Since $k < n$ then $2 \leq l \leq 2n-2$,
- for $s = 1$, we obtain $\zeta = (\psi_1, ..., \psi_{k'})$, $\xi = (\phi_s, ..., \phi_{k"})$ and $\psi_j = \phi_{j-k'+k"}$ for all $1+k' \leq j \leq n$,
- if $k = 1$, then ϕ is a n-simplex, such that $\psi_j = \phi_j$ for all $1 \leq j \leq n$,
- one may consider $k = n$, but in this case we should have $k' < k$ and $k" < k$,
- a k-relaxed moves is a substitution of a k'-simplex ($k' \leq k$) with $k"$-simplex ($k" \leq k$), but keeping fixed the operadic composition into which they are substituted.

Definition 2.7. *Under the same hypotheses of the previous definition, we say that ψ and ϕ are equivalent by a relaxed k-moves.*

Vocabulary.
Let $\mathcal{O}$ be an operad with a finite generating set $\mathcal{T}$, then :

- every multimap in $\mathcal{O}$ has a decomposition (or presentation) in $\mathcal{T}$, then there exists a simplex ψ in $\mathcal{T}$ which is a presenation of this multimap. The above equality relation

$$|\zeta| = |\xi|$$

is either an operad associativity or unity or an equivariance axiom, otherwise is called an *elementary relation*,

- an *elementary sub-simplex* $\hat{\psi}$ of ψ is a sub-simplex of one of two following forms:
  - $\hat{\psi}$ is one side (either left or right) of a specified elementary relation,
  - $\hat{\psi}$ is one side (either left or right) of a specified operad associativity or unity or equivariance axiom involving only the generating multimaps.
    Suppose that we have a n-simplex ψ (for some integer $n \geq 2$) in $\mathcal{O}$ which is a presentation of the multimap $|\psi|$ in $\mathcal{O}$, and ζ is a sub-simplex of ψ such that $|\zeta| = |\xi|$, where ξ is a simplex in $\mathcal{O}$, then the relation $|\zeta| = |\xi|$ is either an elementary relation or an operad associativity or unity or equivariance axiom involving only the generating multimaps, hence, one can obtain a relaxed k-moves of the simplex ψ by substituting the elementary sub-simplex ζ by the other one ξ,
- two simplicies ψ and ϕ are called to be *elementarily equivalent* in $\mathcal{O}$, if ψ and ϕ are equivalent by a relaxed k-moves (for some integer $k \geq 2$), then we write $\psi \overset{k}{\sim} \phi$ (if there is no confusion, we can drop k and write $\psi \sim \phi$) and call this an *elementary equivalence* in $\mathcal{O}$, in other words an eleelementary equivalence is a subsitution of a elementary sub-simplex of one side by the other one,
- two simplicies ψ and ϕ are said to be *connected by a finite sequence of elementary equivalences* in $\mathcal{O}$ if and only if there exist some simplices $\psi_1, ..., \psi_r$ in $\mathcal{O}$ such that $\psi_1 \overset{k_1}{\sim} \ldots \overset{k_{r-1}}{\sim} \psi_r$, and $\psi \overset{k}{\sim} \psi_1 \overset{k_1}{\sim} \ldots \overset{k_{r-1}}{\sim} \psi_r \overset{k_r}{\sim} \phi$ for some integers $k, k_1, \ldots, k_r \geq 2$.

Denotations. Let $\mathcal{O}$ be an operad with a finite generating set $\mathcal{T}$. Let ψ be an n-simplex in $\mathcal{O}$ for some integer $n \geq 2$.

- The set of all relaxed k-moves of ψ in $\mathcal{T}$ is denoted by $\mathcal{W}^k_\psi$,
- The set of all relaxed k-moves for all $k \in \{2, \ldots, n-1\}$ is

$$\mathcal{W}^n_\psi := \bigcup_{k=2}^{n-1} \mathcal{W}^k_\psi,$$

and this is the set of all relaxed n-moves of ψ in $\mathcal{T}$. In other words, $\mathcal{W}^n_\psi$ is the set of all elementary equvalences of ψ.

- The set of all relaxed n-moves in $\mathcal{T}$ is denoted by $\mathcal{W}$, where

$$\mathcal{W} := \bigcup_{\psi\in\mathcal{O}} \mathcal{W}^n_\psi$$

In other words, $\mathcal{W}$ is the set of all elementary equivalences in $\mathcal{T}$.

Definition 2.8. *Let $\mathcal{O}$ be an operad with a finite generating set $\mathcal{T}$, and $\mathcal{W}$ the set of all its elementary equivalences. We say that $\mathcal{W}$ is a strong generating set for $\mathcal{O}$ in $\mathcal{T}$ if any two equivalent simplices in $\mathcal{T}$ are connected by a finite sequence of elementary equivalences in $\mathcal{W}$. Roughly speaking, $\mathcal{W}$ is the set that generate all relations in $\mathcal{O}$.*

Finally, one may understand that a multimap can be built by using the generating multimaps in two ways different thanks to a finite sequence of steps connecting them such that each step is related to the next one by replacing an elementary sub-simplex of one side (either left or right) of a specified elementary relation or a specified operad associativity/unity/equivariance axiom involving only the generating multimaps with the elementary sub-simplex of the other side.

3 Results and proofs

In this section we will present and prove our results related the finite presentations of an operad and its associate algebra. Firstly let us recall what that means. In all the remainder of this paper, let $(\mathcal{O}, 1, \circ)$ be an $\mathcal{S}$-colored operad and $(\mathcal{A}, \mu)$ the associated $\mathcal{O}$-algebra.

Definition 3.1. *We say that $\mathcal{O}$ (respectively $\mathcal{O}$-algebra $\mathcal{A}$) has a finite presentation if $\mathcal{O}$ (respectively $\mathcal{O}$-algebra $\mathcal{A}$) satisfies the two following assertions :*

1. *$\mathcal{O}$ (respectively $\mathcal{O}$-algebra $\mathcal{A}$) has a finite generating set,*
2. *any two simplices in $\mathcal{O}$ (respectively $\mathcal{O}$-algebra $\mathcal{A}$) that are presentation of the same multimap (respectively structure map) are connected by a finite sequence of elementary equivalences.*

Our first result states the following :

Theorem 3.1. *If the operad $\mathcal{O}$ has a finite generating set, then the associated $\mathcal{O}$-algebra $\mathcal{A}$, has a corresponding finite generating set.*

Proof. Let $\mathcal{T} = \{\psi_1, \dots, \psi_d\}$ be a finite generating set for $\mathcal{O}$ where $d \in \mathbb{N}^*$. Then every multimap ψ in $\mathcal{O}$ has a presentation in $\mathcal{T}$ of the form

$$\psi = \psi_{l_1} \circ_{i_1} \cdots \circ_{i_{k-1}} \psi_{l_k},$$

where $l_r, k \in \mathbb{N}^*$ for all integers $1 \le r \le k$, and $\psi_{l_1}, \dots, \psi_{l_k}$ in $\mathcal{T}$ (note that some of the ψ_{l_r} may be repeated). It follows that there exists a l-simplex $(\psi_{l_1}, \dots, \psi_{l_k})$ in $\mathcal{O}$ (especially in $\mathcal{T}$) whose composition is ψ. Then the structure map μ_ψ in $\mathcal{O}$-algebra $\mathcal{A}$ associated to the multimap ψ in $\mathcal{O}$ is $\mu_\psi = \mu_{\psi_{l_1}\circ_{i_1}\cdots\circ_{i_{k-1}}\psi_{l_k}}$, applying the associativity axiom of the $\mathcal{O}$-algebra $\mathcal{A}$, we get $\mu_\psi = \mu_{\psi_{l_1}} \circ_{i_1} \cdots \circ_{i_{k-1}} \mu_{\psi_{l_k}}$. Then μ_ψ has a presentation in the set $\mathcal{T}_\mu = \{\mu_{\psi_1}, \dots, \mu_{\psi_d}\}$. Since ψ is an arbitrary element in $\mathcal{O}$, then this is true for all ψ in $\mathcal{O}$, so each structure map μ_ψ in $\mathcal{A}$ has a presentation in $\mathcal{T}_\mu$, then $\mathcal{T}_\mu$ is a finite generating set for $\mathcal{A}$. ☐

Definition 3.2. *The structure maps of $\mathcal{T}_\mu$ are called generating structure maps.*

Lemma 3.1. *Let $n \ge 2$, and $\psi_1, \dots, \psi_n$ be some multimaps in $\mathcal{O}$. If $(\psi_1, \dots, \psi_n)$ is a n-simplex in $\mathcal{O}$, then its corresponding n-simplex in $\mathcal{A}$ is $(\mu_{\psi_1}, \dots, \mu_{\psi_n})$.*

Proof. We will lead an induction proof. For $n = 1$, a 1-simplex ψ in $\mathcal{O}$ is a generating multimap, and its composition is itself $|\psi| = \psi$, since ψ is a generating multimap in $\mathcal{O}$, then by the previous theorem, its associated structure maps μ_ψ in $\mathcal{A}$ is a generating structure map, hence the corresponding 1-simplex of ψ in $\mathcal{A}$ is the 1-simplex μ_ψ, and its composition is defined as itself $|\mu_\psi| = \mu_{|\psi|} = \mu_\psi$, in other words, one can define the 1-simplices in $\mathcal{A}$ as the generating structure maps.
For $n = 2$, a 2-simplex is an iterated operadic composition in $\mathcal{O}$ of the form $\psi = (\psi_1, i, \psi_2)$ for some integer $i \ge 1$, and its composition is $|\psi| = \psi_1 \circ_i \psi_2$, consider $\psi_1 \in \mathcal{O}\begin{pmatrix} b \\ a_1, \dots, a_n \end{pmatrix}$, and $\psi_2 \in \mathcal{O}\begin{pmatrix} a_i \\ c_1, \dots, c_m \end{pmatrix}$, are two generating multimaps in $\mathcal{O}$, then by the operadic composition $\circ_i$ defined by $\mathcal{O}$, we can assert that $\psi_1 \circ_i \psi_2 \in \mathcal{O}\begin{pmatrix} b \\ \underline{a} \circ_i \underline{c} \end{pmatrix}$, then its associated structure map is defined by

$$\mu_\psi = \mu_{\psi_1\circ_i\psi_2} := \prod_{k=1}^{i-1} \mathcal{A}_{a_i} \times \prod_{j=1}^{m} \mathcal{A}_{c_j} \times \prod_{k=i+1}^{n} \mathcal{A}_{a_i} \xrightarrow{\mu_\psi} \mathcal{A}_b$$

Follows, [1], the associativity axiom in $\mathcal{A}$ states that

$$\mu_\psi = \mu_{\psi_1 \circ_i \psi_2} = \mu_{\psi_1} \circ_i \mu_{\psi_2}$$

where μ_{ψ_1} and μ_{ψ_2} are generating structures maps in $\mathcal{A}$ (1-simplices) corresponding to the generating multimaps (1-simplices) ψ_1 and ψ_2 respectively in $\mathcal{O}$. Hence, the corresponding 2-simplex of ψ in $\mathcal{A}$ is $\mu_\psi = (\mu_{\psi_1}, i, \mu_{\psi_2})$ for some integer $i \geq 1$, and its composition is defined in $\mathcal{A}$ as

$$|\mu_\psi| = \mu_{|\psi|} = \mu_{\psi_1 \circ_i \psi_2} = \mu_{\psi_1} \circ_i \mu_{\psi_2}$$

Then the statement holds for $n = 2$.

Assume that the statement holds for any given integer $p, q \leq n$, and let $\psi = (\psi_1, \ldots, \psi_{n+1})$ be a $(n+1)$-simplex, then ψ is an iterated operadic composition in $\mathcal{O}$ of the form $\psi = (\phi, i, \varphi)$ for some integer $i \geq 1$, where ϕ is a p-simplex for some integer $p \geq 1$, and φ is a q-simplex for some integer $q \geq 1$ such that $p + q = n + 1$, without loss of generality we can suppose that $\phi = (\psi_1, \ldots, \psi_r)$ and $\varphi = (\psi_{r+1}, \ldots, \psi_{n+1})$ then by induction on n, the corresponding p-simplex of ϕ in $\mathcal{A}$ is $\mu_\phi = (\mu_{\psi_1}, \ldots, \mu_{\psi_r})$, its composition $|\mu_\phi| = \mu_{|\phi|} = \mu_{\psi_1} \circ_{i_1} \cdots \circ_{i_{r-1}} \mu_{\psi_r}$, and the corresponding q-simplex of φ in $\mathcal{A}$ is $\mu_\varphi = (\mu_{\psi_{r+1}}, \ldots, \mu_{\psi_{n+1}})$, its composition $|\mu_\varphi| = \mu_{|\varphi|} = \mu_{\psi_{r+1}} \circ_{i_{r+1}} \cdots \circ_{i_n} \mu_{\psi_{n+1}}$, by definition of the $(n+1)$-simplex ψ, the next operadic composition $|\psi| = |\phi| \circ_i |\varphi|$ is well defined in $\mathcal{O}$, then $\mu_{|\psi|} = \mu_{|\phi| \circ_i |\varphi|}$, and by the associativity axiom in $\mathcal{A}$, we get

$$|\mu_\psi| = \mu_{|\psi|} = \mu_{|\phi| \circ_i |\varphi|} = \mu_{|\phi|} \circ_{i_1} \mu_{|\varphi|}$$

Hence the $n+1$-simplex in $\mathcal{A}$ corresponding the $n+1$-simplex $\psi = (\psi_1, \ldots, \psi_{n+1})$ in $\mathcal{O}$ is $\mu_\psi = (\mu_{\psi_1}, \ldots, \mu_{\psi_{n+1}})$.
Then the statement holds for $n + 1$. □

Corollary 3.1. *Let $\widehat{\psi}$ be a sub-simplex of the n-simplex ψ in $\mathcal{O}$, then the corresponding sub-simplex of $\widehat{\psi}$ in $\mathcal{A}$ is $\mu_{\widehat{\psi}}$.*

Proof. Given a n-simplex ψ in $\mathcal{O}$ for an integer $n \geq 1$, then its corresponding n-simplex in $\mathcal{A}$ is μ_ψ. Once again we will lead an induction proof ; for $n = 1$, ψ is a 1-simplex, then ψ is a generating multimap, and a sub-simplex of ψ is defined to be itself $\widehat{\psi} = \psi$, then the corresponding sub-simplex of $\widehat{\psi} = \psi$ in $\mathcal{A}$ is $\mu_{\widehat{\psi}} = \mu_\psi$ that is a generating structure map. Then the statement holds for $n = 1$.
Suppose that the statement holds for any integers $p, q \geq 1$, and ψ is a $n+1$-simplex for an integer $n \geq 1$, then according to the definition of a simplex ψ can be written as the form $\psi = (\phi, i, \varphi)$ for some integer $i \geq 1$, with ϕ is p-simplex and φ is q-simplex such that $p + q = n + 1$, and the operadic composition in $\mathcal{O}$, $|\psi| = |\phi| \circ_i |\varphi|$, then a sub-simplex $\widehat{\psi}$ of ψ is defined to be a sub-simplex of either ϕ, or of φ, or ψ itself, hence, if $\widehat{\psi}$ is a sub-simple of either ϕ, or of φ, by induction on n, the corresponding sub-simplex of $\widehat{\psi}$ in $\mathcal{A}$ is $\mu_{\widehat{\psi}}$ which is a sub-simplex of either μ_ϕ, or of μ_φ, otherwise, if $\widehat{\psi}$ is defined to be ψ itself, then the corresponding sub-simplex of $\widehat{\psi} = \psi$ in $\mathcal{A}$, is the corresponding simplex of ψ in $\mathcal{A}$ which is $\mu_{\widehat{\psi}} = \mu_\psi$.
Then the statement holds for $n + 1$. □

Corollary 3.2. *Let ψ and ϕ be two simplices in $\mathcal{O}$, then ψ and ϕ are equivalent in $\mathcal{O}$ if and only if their associated simplices μ_ψ and μ_ϕ are equivalent in $\mathcal{A}$. More presicely*

$$|\psi| = |\phi| \Leftrightarrow |\mu_\psi| = |\mu_\phi|$$

Proof. Suppose $\psi = (\psi_1, \ldots, \psi_r)$ and $\phi = (\phi_1, \ldots, \phi_s)$ are two simplices in $\mathcal{O}$ for some integers $r, s \geq 1$ such that $|\psi| = |\phi|$.
Then ther exist $i_1, \ldots i_{r-1}, j_1, \ldots, j_{s-1} \in \mathbb{N}^*$ such that $|\psi| = \psi_1 \circ_{i_1} \cdots \circ_{i_{r-1}} \psi_r$, and $|\phi| = \phi_1 \circ_{j_1} \cdots \circ_{j_{s-1}} \phi_s$, since the corresponding simplex of ψ in $\mathcal{A}$ is $\mu_\psi = (\mu_{\psi_1}, \ldots, \mu_{\psi_r})$ with composition $|\mu_\psi| = \mu_{|\psi|} = \mu_{\psi_1} \circ_{i_1} \cdots \circ_{i_{r-1}} \mu_{\psi_r}$, and the corresponding simplex of ϕ in $\mathcal{A}$ is $\mu_\phi = (\mu_{\phi_1}, \ldots, \mu_{\phi_s})$ with composition $|\mu_\phi| = \mu_{|\phi|} = \mu_{\phi_1} \circ_{j_1} \cdots \circ_{j_{s-1}} \mu_{\phi_s}$, then the relation $|\psi| = |\phi|$ implies

$$|\mu_\psi| = \mu_{|\psi|} = \mu_{\psi_1 \circ_{i_1} \cdots \circ_{i_{r-1}} \mu_{\psi_r}} = \mu_{\phi_1 \circ_{j_1} \cdots \circ_{j_{s-1}} \mu_{\phi_s}} = \mu_{|\phi|} = |\mu_\phi|$$

Conversely, we know that the construction of a simplex in $\mathcal{A}$ requires the existence of a simplex in $\mathcal{O}$. Let ψ and ϕ be two simplices in $\mathcal{O}$ and their associated simplices respectively μ_ψ and μ_ϕ are equivalent in $\mathcal{A}$, i.e $|\mu_\psi| = \mu_{|\psi|} = \mu_{|\phi|} = |\mu_\phi|$, since each structure map has one entry in $\mathcal{O}$, let it be for example $\mathcal{O}\begin{pmatrix} a \\ b_1, \ldots, b_m \end{pmatrix}$ with $(a, \underline{b}) \in \mathcal{S} \times Prof(\mathcal{S})$ for some $m \in \mathbb{N}^*$, then $|\psi|, |\phi| \in \mathcal{O}\begin{pmatrix} a \\ b_1, \ldots, b_m \end{pmatrix}$, therefore they have the same structure map, hence $|\psi| = |\phi|$, so ψ and ϕ are two simplices equivalent in $\mathcal{O}$. This finishes the proof. □

Our second lemma corresponding to the k-moves states that

Lemma 3.2. *Given a n-simplex ψ in $\mathcal{O}$ with $\psi = (\psi_1, \ldots, \psi_n)$, for any $k < n$, and any $\phi \in \mathcal{W}^k_\psi$,* *one have*

$$\mu_\phi \in \mathcal{W}^k_{\mu_\psi}$$

in other words, if ψ and ϕ are equivalent by a relaxed k-moves in $\mathcal{O}$, then μ_ψ and μ_ϕ are equivalent by a relaxed k-moves in $\mathcal{O}$-Algebra $\mathcal{A}$.

In fact

$$\phi \in \mathcal{W}^k_\psi \Rightarrow \mu_\phi \in \mathcal{W}^k_{\mu_\psi}$$

Proof. Let $\phi \in \mathcal{W}^k_\psi$, then there exist $1 \leq k', k" \leq k$, $1 \leq s \leq n - k'$, and $(n - k' + k")$-simplex ϕ with

1. $\psi_j = \phi_j$ for all $1 \leq j \leq s - 1$,
2. $\psi_j = \phi_{j-k'+k"}$ for all $s + k' \leq j \leq n$,
3. $\zeta = (\psi_s, \ldots, \psi_{s+k'-1})$ is a sub-simplex of ψ, and $\xi = (\phi_s, \ldots, \phi_{s+k"-1})$ is a sub-simplex of ϕ such that $|\zeta| = |\xi|$, which is compatible with the operadic composition $\circ_i$ of $\psi_{s+k'} = \phi_{s+k"}$.

We conclude by the Lemma 3.1 and the Corollary 3.2

1. $\mu_{\psi_j} = \mu_{\phi_j}$ for all $1 \leq j \leq s - 1$,
2. $\mu_{\psi_j} = \mu_{\phi_{j-k'+k"}}$ for all $s + k' \leq j \leq n$,
3. $\mu_\zeta = (\mu_{\psi_s}, \ldots, \mu_{\psi_{s+k'-1}})$ is a sub-simplex of μ_ψ, and $\mu_\xi = (\mu_{\phi_s}, \ldots, \mu_{\phi_{s+k"-1}})$ is a sub-simplex of μ_ϕ such that $\mu_{|\zeta|} = \mu_{|\xi|}$, which is compatible with the operadic composition $\circ_i$ of $\mu_{\psi_{s+k'}} = \mu_{\phi_{s+k"}}$.

Then μ_ψ and μ_ϕ are equivalent by a relaxed k-moves in $\mathcal{O}$-Algebra $\mathcal{A}$. $\square$

As a direct consequence, now we can announce the following :

Theorem 3.2. *Under the same hypotheses of Definition 2.6, consider ζ and ξ two simplices in $\mathcal{O}$ such that $|\zeta| = |\xi|$ is an elementary relation in $\mathcal{O}$, then its corresponding elementary relation in $\mathcal{A}$ is $\mu_{|\zeta|} = \mu_{|\xi|}$.*

Proof. Let $|\zeta| = |\xi|$ be an elementary relation in $\mathcal{O}$, then ζ and ξ are two simplices that are equivalent in $\mathcal{O}$, by the Corollary 3.2, their corresponding simplices in $\mathcal{A}$ are equivalent, hence $|\mu_\zeta| = \mu_{|\zeta|} = \mu_{|\xi|} = |\mu_\xi|$, and the previous Lemma 3.2 provides that this relation is an elementary relation in $\mathcal{A}$ which is the corresponding of the elementary relation $|\zeta| = |\xi|$ in $\mathcal{O}$. $\square$

Remarks

- For the operad $\mathcal{WD}$ (resp. $\mathcal{UWD}$) , we know from the finite presentation theorem in Chapter 5, in [1] (resp. in Chapter 10, in [1]) that one may substitute a sub-simplex within a simplex presentation by another, only by allowing substitution of an elementary sub-simplex of one side of an elementary relation or an operad associativity/unity/equivariance axiom involving only generating wiring diagrams by the elementary sub-simplex of the other side,
- in theorems 3.2, we saw that each elementary relation in $\mathcal{O}$ has a corresponding elementary relation in $\mathcal{A}$. Now we will prove the corresponding associativity/unity/equivariance axiom for generating multimaps in $\mathcal{A}$.

Proposition 3.1. *The associativity axiom holds in $\mathcal{A}$ (for the definition see 2.11 and 2.12 in [1]).*

Proof. For some integers $n \geq 2$, $m, l \geq 1$, and $1 \leq i < j \leq n$ where $|\underline{c}| = n$, $|\underline{b}| = m$, and $|\underline{a}| = l$. Let $f \in \mathcal{O}\binom{d}{\underline{c}}$, $g \in \mathcal{O}\binom{c_i}{\underline{a}}$, and $h \in \mathcal{O}\binom{c_j}{\underline{b}}$, be some generating multimaps in $\mathcal{O}$, then the horizontal associativity in $\mathcal{O}$ states that

$$(f \circ_j h) \circ_i g = (f \circ_i g) \circ_{j-1+l} h$$

the corresponding equality in $\mathcal{A}$ of this equality is given by

$$\mu_{(f\circ_j h)\circ_i g} = \mu_{(f\circ_i g)\circ_{j-1+l} h}$$

by the associativity axiom in $\mathcal{A}$, we obtain

$$(\mu_f \circ_j \mu_h) \circ_i \mu_g = (\mu_f \circ_i \mu_g) \circ_{j-1+l} \mu_h$$

Since f, g and h are generating multimaps in $\mathcal{O}$, then μ_f, μ_g, and μ_h are generating structure maps in $\mathcal{A}$ corresponding respectively to f, g and h. Hence the last equality is the corresponding associativity of the horizontal associativity in $\mathcal{A}$.

Suppose now $n, m \geq 1$, $1 \leq i \leq n$, and $1 \leq j \leq m$.

Let $f \in \mathcal{O}\binom{d}{\underline{c}}$, $g \in \mathcal{O}\binom{c_i}{\underline{b}}$, and $h \in \mathcal{O}\binom{b_j}{\underline{a}}$, are some generating multimaps in $\mathcal{O}$, then the vertical associativity in $\mathcal{O}$ states that

$$f \circ_i (g \circ_j h) = (f \circ_i g) \circ_{i-1+j} h$$

the corresponding equality in $\mathcal{A}$ of this equality is given by

$$\mu_{f\circ_i(g\circ_j h)} = \mu_{(f\circ_i g)\circ_{i-1+j} h}$$

by the associativity axiom in $\mathcal{A}$, we have

$$\mu_f \circ_i (\mu_g \circ_j \mu_h) = (\mu_f \circ_i \mu_g) \circ_{i-1+j} \mu_h$$

where μ_f, μ_g, and μ_h are generating structure maps in $\mathcal{A}$, corresponding respectively to the generating multimaps f, g and h in $\mathcal{O}$. Then the last equality is the corresponding associativity of the vertical associativity in $\mathcal{A}$. □

Remark: By the operad associativity in $\mathcal{O}$, we mean both the horizontal and the vertical associativity, and by the $\mathcal{O}$-operad associativity in $\mathcal{A}$, we mean their corresponding in $\mathcal{A}$.

Proposition 3.2. *The unity axiom holds in $\mathcal{A}$ (for the definition see 2.13 and 2.14 in [1]).*

Proof. Let $f \in \mathcal{O}\binom{d}{\underline{c}}$ be a generating multimap, $1_d \in \mathcal{O}\binom{d}{\underline{d}}$ and 1_d the d-colored unit in $\mathcal{O}$, and $1_{c_i} \in \mathcal{O}\binom{c_i}{\underline{c_i}}$ the c_i-colored unit in $\mathcal{O}$.

The left unity in $\mathcal{O}$ states $1_d \circ_1 f = f$. Then, the corresponding equality in $\mathcal{A}$ of this equality is given by

$$\mu_{1_d \circ_1 f} = \mu_f$$

by associtivity axiom in $\mathcal{A}$, we get

$$\mu_{1_d} \circ_1 \mu_f = \mu_f$$

On the other hand, the right unity in $\mathcal{O}$ states that

$$f \circ_i 1_{c_i} = f.$$

the corresponding equality in $\mathcal{A}$ of this equality is given by

$$\mu_{f \circ_i 1_{c_i}} = \mu_f$$

Once again, the associtivity axiom in $\mathcal{A}$, assert

$$\mu_f \circ_i \mu_{1_{c_i}} = \mu_f$$

where μ_f is a generating structure map corresponding to the generating multimap f in $\mathcal{O}$, and $\mu_{1_{c_i}}$, μ_{1_d} are the identity maps in $\mathcal{A}$ corresponding respectively to the c_i-colored and d-colored unit in $\mathcal{O}$. Then the corresponding unity axiom in $\mathcal{A}$ holds. □

Proposition 3.3. *The equivariance axiom holds in $\mathcal{A}$ (for the definition see 2.15 in [1]).*

Proof. For some integers $n, m \geq 1$ and $1 \leq i \leq n$ where $|\underline{c}| = n$, $|\underline{b}| = m$, $\sigma \in \mathfrak{S}_n$, and $\tau \in \mathfrak{S}_m$. Let $f \in \mathcal{O}\binom{d}{\underline{c}}$, and $g \in \mathcal{O}\binom{c_{\sigma(i)}}{\underline{b}}$, be generating multimaps in $\mathcal{O}$, then the equivariance axiom in O states

$$f^\sigma \circ_i g^\tau = (f \circ_{\sigma(i)} g)^{\sigma \circ_i \tau}$$

the corresponding equality in $\mathcal{A}$ of this equality is given by

$$\mu_{f^\sigma \circ_i g^\tau} = \mu_{(f \circ_{\sigma(i)} g)^{\sigma \circ_i \tau}}$$

by the associativity axiom in $\mathcal{A}$, we obtain

$$\mu_{f^\sigma} \circ_i \mu_{g^\tau} = \mu_{(f \circ_{\sigma(i)} g)^{\sigma \circ_i \tau}}$$

where $f^\sigma \in \mathcal{O}\binom{d}{\underline{c}\sigma}$, $g^\tau \in \mathcal{O}\binom{c_{\sigma(i)}}{\underline{b}\tau}$ and $(f \circ_{\sigma(i)} g)^{\sigma \circ_i \tau} \in \mathcal{O}\binom{d}{(\underline{c} \circ_{\sigma(i)} \underline{b})(\sigma \circ_i \tau)}$.
Since f, g are generating multimap in $\mathcal{O}$, then f^σ, g^τ are too, hence μ_{f^σ}, μ_{g^τ} are generating structure maps in $\mathcal{A}$ corresponding respectively to the generating multimaps f^σ, g^τ in $\mathcal{O}$.
Then the last equality is the equivariance axiom in $\mathcal{A}$ corresponding to the equivariance axiom in $\mathcal{O}$. □

Vocabulary.
Let $\mathcal{O}$ be an operad equipped with a finite generating set $\mathcal{T}$, and $\mathcal{A}$ the $\mathcal{O}$-algebra equipped with the corresponding finite generating set $\mathcal{T}_\mu$.

- every structure map in $\mathcal{A}$ has a presentation in $\mathcal{T}_\mu$, then there exists a simplex in $\mathcal{T}_\mu$ which is presentation of this structure map. The equality relation defined in the Theorem 3.1

  $$|\mu_\zeta| = \mu_{|\zeta|} = \mu_{|\xi|} = |\mu_\xi|$$

  is either an $\mathcal{O}$-operad associativity or unity or an equivariance axiom, or an *elementary relation* in $\mathcal{A}$,

- an *elementary sub-simplex* $\mu_{\widehat{\psi}}$ of μ_ψ is a sub-simplex of one of two following forms:
  - $\mu_{\widehat{\psi}}$ is one side (either left or right) of a specified elementary relation in $\mathcal{A}$,
  - $\mu_{\hat{\psi}}$ is one side (either left or right) of a specified $\mathcal{O}$-operad associativity or unity or equivariance axiom involving only the generating sturture maps.

    Suppose that we have a n-simplex μ_ψ (for some integer $n \geq 2$) in $\mathcal{A}$ which is a presentation of the structure map $|\mu_\psi|$ in $\mathcal{A}$, and μ_ζ is a sub-simplex of μ_ψ such that $|\mu_\zeta| = |\mu_\xi|$, where μ_ξ is a simplex in $\mathcal{A}$, then the relation $|\mu_\zeta| = |\mu_\xi|$ is either an elementary relation in $\mathcal{A}$ or an $\mathcal{O}$-operad associativity or unity or equivariance axiom involving only the generating structur maps, hence, one can obtain a relaxed k-moves of the simplex μ_ψ by substituting the elementary sub-simplex μ_ζ by the other one μ_ξ,

- two simplicies μ_ψ and μ_ϕ are called to be *elementarily equivalent* in $\mathcal{A}$, if μ_ψ and μ_ϕ are equivalent by a relaxed k-moves (for some integer $k \geq 2$), then we write $\mu_\psi \overset{k}{\sim} \mu_\phi$ (if there is no confusion, we can drop k and write $\mu_\psi \sim \mu_\phi$) and call this an *elementary equivalence* in $\mathcal{A}$, in other words an elementary equivalence is a subsitution of a elementary sub-simplex of one side by the other one.
- two simplicies μ_ψ and μ_ϕ are said to be *connected by a finite sequence of elementary equivalences* in $\mathcal{A}$ if and only if there exist some simplicies $\mu_{\psi_1}, \ldots, \mu_{\psi_r}$ in $\mathcal{A}$ such that $\mu_{\psi_1} \overset{k_1}{\sim} \ldots \overset{k_{r-1}}{\sim} \mu_{\psi_r}$, and $\mu_\psi \overset{k}{\sim} \mu_{\psi_1} \overset{k_1}{\sim} \ldots \overset{k_{r-1}}{\sim} \mu_{\psi_r} \overset{k_r}{\sim} \mu_\phi$ for some integers $k, k_1, \ldots, k_r \geq 2$.

Denotations. Let $\mathcal{O}$ be an operad with a finite generating set $\mathcal{T}$ and $\mathcal{A}$ its associated $\mathcal{O}$-algebra with the corresponding finite generating set $\mathcal{T}_\mu$. Let ψ be an n-simplex in $\mathcal{O}$ and μ_ψ its corresponding n-simplex in $\mathcal{A}$.

- The set of all relaxed k-moves of μ_ψ in $\mathcal{T}_\mu$ is denoted by $\mathcal{W}^k_{\mu_\psi}$,
- The set of all relaxed k-moves for all $k \in \{2, \ldots, n-1\}$ is
$$\mathcal{W}^n_{\mu_\psi} := \bigcup_{k=2}^{n-1} \mathcal{W}^k_{\mu_\psi},$$
and this is the set of all relaxed n-moves of μ_ψ in $\mathcal{T}_\mu$. In other words, $\mathcal{W}^n_{\mu_\psi}$ is the set of all elementary equvalences of μ_ψ in $\mathcal{A}$.
- The set of all relaxed n-moves in $\mathcal{T}_\mu$ is denoted by $\mathcal{W}_\mu$, where
$$\mathcal{W}_\mu := \bigcup_{\mu_\psi \in \mathcal{A}} \mathcal{W}^n_{\mu_\psi}$$
In other words, $\mathcal{W}_\mu$ is the set of all elementary equivalences in $\mathcal{T}_\mu$.

Lemma 3.3. *Let ψ and ϕ be two simplices elementarily equivalent in $\mathcal{O}$, then their corresponding simplices μ_ψ and μ_ϕ are elementarily equivalent in $\mathcal{A}$. In other words*
$$\psi \sim \phi \Rightarrow \mu_\psi \sim \mu_\phi$$

Proof. Let ψ and ϕ be two simplices elementarily equivalent in $\mathcal{O}$, then ψ and ϕ are equivalent by a relaxed k-moves, then $\phi \in \mathcal{W}^k_\psi$, by the Lemma 3.2, we get $\mu_\phi \in \mathcal{W}^k_{\mu_\psi}$, where μ_ψ and μ_ϕ are the corresponding simplices of ψ and ϕ respectively in $\mathcal{A}$, in other words, μ_ψ and μ_ϕ are equivalent by a relaxed k-moves in $\mathcal{A}$, then they are elementarily equivalent in $\mathcal{A}$. □

Theorem 3.3. *Let ψ and ϕ be two simplices connected by a finite sequence of elementary equivalences in $\mathcal{O}$, then their corresponding simplices μ_ψ and μ_ϕ are connected by a finite sequence of elementary equivalences in $\mathcal{A}$. More precisely if $\psi_1, \ldots, \psi_r$ is a finite sequence of elementary equivalences in $\mathcal{O}$, for an integer $r \geq 1$ such that*
$$\psi \sim \psi_1 \sim \cdots \sim \psi_r \sim \phi$$
Then
$$\mu_\psi \sim \mu_{\psi_1} \sim \cdots \sim \mu_{\psi_r} \sim \mu_\phi$$
where $\mu_{\psi_1}, \ldots, \mu_{\psi_r}$ is the corresponding finite sequence of elementary equivalences of $\psi_1, \ldots, \psi_r$ in $\mathcal{A}$.

Proof. Let ψ and ϕ be two simplices connected by a finite sequence of elementary equivalences in $\mathcal{O}$, then there exist simplices $\psi_1, \ldots, \psi_r$ such that $\psi \sim \psi_1 \sim \cdots \sim \psi_r \sim \phi$, since $\psi \sim \psi_1$, then the Lemma 3.3 assert that $\mu_\psi \sim \mu_{\psi_1}$, similraly $\mu_{\psi_1} \sim \mu_{\psi_2}, \ldots, \mu_{\psi_r} \sim \mu_\phi$.
Hence
$$\mu_\psi \sim \mu_{\psi_1} \sim \cdots \sim \mu_{\psi_r} \sim \mu_\phi$$
□

Lemma 3.4. *Let $\mathcal{O}$ be an operad and $\mathcal{A}$ its associated algebra, let $\mathcal{T}$ be a finite generating set for $\mathcal{O}$, and $\mathcal{T}_\mu$ its corresponding finite generating set of $\mathcal{A}$, if $\mathcal{W}$ the set of all elementary equivalences in $\mathcal{T}$ is a strong generating set of $\mathcal{O}$ in $\mathcal{T}$, then $\mathcal{W}_\mu$ the set of all elementary equivalences in $\mathcal{T}_\mu$ is a strong generating set of $\mathcal{A}$ in $\mathcal{T}_\mu$.*

Proof. Let $\mu_\psi = (\mu_{\psi_1}, \ldots, \mu_{\psi_r})$ and $\mu_\phi = (\mu_{\phi_1}, \ldots, \mu_{\phi_s})$ be two simplices in $\mathcal{T}_\mu$ for some integers $r, s \geq 1$ that are equivalent, then their composition are equal in $\mathcal{A}$, i.e $|\mu_\psi| = |\mu_\phi|$, the Corollary 3.2 assert that the two simplices $\psi = (\psi_1, \ldots, \psi_r)$ and $\phi = (\phi_1, \ldots, \phi_s)$ are equivalent in $\mathcal{O}$, i.e $|\psi| = |\phi|$, since $\mathcal{W}$ is a strong generating set of $\mathcal{O}$ in $\mathcal{T}$, then ψ and ϕ are connected by a finite sequence of elementary equivalences in $\mathcal{W}$, then there exist $\varphi_1, \ldots, \varphi_l$ for an integer $l \geq 1$ such that $\psi \sim \varphi_1 \sim \cdots \sim \varphi_l \sim \phi$, by the previous Lemma 3.3, we get $\mu_\psi \sim \mu_{\varphi_1} \sim \cdots \sim \mu_{\varphi_l} \sim \mu_\phi$, where $\mu_{\varphi_1}, \ldots, \mu_{\varphi_l}$ is a finite sequence of elementary equivalences in $\mathcal{W}_\mu$. This finishes the proof. □

As we have collected all necessary tools, we can now formulate and prove our following main theorem :

Theorem 3.4. *Suppose that $\mathcal{O}$ has a finite presentation, then $\mathcal{A}$ has a corresponding finite presentation one.*

Proof. Firstly, we have to prove that every structure map can be expressed as a finite iterated operadic composition in terms of generating structue maps in $\mathcal{A}$.
Indeed, let μ_ψ be a simplex in $\mathcal{A}$ for a fixed simplex ψ in $\mathcal{O}$ whose composition is $\mu_{|\psi|}$, let $\mathcal{T} = \{\psi_1, \dots, \psi_d\}$ be a finite generating set for $\mathcal{O}$ where $d \in \mathbb{N}^*$, then there exist $\psi_1, \dots, \psi_l \in \mathcal{T}$ for $l \in \mathbb{N}^*$ sucht that $\psi = (\psi_1, \dots, \psi_l)$ and

$$|\psi| = \psi_1 \circ_{i_1} \cdots \circ_{i_{l-1}} \psi_l \text{ fo some integers } \imath_1, \dots, i_{l-1} \geq 1$$

By Theorem 3.1, the corresponding finite generating set of $\mathcal{T}$ in $\mathcal{A}$ is $\mathcal{T}_\mu = \{\mu_{\psi_1}, \dots, \mu_{\psi_d}\}$, and by Lemma 3.1 the corresponding l-simplex of $(\psi_1, \dots, \psi_l)$ in $\mathcal{A}$ is $(\mu_{\psi_1}, \dots, \mu_{\psi_l})$ whose composition is given by

$$|\mu_\psi| = \mu_{|\psi|} = \mu_{\psi_1 \circ_{i_1} \cdots \circ_{i_{l-1}} \psi_l} = \mu_{\psi_1} \circ_{i_1} \cdots \circ_{i_{l-1}} \mu_{\psi_l}$$

Since this is true for every structure map in $\mathcal{A}$, then every structure map can be expressed as a finite iterated operadic composition in terms of generating structue maps in $\mathcal{T}_\mu$ which is a finite generating set of $\mathcal{A}$.

Secondly, it remains to prove that if a structure map can be operadically generated by the generating structure maps in two different ways, then there exists a finite sequence of elementary equivalences in $\mathcal{W}_\mu$ from the first iterated operadic composition to the other one.
Let $\mu_{|\varphi|}$ be a structure map in $\mathcal{A}$, for a fixed $|\varphi|$ in $\mathcal{O}$ which is generated by the generating structure maps in two different ways, then there exist two simplices in $\mathcal{O}$, $\psi = (\psi_1, \dots, \psi_r)$ in which the composition is $|\psi|$ and $\phi = (\phi_1, \dots, \phi_s)$ in which the composition is $|\phi|$ for some integers $r, s \geq 1$ such that

$$\mu_{|\varphi|} = \mu_{|\psi|} = \mu_{|\phi|}$$

By the corollary 3.2, we get $|\psi| = |\phi|$, then ψ and ϕ are equivalent in $\mathcal{T}$, since $\mathcal{W}$ is a strong generating set for $\mathcal{O}$ in $\mathcal{T}$, then there exist a finite sequence of elementary equivalences $\varphi_1, \dots, \varphi_m$ in $\mathcal{W}$, such that $\psi \sim \varphi_1 \sim \cdots \sim \varphi_m \sim \phi$ for some integer $m \geq 1$, the Theorem 3.3 assert that μ_ψ and μ_ϕ are connected by a finite sequence of elementary equivalences in $\mathcal{W}_\mu$

$$\mu_\psi \sim \mu_{\varphi_1} \sim \cdots \sim \mu_{\varphi_m} \sim \mu_\phi$$

By the Lemma 3.4, $\mathcal{W}_\mu$ is a strong generating set of $\mathcal{A}$ in $\mathcal{T}_\mu$, hence μ_ψ and μ_ϕ are connected by finite sequence of elementary equivalences in $\mathcal{W}_\mu$.
This fineshes the proof □

Our approach's advantage that it allow us to get a finite presentation of the $\mathcal{O}$-algebra $\mathcal{A}$ directly out of the finite presentation of the operad $\mathcal{O}$.

Unfortunately, we wanted to prove that any multimap (resp. structure map) in the $\mathcal{O}$ operad with a finite generating set has a stratified presentation which is a simplex presentation where the same generating multimaps within the simplex must appear in a consecutive serie, for example, suppose $\mathcal{O}$ an operad with the finite generating set $\mathcal{T} = (\psi_1, \psi_2, \psi_3)$, consider the following 9-simplex $(\psi_1, \psi_2, \psi_3, \psi_1, \psi_3, \psi_2, \psi_3, \psi_1, \psi_3)$ then the stratified 9-simplex is given by $(\psi_1, \psi_1, \psi_1, \psi_2, \psi_2, \psi_3, \psi_3, \psi_3, \psi_3)$. D.Yau had proved that every wiring diagram in $\mathcal{WD}$ (resp. undirected wiring diagram in $\mathcal{UWD}$) has a stratified presentation (see Theorem 5.11 and 10.12 in [1]), but we couldn't prove that for a such simplex in $\mathcal{O}$.

4 Applications

As an application of our main theorem, we will consider the operad of wiring diagrams, denoted $\mathcal{WD}$, and that of undirected wiring diagrams, denoted $\mathcal{UWD}$. We suppose here that the reader is familiar with the basic tools, vocabulary of the operad of wiring diagrams (directed or not) and its structure. If not, we recommend, [1]. Let us recall that a multimap in $\mathcal{WD}$ (respectively in $\mathcal{UWD}$) is called a wiring diagram (respectively undirected wiring diagram), and a generating multimap in $\mathcal{WD}$ (respectively in $\mathcal{UWD}$) is called a generating wiring diagram (respectively a generating undirected wiring diagram). In fact, D. Yau has especially proved that both $\mathcal{WD}$ and $\mathcal{UWD}$ have a finite presentation (Theorem 5.22 in [1], for $\mathcal{WD}$), (Theorem 10.19 in [1], for $\mathcal{UWD}$).

In the chapter 3 in [1] D. Yau described the set

$$\mathcal{T}^{\mathcal{WD}} = \{\epsilon, \delta, \tau, \theta, \lambda, \sigma_*, \sigma^*, \omega\}$$

of 8 wiring diagrams which is a finite generating set for the operad $\mathcal{WD}$. In the chapter 8 in [1] he had described the set

$$\mathcal{T}^{\mathcal{UWD}} = \left\{\epsilon, \omega_*, \tau_f, \theta_{(X,Y)}, \lambda_{(X,x_\pm)}, \sigma^{(X,x_1,x_2)}\right\}$$

of 6 undirected wiring diagrams which is a finite generating set for the operad $\mathcal{UWD}$. Then every wiring diagram (resp. undirected wiring diagram) has a presentation in $\mathcal{T}^{\mathcal{WD}}$ (resp. $\mathcal{T}^{\mathcal{UWD}}$)

as a finite iterated operadic composition. That is the first assumption of our theorem.

For the wirings diagrams, in the chapter 3 in [1] D. Yau described 28 elementary relations in $\mathcal{T}^{\mathcal{WD}}$, and in the chapter 5 in [1] he had proved that any two equivalent simplices in $\mathcal{WD}$ are either equal or are connected by a finite sequence of elementary equivalences where each elementary equivalence is induced by either one of the 28 elementary relations or an operad associativity/unity/equivariance axiom for the generating wiring diagrams. Following our denotations, the set $\mathcal{W}^{\mathcal{WD}}$ of all elementary equivalences in $\mathcal{T}^{\mathcal{WD}}$ is a strong generating set of $\mathcal{WD}$. That is the second assumption of our theorem.
Hence, by our main theorem 3.4, the $\mathcal{WD}$-algebra has a corresponding finite presentation.
By using our approach in Theorem 3.1, the cor-reseponding finite generating set for the $\mathcal{WD}$-algebra is given by

$$\mathcal{T}_{\mu}^{\mathcal{WD}} = \{\mu_{\epsilon}, \mu_{\delta}, \mu_{\tau}, \mu_{\theta}, \mu_{\lambda}, \mu_{\sigma_*}, \mu_{\sigma^*}, \mu_{\omega}\}.$$

which elements are called the generating structure maps corresponding to the 8 generating wiring diagrams. Then every structure map can be obtained from finitely many generating structure maps via some iterated operadic compositions. These generating structure maps are exactly those defined by D. Yau (see definition 6.9 in [1]). For example the first generating wiring diagram in $\mathcal{WD}$ is the empty wiring diagram $\epsilon \in \mathcal{WD}\binom{\emptyset}{}$ (see Definition 3.1 in [1]), then its corresponding generating structure map in the $\mathcal{WD}$-algebra is defined to be $\mu_{\epsilon} : * \longrightarrow \mathcal{A}_{\emptyset}$ (see Definition 6.9 in [1]).

In other hand, by applying Theorem 3.2 to the operad $\mathcal{WD}$, we find that every elementary relation in $\mathcal{WD}$ has a corresponding elementary relations in the $\mathcal{WD}$-algebra. Then every one of the 28 elementary relations in $\mathcal{WD}$ (see definition 3.43 in [1]) has a corresponding elementary relation in the associated $\mathcal{WD}$-algebra that are exactly the generating axioms defined by D. Yau (see definition 6.9 in [1]). For example, the first elementary relation in $\mathcal{WD}$ (see proposition 3.15 in [1]) says that if $\tau_{b,a} \in \mathcal{WD}\binom{a}{b}$ and $\tau_{c,b} \in \mathcal{WD}\binom{b}{c}$ are two consecutive name changes, then they can be composed into one name change $\tau_{c,a} \in \mathcal{WD}\binom{a}{c}$, i.e

$$\tau_{b,a} \circ \tau_{c,b} = \tau_{c,a}$$

By using our Theorem 3.2, its corresponding elementary relation in $\mathcal{WD}$ is given by

$$\mu_{\tau_{b,a}} \circ \mu_{\tau_{c,b}} = \mu_{\tau_{c,a}}$$

where $\mu_{\tau_{x,y}} := \mathcal{A}_x \longrightarrow \mathcal{A}_y$.
This last equality is exactly the first generating axiom in $\mathcal{WD}$-algebra defined by D.Yau (see definition 6.9 in [1]). The corresponding operad associativity, unity and equivariance axiom can be obtained immediately from the propositions 3.1, 3.2 and 3.3.

In [1] D.Yau had proved that (Theorem 5.22) any two equivalent simplices in $\mathcal{WD}$ are either equal or are connected by a finite sequence of elementary equivalences in $\mathcal{WD}$ which each elementary equivalence is induced by either one of the 28 elementary equivalences or an operad associativity/unity/equivariance axiom for the generating wiring diagrams in $\mathcal{WD}$, hence, by our Theorem 3.3 any two equivalent simplices in $\mathcal{WD}$-algebra are either equal or are connected by a finite sequence of elementary equivalences in $\mathcal{WD}$-algebra, which each elementary equivalence in $\mathcal{WD}$-algebra is induced by either one of the 28 generating axioms or an $\mathcal{WD}$-algebra operad associativity/unity/equivariance axiom for the generating structures maps. Following our denotations, the set $\mathcal{W}_{\mu}^{\mathcal{WD}}$ of all elementary equivalences in $\mathcal{T}_{\mu}^{\mathcal{WD}}$ is a strong generating set of $\mathcal{WD}$-algebra. Then, $\mathcal{WD}$-algebra has a finite presentation.

The same can be done for the operad $\mathcal{UWD}$, of undirected wiring diagrams by considering its finite generating set

$$\mathcal{T} = \left\{\epsilon, \omega_*, \tau_f, \theta_{(X,Y)}, \lambda_{(X,x_{\pm})}, \sigma^{(X,x_1,x_2)}\right\},$$

whose corresponding finite generating set for the $\mathcal{WD}$-algebra is

$$\mathcal{T}_{\mu} = \left\{\mu_{\epsilon}, \mu_{\omega_*}, \mu_{\tau_f}, \mu_{\theta_{(X,Y)}}, \mu_{\lambda_{(X,x_{\pm})}}, \mu_{\sigma^{(X,x_1,x_2)}}\right\}.$$

and the elementary relations in $\mathcal{UWD}$ is given by D. Yau (see definition 8.26 in [1]), and their corresponding elementary relations in $\mathcal{UWD}$-algebra that are called by D. Yau generating axioms (see definition 11.1 in [1]).
For more application, the reader can see the operad of normal wiring diagrams and strict wiring diagrams and their algebras.

5 Future Work

As an extension of our study, we will try to investigate our result and discribe the operad with two inputs and one output, a such operad is of the form $\mathcal{O}\binom{Z}{X \quad Y}$, where $X, Y, Z \in \mathcal{S}$.

Let ψ be a multimap in $\mathcal{O}\binom{Z}{X \quad Y}$, by the $\circ_i$-composition in $\mathcal{O}\binom{Z}{X \quad Y}$, we can decompose ψ as follows :

$$\mathcal{O}\binom{Z}{I} \times \mathcal{O}\binom{I}{I\ I} \times \mathcal{O}\binom{I}{X} \times \mathcal{O}\binom{I}{Y}$$

$$\xrightarrow{(\circ_1;Id;Id)} \mathcal{O}\binom{Z}{I\ I} \times \mathcal{O}\binom{I}{X} \times \mathcal{O}\binom{I}{Y}$$

$$\xrightarrow{(\circ_1;Id)} \mathcal{O}\binom{Z}{X\ I} \times \mathcal{O}\binom{I}{Y} \xrightarrow{\circ_2} \mathcal{O}\binom{C}{X\ Y}$$

Here, I is a specific element in $\mathcal{S}$.
For a specific elements X, Y and Z in $\mathcal{S}$, we will try to find a finite generating set $\mathcal{T}$ for this operad.
By putting

$$\psi^Z \in \mathcal{O}\binom{Z}{I}$$

$$\psi_I \in \mathcal{O}\binom{I}{I\ I}$$

$$\psi_X \in \mathcal{O}\binom{I}{X}$$

We obtain

$$\psi = ((\psi^Z \circ_1 \psi_I) \circ_1 \psi_X) \circ_2 \psi_Y$$

By the associativity axiom in $\mathcal{O}$, we get also

$$\psi = ((\psi^Z \circ_1 \psi_I) \circ_2 \psi_Y) \circ_1 \psi_X$$

$$\psi = \psi^Z \circ_1 ((\psi_I \circ_1 \psi_X) \circ_2 \psi_Y)$$

$$\psi = \psi^Z \circ_1 ((\psi_I \circ_2 \psi_Y) \circ_1 \psi_X)$$

$$\psi = (\psi^Z \circ_1 (\psi_I \circ_1 \psi_X)) \circ_2 \psi_Y$$

$$\psi = (\psi^Z \circ_1 (\psi_I \circ_2 \psi_Y)) \circ_1 \psi_X$$

These equalities give all the presentation possible of ψ, then there is 6 simplices, in fact, six 4-simplices that can be a presentation of ψ. from these equalities, we check the following elementary relations in $\mathcal{O}\binom{Z}{X\ Y}$:

$$(\psi^Z \circ_1 \psi_I) \circ_1 \psi_X = \psi^Z \circ_1 (\psi_I \circ_1 \psi_X)$$

$$(\psi^Z \circ_1 \psi_I) \circ_2 \psi_Y = \psi^Z \circ_1 (\psi_I \circ_2 \psi_Y)$$

$$(\psi_I \circ_1 \psi_X) \circ_2 \psi_Y = (\psi_I \circ_2 \psi_Y) \circ_1 \psi_X$$

The six 4-simplices are :

$$\psi_1 = (((\psi^Z, 1, \psi_I), 1, \psi_X), 2, \psi_Y)$$
$$\psi_2 = (((\psi^Z, 1, \psi_I), 2, \psi_Y), 1, \psi_X)$$
$$\psi_3 = (\psi^Z, 1, ((\psi_I, 1, \psi_X), 2, \psi_Y))$$
$$\psi_4 = (\psi^Z, 1, ((\psi_I, 2, \psi_Y), 1, \psi_X))$$
$$\psi_5 = ((\psi^Z, 1, (\psi_I, 1, \psi_X)), 2, \psi_Y)$$
$$\psi_6 = ((\psi^Z, 1, (\psi_I, 2, \psi_Y)), 1, \psi_X)$$

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