Multiple Solutions for Liénard Type Generalized Equations

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Abstract: - Two-point boundary value problems for second-order ordinary differential equations of Lie´nard type are studied. A comparison is made between equations $x'' + f(x)\, x'^2 + g(x) = 0$ and $x'' + f(x)\, x' + g(x) = 0$. In our approach, the Dirichlet boundary conditions are considered. The estimates of the number of solutions in both cases are obtained. These estimates are based on considering the equation of variations around the trivial solution and some additional assumptions. Examples and visualizations are supplied.

Key-Words: - Lie´nard equation, ordinary differential equation, phase plane, boundary value problems, multiplicity of solutions, number of solutions.

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1 Introduction

Two-point boundary value problems (BVP) often arise in various types of mathematical models of real-world phenomena. Consequently, the literature on BVP for ordinary differential equations (ODE) is im- mense. We refer to the books and chapters in the books, [1], [3], [10], [11], where BVPs for ODE were studied. Numerous articles are devoted to this subject.

The first results on BVP for ODE were focused on the existence of solutions and uniqueness. The Dirichlet and the Neumann problems shared the main attention. Then the following issues were studied: dependence of solutions on parameters; different boundary conditions; periodic boundary conditions; multipoint boundary conditions; functional boundary conditions; BVP for higher order ODE; BVP for systems of ODE; BVP for fractional differential equations; BVP for time scales; the multiplicity of solutions.

The last issue is of great importance. It is a remarkable feature of nonlinear ODE to have multiple solutions. Estimates of the number of solutions were the focus of study in many papers.

In this article, we wish to contribute to the special approach that is applied in the studies of multiple solutions to BVP. In the early papers of [10], the following idea was used.

Imagine that the boundary value problem has a solution $\xi(t)$. Suppose the variational equation around $\xi(t)$ is oscillatory. If some other solution, satisfying the first of boundary conditions exists, and it is not oscillatory, then between those two solutions may exist other solutions of the same BVP.

More precisely, suppose that the problem

$$x'' + g(x) = 0, \quad g \in C^1 \tag{1}$$

$$x(a) = 0, \qquad x(b) = 0 \tag{2}$$

is to be studied. Let $g(0) \equiv 0$. Then there is the trivial solution $\xi(t) \equiv 0$ for the problem. Suppose that:

(A1) There exist solutions $x_+(t)$ **(resp.:** $x_-(t)$**) to the Cauchy problem** $x''+g(x) = 0,\ x(a) = 0,\ x'(0) > 0$ **(resp.:** $x''(0) < 0$**) such that** $x_\pm(t)$ **do not vanish in the interval** $(a, b]$.

Does this problem have a solution other than the trivial one?

It has, if the equation of variations

$$y'' + g_x(0)\, y = 0 \tag{3}$$

is oscillatory.

To be definite, the following is true: if solutions

$y_+(t)$ and $y_-(t)$ of the Cauchy problems

$$y'' + g_x(0)\, y = 0, y(a) = 0, y'(a) = \pm 1 \quad (4)$$

have i zeros in the interval (a, b), and the condition **(A1**) holds, then the BVP (1), (2) has at least $2i$ more solutions.

The conditions hold for the problem

$$x'' + ax - bx^3 = 0, x(a) = 0, x(b) = 0, \quad (5)$$

for instance.

This approach was used in the study of BVPs in the articles, [5], [6], [7], [8], [9].

In the recent work, [4], the two-point BVPs were considered for the equation

$$x'' + f(x)x'^2 + g(x) = 0, \quad (6)$$

which can be classified as the Liénard equation with the quadratic dependence on x′. The function $g(x) = ax - bx^3$ was specified. Several choices of $f(x)$ were tested. The conclusion was that for the problem (6), (2) the number of solutions is not less than that for the problem (5).

The method of investigation was based on the variable transformation proposed by M. Sabatini in [12]. After applying this transformation the equation (6) reduces to the conservative equation of the form $u'' + h(u) = 0$. Sabatini's approach is briefly described below.

It seems natural to consider the formally similar equation

$$x'' + f(x)x' + g(x) = 0 \quad (7)$$

and to compare the results, concerning the multiplicity of solutions to BVPs (6), (2) and (7), (2). This is the main thrust of this paper. In our considerations functions $f(x)$ and $g(x)$ are continuously differentiable in $[a, b]$.

After the introduction brief description of Sabatini's transformation and the results for equation (6) follow.

Then the BVP (7), (2) is considered and the multiplicity results are formulated.

In the Examples section, several BVPs are studied for different choices of $f(x)$. The function $g(x)$ is cubic, that is, $g(x) = ax - bx^3$, where a and b are constants.

2 Equation $x'' + f(x)x'^2 + g(x) = 0$

2.1 Reduction to Shorter Equation

It is known that this equation by the variable change

$$u = \int_0^x e^{F(s)} ds, F(x) = \int_0^x f(s) ds \quad (8)$$

is reduced to the conservative equation

$$u'' + h(u) = 0, \quad h(u) = g(x(u))\, e^{F(x(u))}. \quad (9)$$

The differential equation (9) is of the same form as equation (1). The boundary conditions (2) in the new variable are the same

$$u(a) = 0, u(b) = 0. \quad (10)$$

The BVP (9), (10) can be qualitatively studied by the same method, making use of the fact that zeros of $x(t)$ and the respective $u(t)$ coincide.

The result of this study is the main conclusion: BVP

$$x'' + f(x)x'^2 + g(x) = 0, x(a) = 0, x(b) = 0, \quad (11)$$

generally, has at least the same number of solutions, as the BVP

$$x'' + g(x) = 0, \quad x(a) = 0, x(b) = 0 \quad (12)$$

had.

2.2 Alternative Approach

The problem (11) could be studied directly. It has the trivial solution $\xi(t) \equiv 0$. The variational equation around the trivial solution is

$$y'' + g_x(0)\, y = 0. \quad (13)$$

Suppose that the solution $y(t)$ with the initial conditions $y(a) = 0, y'(a) = 1$ has exactly i zeros in the interval (a, b) and $y(b) \neq 0$. Assume also that there exist
solutions $x_+(t)$ (resp.: $x_-(t)$) to the Cauchy problem $x'' + f(x)x'^2 + g(x) = 0, x(a) = 0, x'(a)>0$ (resp.: $x'(a)< 0$) such that $x_\pm(t)$ do not vanish in the interval $(a, b]$ (refer to this assumption as **A1$_{\text{quadr.}}$**) Then there exist at least $2i$ nontrivial solutions of the BVP (11). The condition **A1$_{\text{quadr.}}$** fulfils for $g(x) = ax - bx^3$.

3 Dissipative Equation

Equation (9) is conservative, so equation (6) also is. Our intent now is to consider the dissipative equation

$$x'' + f(x)x' + g(x) = 0, \quad (14)$$

together with the boundary conditions (2).

Suppose that $g(0) = 0$, so the problem has the trivial solution $\xi(t) \equiv 0$. The variational equation at $\xi(t)$ is

$$y'' + (f_x x' + g_x)\,|_{x\equiv 0}\, y + f\,|_{x\equiv 0}\, y' = 0 \quad (15)$$

or, the same,

$$y'' + f(0)\, y' + g_x(0)\, y = 0. \quad (16)$$

Formulate the condition:

(B1) There exist solutions $x_+(t)$ **(resp.:** $x_-(t)$**) to the Cauchy problem** $x'' + f(x)x' + g(x) = 0, x(a) = 0,$

$x'(a) > 0$ **(resp.:** $x'(a) < 0$**) such that** $x(t)$ **is positive (resp.: negative) in the interval** $(a, b]$.

The following result can be proved.

Theorem 1 *Let the condition* ***(B1)*** *hold. Suppose that the solution y(t) of the Cauchy problem*

$$y'' + f(0)\, y' + g_x(0)y = 0,\; y(a) = 0,\; y'(a) = 1 \quad (17)$$

has n zeros in the interval (a, b). Then the BVP (14), (2) has at least 2n nontrivial solutions.

Scheme of the proof. To avoid many technicalities and discussion about types of non-extendibility of solutions of the second order equations, we make one additional assumption: solutions of (14), $x(a) = 0,\ x'(a) = \alpha,\ \alpha \in [0, \alpha_{max}]$ extend to the interval $[a, b]$, where α_{max} is the initial value for the first derivative of a solution $x(t)$ from the condition **(B1).**

Consider the set S of solutions of the problem (14), $x(a) = 0,\ x'(a) = \alpha,\ \alpha \in [0, \alpha_{max}]$. Due to the extendibility of solutions, this set is bounded in $C^1[a, b]$ (Theorem 15.1 in [10]). Then there exists a number $\delta > 0$ such that for any solution $x(t)$ of S the arbitrary consecutive t_1 and t_2 are separated by this δ, that is, $|t_1 - t_2| > \delta$. The number δ is dependent on S only, not on the choice of a solution. This was proved in [2], with reference to Valle`e Poussin Theorem, [13]. All solutions in S cross the zero solution transversally. From these facts and the continuous dependence on the initial data follows that zeros $t_i(\alpha)$ of solutions in S are continuous functions of α. Any zero escapes the interval $(a, b]$ when α goes from zero value to α_{max}. On any occasion, $t_i(\alpha_i) = b$, a new solution to the boundary value problem (14), (2) emerges. This process, when performed up and down (for α positive and negative) yields at least $2n$ nontrivial solutions.

4 Conclusion

The estimations of the number of solutions to the BVP for equations

$$x'' + f(x)x'^2 + g(x) = 0 \quad (18)$$

and

$$x'' + f(x)x' + g(x) = 0 \quad (19)$$

can be produced using the equations of variations (at the trivial solution $\xi(t) \equiv 0$)

$$y'' + g_x(0)\, y = 0, \quad (20)$$

for equation (18), and

$$y'' + f(0)\, y' + g_x(0)\, y = 0, \quad (21)$$

for equation (19).

Since equations of variations are different, **the estimates of the number of solutions to both BVPs are also different.**

5 Examples

In this section, examples are constructed for the specific choice of $g(x) = ax - bx^3$.

5.1 Conservative Equation

Consider the problem

$$x'' + \mu x'^2 + (ax - bx^3) = 0, \quad (22)$$

$$x(0) = 0, \qquad x(1) = 0. \quad (23)$$

The phase portrait of equation (22), $a = 50,\ b = 25$, $\mu = 1$ is depicted in Fig. 1. It corresponds to $f(x) = \mu$ and $g(x) = ax - bx^3$ in (18).

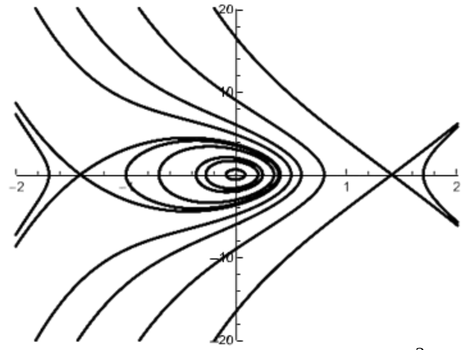

Fig. 1: The phase portrait of $x'' + \mu x'^2 + (ax - bx^3) = 0$, $a = 50$, $b = 25$, $\mu = 1$.

The equation of variations around the trivial solution $\xi(t) \equiv 0$ is

$$y'' + ay = 0. \quad (24)$$

Let $a = 50,\ b = 25$. The equation of variations is $y'' + 50y = 0$. The solution of $y(0) = 0,\ y'(0) = 1$ is $y(t) = \frac{1}{\sqrt{50}} \sin \sqrt{50}\, t$. It has exactly two zeros in the interval $(0, 1)$ and it is not zero at $t = 1$. The condition **(A1)** is fulfilled also ([4]) and BVP (18), (2) has at least four nontrivial solutions: three nontrivial solutions for the initial conditions (see Fig.2, $\alpha \approx -3.11,\ \alpha \approx -3.6605,\ \alpha \approx 3.1$) in the region bounded by a homoclinic trajectory and one solution outside this region (see Fig.2, $\alpha \approx 16.4802$).

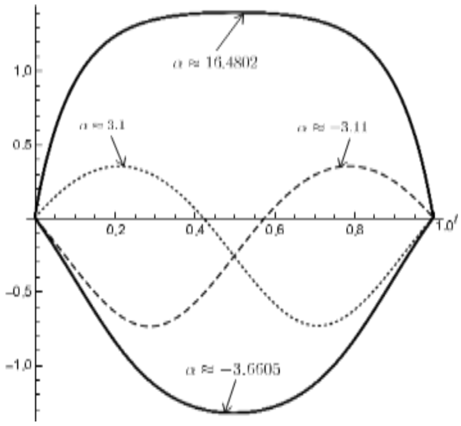

Fig. 2: *Graphs x(t) for solutions of the problem* $x''+x'^2 + (50x - 25x^3) = 0$, $x(0) = 0$, $x(1) = 0$, $\alpha \approx 16.4802$, $\alpha \approx 3.1$, $\alpha \approx -3.11$, $\alpha \approx -3.6605$.

Consider the equation

$$x'' + \mu (x^2 - 1) x'^2 + g(x) = 0, \qquad (25)$$

with the conditions (23), where $\mu = 1$ and $g(x) = 50x - 25x^3$. The phase portrait is depicted in Fig.3.

For equation (25) the number of solutions for the Dirichlet problem is the same as for equation $x'' + \mu x'^2 + (ax - bx^3) = 0$. The Dirichlet problem (25), (23) has three solutions in the region bounded by a homoclinic trajectory ($\alpha \approx 3.367$, $\alpha \approx 4.214$, $\alpha \approx -3.378$) and one solution outside this region ($\alpha \approx -12.3317$, see in Fig. 4).

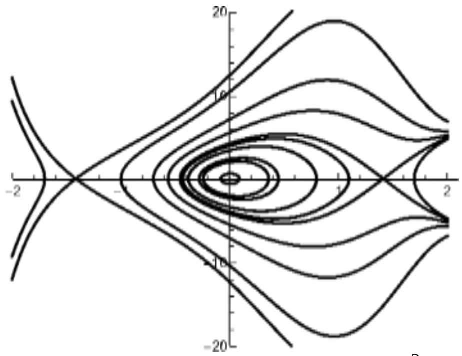

Fig. 3: *The phase portrait of* $x'' + (x^2 - 1) x'^2 + 50x - 25x^3 = 0$.

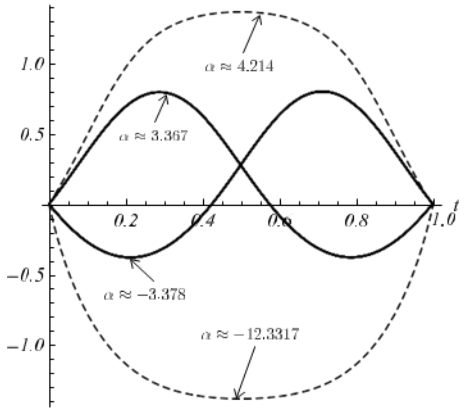

Fig. 4: *Graphs x(t) for solutions of the* $x'' + (x^2 - 1) x'^2 + 50x - 25x^3$, $x(0) = 0$, $x(1) = 0$, $\alpha \approx 3.367$, $\alpha \approx 4.214$, $\alpha \approx -3.378$, $\alpha \approx -12.3317$.

5.2 Dissipative Equation

Consider the problem

$$x'' + \mu x' + (ax - bx^3) = 0, \; x(0) = 0, \; x(1) = 0. \qquad (26)$$

It corresponds to $f(x) = \mu$ and $g(x) = ax - bx^3$ in equation (19). The equation of variations around the trivial solution $\xi(t) \equiv 0$ is

$$y'' - \mu y' + ay = 0. \qquad (27)$$

Let again $a = 50$, $b = 25$, and μ be an arbitrary positive number. The solution $y(t)$ of (27) is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}, \qquad (28)$$

where λ' s are solutions of the characteristic equation

$$\lambda^2 - \mu \lambda + a = 0. \qquad (29)$$

The roots of (29) are

$$\lambda = \frac{1}{2}\mu \pm \sqrt{\frac{1}{4}\mu^2 - a}. \qquad (30)$$

The complex roots (then $\frac{1}{4}\mu^2 - a < 0$) correspond to the oscillatory solutions

$$y(t) = C_1 e^{0{,}5\mu t} \sin \sqrt{a - \frac{1}{4}\mu^2}\, t + C_2 e^{0{,}5\mu t} \cos \sqrt{a - \frac{1}{4}\mu^2}\, t. \qquad (31)$$

The solution of (31) with the initial conditions $y(0) = 0$, $y'(0) = 1$ is

$$y(t) = C_1 e^{0{,}5\mu t} \sin \sqrt{a - \frac{1}{4}\mu^2}\, t, \qquad (32)$$

where C_1 is the appropriate positive constant.

The estimate for the number of solutions to the BVP (25) depends on the number of zeros of the function (32) in the interval $(0,1)$.

Lemma 2 *The number of zeros of the function (32) in the interval (0, 1) is not greater than the number of zeros of the function* $\sin \sqrt{a}\, t$.

This means that the estimate of the number of solutions to BVP for the dissipative equation is not better (better=more solutions) than the estimate for the conservative equation.
On the other hand, it may be worst. Since μ in (32) can be chosen so that $\sqrt{a-\frac{1}{4}\mu^2}$ is arbitrarily small, it is possible that

$$\sqrt{a-\frac{1}{4}\mu^2} < \pi. \tag{33}$$

Then the function (32) does not vanish in *(0, 1]*. If additionally, the condition **(B1)** holds, the BVP may have no nontrivial solutions.

Theorem 3 *Suppose that*

$$i\frac{\pi}{\sqrt{a}} < 1 < (i+1)\frac{\pi}{\sqrt{a}} \tag{34}$$

for some positive integer i. Then problem (22), and (23) has at least 2i nontrivial solutions.
Suppose that

$$\frac{\pi}{\sqrt{a-\frac{1}{4}\mu^2}} > 1 \tag{35}$$

Then problem (26) may have less nontrivial solutions than problems (22), (23) with the same a, μ.

If $a = 50$, then to satisfy the inequality (33), choose $12.67129 < \mu < 14.14215$. Suppose $\mu = 14$. It appears (look at the phase portrait in Fig. 5) that the condition **(B1)** for the dissipative equation holds. Fig. 6 shows that there are no nontrivial solutions for the BVP.

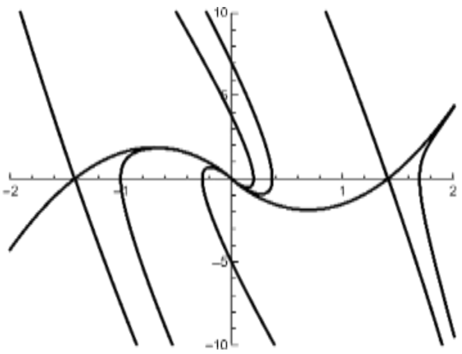

Fig. 5: The phase portrait of $x'' + \mu x' + 50x - 25x^3 = 0$, $\mu = 14$.

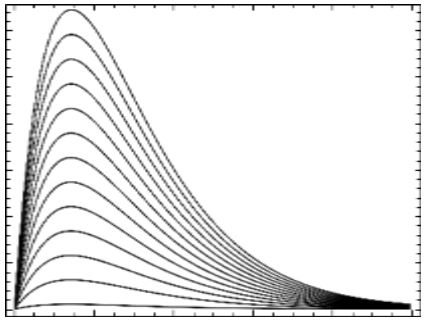

Fig. 6: *Graphs x(t) for solutions of the problem* $x'' + \mu x' + 50x - 25x^3 = 0$, $\mu = 14$, $\alpha \approx 3.367$, $\alpha \approx 4.214$, $0.1 < \alpha < 6.1$, *step 0.5.*

Consider the example where in problem (26) $a = 50$, $b = 25$, $\mu = 1$. The phase portrait is depicted in Fig. 7. The variational equation

$$y'' - y' + ay = 0 \tag{36}$$

along with the initial conditions $y(0) = 0$, $y'(0) = 1$ has a solution $y(t)$ with exactly two zeros in the interval $(0,1)$ and $y(1) \neq 0$. Therefore, by Theorem 3, the corresponding BVP (22), (23) must have at least 4 nontrivial solutions. It has four, as may be concluded, looking at Fig. 7.

All four nontrivial solutions are depicted in Fig. 8.

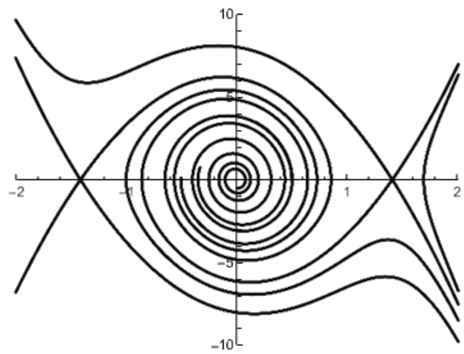

Fig. 7: The phase portrait of $x'' + x' + 50x - 25x^3 = 0$

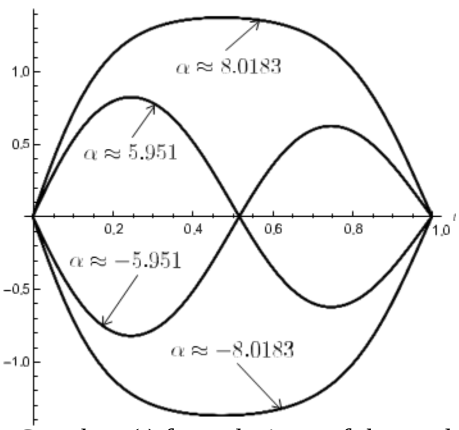

Fig. 8: *Graphs x(t) for solutions of the problem* $x'' + x' + 50x - 25x^3 = 0$, $\alpha \approx 5.951$, $\alpha \approx 8.0183$, $\alpha \approx -5.951$, $\alpha \approx -8.0183$

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The authors have no conflicts of interest to declare that are relevant to the content of this article.

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