Application of the Homotopy Perturbation Method for Differential Equations

NECIB ABDELHALIM, REZZOUG IMAD, BENBRAHIM ABDELOUAHAB
Department of Mathematics and Computer Science
University of Oum El Bouaghi, Algeria.
Laboratory of Dynamics systems and control
ALGERIA.

Abstract: In this paper and in the first part of it, homotopy perturbation method is applied to solve second order differential equation with non-constant coefficients. The method yields solutions in convergent series forms with easily computable terms (the convergence of this series is demonstrated in this paper). The result shows that this method is very convenient and can be applied to large class of problems. As for the second part, we found a solution of Telegraph equation using the Laplace transform and Stehfest algorithm method. Next, we used method of Homotopy perturbation. Finally, we give some examples for illustration.

Key-Words: Ordinary/Partial differential equations, Laplace transform method, Stehfest algorithm, Homotopy Perturbation Method.

Received: June 28, 2022. Revised: February 21, 2023. Accepted: March 8, 2023. Published: March 23, 2023.

1 Introduction

Homotopy perturbation method (HPM) is a semi-analytical technique for solving linear as well as non-linear ordinary/partial differential equations. The method may also be used to solve a system of coupled linear and nonlinear differential equations. The HPM was proposed by J. He in 1999 [10]. This method was developed by making use of artificial parameters [11]. Interested readers may go through Refs. [[8], [5]] for further details.

Almost all traditional perturbation methods are based on small parameter assumption. Liu [11] proposed artificial parameter method and Liao [[13], [12]] contributed homotopy analysis method to eliminate small parameter assumption. Further, He [10] developed an effective technique viz. The HPM method in which no small parameter assumptions are required.

In this paper and in the first part of it, He's homotopy perturbation method is applied to solve linear and nonlinear second order differential equations with non-constant coefficients. The method yields solutions in convergent series forms with easily computable terms, for the information that the proof of the convergence of this series has not been demonstrated in the articles which have been published before, we in this article have demonstrated this convergence in a simple way.

In the second part of this paper is to establish the solution of Telegraph equations with Dirichlet boundary conditions. The proofs are based on a Homotopy perturbation method and Laplace transformation technique with Stehfest algorithm. Furthermore, some examples are given to compare between the approximate and the exact solutions and to show the efficiency of this method.

2 Homotopy-perturbation method

The Homotopy-perturbation method was proposed by Ji-Huan He in 1998 [5] and was developed and improved by himself [9, 7, 4]. To illustrate the basic ideas of this method, we consider the following non-linear functional equation

$$A(u) - f(r) = 0; r \in \Omega, \tag{1}$$

with the boundary condition of

$$B(u; \frac{\partial u}{\partial \eta}) = 0; r \in \Gamma, \tag{2}$$

where A represents a general differential operator, B is a boundary operator, Γ is the boundary of the domain Ω, and $f(r)$ is known analytic function. The operator A can be decomposed into two parts viz. linear L and nonlinear N. Therefore, Eq. (1) may be written in the following form

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

Using homotopy technique, proposed by He [10] and Liao [[13], [12]], we construct a homotopy $v(r; p) : \Omega \times [0; 1] \to R.$ to Eq. (3) which satisfies

$$H(v; p) = \Theta = 0, \tag{4}$$

$\Theta = L(v) - L(u_0) + p\left[L(u_0) + N(v) - f(r)\right],$
and

$$H(v;p) = \Theta = 0; p \in [0;1]; r \in \Omega, \tag{5}$$

$\Theta = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)],$

where $p \in [0;1]$ is the embedding parameter (also called as an artificial parameter), u_0 is an initial approximation of Eq. (1) which satisfies the given conditions.

Obviously, considering Eqs. (5) and (4) we have

$$H(v;0) = L(v) - L(u_0) = 0, \tag{6}$$

$$H(v;1) = A(v) - f(r) = 0. \tag{7}$$

As p changes from zero to unity, $v(r;p)$ changes from $u_0\left(r\right)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are homotopic to each other. Due to the fact that $p \in [0;1]$ is a small parameter, we consider the solution of Eq. (5) as a power series in p as below

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{8}$$

and the best approximation is

$$u = p \to 1 \lim v = v_0 + v_1 + v_2 + \cdots \tag{9}$$

The combination of the perturbation method and the Homotopy method is called the Homotopy-perturbation method (HPM), which has eliminated the limitations of traditional perturbation techniques. The series (9) is convergent for several cases. Certain criteria are suggested for the convergence of the series (9), in [4].

3 Homotopy perturbation method for a second-order differential equation with non-constant coefficients

3.1 Statement of the problem

We consider the following problem

$$u'' = A\left(t\right)\frac{du}{dt} + B\left(t\right)u + C\left(t\right), \tag{10}$$

$$u\left(\alpha\right) = a, \tag{11}$$

$$\frac{du\left(\alpha\right)}{dt} = b, \tag{12}$$

where $A\left(t\right), B\left(t\right)$ and $C\left(t\right)$ are continuous functions on I an interval contains α.

3.1.1 Homotopy-perturbation method

We can construct the following homotopy

$$v\left(t,p\right) : I \times [0,1] \to R; \frac{d^2 v}{dt^2} = \Theta, \tag{13}$$

$\Theta = p\left(A\left(t\right)\frac{dv}{dt} + B\left(t\right)v + C\left(t\right)\right),$

suppose that the solution of (13) is written as the following series

$$h\left(t\right) = \sum_{i=0}^{\infty} p^i v_i\left(t\right), \tag{14}$$

we put (14) in (13), we get

$$\frac{d^2}{dt^2}\left(\sum_{i=0}^{\infty} p^i v_i\right) = p\begin{pmatrix} A\left(t\right)\frac{d}{dt}\left(\sum_{i=0}^{\infty} p^i v_i\right) \\ +B\left(t\right)\left(\sum_{i=0}^{\infty} p^i v_i\right) + C\left(t\right)\end{pmatrix}. \tag{15}$$

By conformity, we find
$p^0 : \frac{d^2 v_0}{dt^2} = 0$
$v_0\left(\alpha\right) = a$
$\frac{dv_0(\alpha)}{dt} = b.$
Then
$v_0 = a - b\alpha + bt$
$p^1 : \frac{d^2 v_1}{dt^2} = \left(A\left(t\right)\frac{d}{dt}v_0 + B\left(t\right)v_0 + C\left(t\right)\right)$
$v_1\left(\alpha\right) = 0$
$\frac{dv_1(\alpha)}{dt} = 0.$
Then
$v_1 = \int_\alpha^t \int_\alpha^\xi \left(A\left(t\right)\frac{d}{dt}v_0 + B\left(t\right)v_0 + C\left(t\right)\right)dtd\xi$
$p^2 : \frac{d^2 v_2}{dt^2} = A\left(t\right)\frac{d}{dt}v_1 + B\left(t\right)v_1$
$v_2\left(\alpha\right) = 0$
$\frac{dv_2(\alpha)}{dt} = 0.$
Then
$v_2 = \int_\alpha^t \int_\alpha^\xi \left(A\left(t\right)\frac{d}{dt}v_1 + B\left(t\right)v_1\right)dtd\xi$
$\cdots$
$p^n : \frac{d^2 v_n}{dt^2} = A\left(t\right)\frac{d}{dt}\left(v_{n-1}\right) + B\left(t\right)v_{n-1}$
$v_{n-1}\left(\alpha\right) = 0$
$\frac{dv_{n-1}(\alpha)}{dt} = 0.$
Then
$v_n = \int_\alpha^t \int_\alpha^\xi \left(A\left(t\right)\frac{d}{dt}v_{n-1} + B\left(t\right)v_{n-1}\right)dtd\xi.$
Where

$$v_0 = a - b\alpha + bt \tag{16}$$

$$v_1 = \int_\alpha^t \int_\alpha^\xi \left(A\left(t\right)\frac{d}{dt}v_0 + B\left(t\right)v_0 + C\left(t\right)\right)dtd\xi \tag{17}$$

$$v_n = \int_\alpha^t \int_\alpha^\xi \left(A\left(t\right)\frac{d}{dt}v_{n-1} + B\left(t\right)v_{n-1}\right)dtd\xi, \forall n \geq 2. \tag{18}$$

When $p \to 1$, (14) gives the approximate solution of the problem (10)-(12), i.e.

$$h(t) = \sum_{i=0}^{\infty} v_i(t) . \tag{19}$$

3.1.2 Study of the convergence of the previous series

Theorem 3.1 *If* $t \in I\sup|A(t)| < \infty$, $t \in I\sup|B(t)| < \infty$ *and* $t \in I\sup|C(t)| < \infty$, *then the series defined by (16)-(19) is converging towards* $h(t)$. *Such that* $h(t)$ *is solution of the problem (10)-(12).*

Proof. We pose
$I^0 v = v(t)$
$I^1 v = \int_\alpha^t v(\xi)\,d\xi$
$I^{n+1} v = I^1(I^n v)$
and
$k = Max\{t \in ISup|A(t)|, t \in ISup|B(t)|\}$
$M = t \in I\sup\left\{\left|\frac{d}{dt}v_1(t)\right| + |v_1(t)|\right\}$
with
$\left|\frac{d}{dt}v_1(t)\right| + |v_1(t)| = \left|\int_\alpha^t \begin{pmatrix} A(\xi)b + B(\xi) \\ (a - b\alpha + b\xi) + C(\xi) \end{pmatrix} d\xi\right| + \left|\int_\alpha^t \int_\alpha^\xi \begin{pmatrix} A(t)b + B(t) \\ (a - b\alpha + bt) + C(t) \end{pmatrix} dt d\xi\right|$.
We have
$v_2(t) = I^2\left(A(t)\frac{d}{dt}v_1(t) + B(t)v_1(t)\right)$.
Where
$v_2(t) \le kI^2\left(1I^0\left(\left|\frac{d}{dt}v_1(t)\right| + |v_1(t)|\right)\right)$
$v_3(t) = I^2\left(A(t)\frac{d}{dt}v_2(t) + v_2(t)\right)$

$$= I^2 \begin{pmatrix} A(t)\frac{d}{dt}I^2\begin{pmatrix} A(t)\frac{d}{dt}v_1(t) \\ +B(t)v_1(t)\end{pmatrix} \\ +B(t)I^2\begin{pmatrix} A(t)\frac{d}{dt}v_1(t) \\ +B(t)v_1(t)\end{pmatrix}\end{pmatrix}$$

$$= I^2 \begin{pmatrix} A(t)I^1\begin{pmatrix} A(t)\frac{d}{dt}v_1(t) \\ +B(t)v_1(t)\end{pmatrix} \\ +B(t)I^2\begin{pmatrix} A(t)\frac{d}{dt}v_1(t) \\ +B(t)v_1(t)\end{pmatrix}\end{pmatrix}.$$

Where
$|v_3(t)| \le k^2 I^3(1I^0 + 1I^1)\left(\left|\frac{d}{dt}v_1(t)\right| + |v_1(t)|\right)$
$v_4(t) = I^2\left(A(t)\frac{d}{dt}v_3(t) + B(t)v_3(t)\right)$

$$= I^2\left(A\frac{d}{dt}\left(I^2\begin{pmatrix} AI^1\left(A\frac{d}{dt}v_1 + Bv_1\right) \\ +BI^2\left(A\frac{d}{dt}v_1 + Bv_1\right)\end{pmatrix}\right)\right) + I^2B\left(I^2\begin{pmatrix} AI^1\left(A\frac{d}{dt}v_1 + Bv_1\right) \\ +BI^2\left(A\frac{d}{dt}v_1 + Bv_1\right)\end{pmatrix}\right)$$

$$= I^2\left(AI^1\begin{pmatrix} AI^1\left(A\frac{d}{dt}v_1 + Bv_1\right) \\ +BI^2\left(A\frac{d}{dt}v_1 + Bv_1\right)\end{pmatrix}\right) + I^2B\left(I^2\begin{pmatrix} AI^1\left(A\frac{d}{dt}v_1 + Bv_1\right) \\ +BI^2\left(A\frac{d}{dt}v_1 + Bv_1\right)\end{pmatrix}\right).$$

Where
$|v_4(t)| \le k^3 I^4(1I^0 + 2I^1 + 1I^2)\left(\left|\frac{d}{d\xi}v_1(t)\right| + |v_1(t)|\right)$
$v_5(t) = I^2\left(A(t)\frac{d}{dt}v_4(t) + B(t)v_4(t)\right)$.
Where
$|v_5(t)| \le k^4 I^5(1I^0 + 3I^1 + 3I^2 + 1I^2)\left(\left|\frac{d}{dt}v_1(t)\right| + |v_1(t)|\right)$.
...

$$|v_n(t)| \le k^{n-1}I^n\sum_{p=0}^{n-2}C_{n-2}^p I^p\begin{pmatrix}\left|\frac{d}{d\xi}v_1(t)\right| \\ +|v_1(t)|\end{pmatrix}$$

$$\le k^{n-1}\sum_{p=0}^{n-2}C_{n-2}^p I^{n+p}\begin{pmatrix}\left|\frac{d}{d\xi}v_1(t)\right| \\ +|v_1(t)|\end{pmatrix}$$

$$\le k^{n-1}\sum_{p=0}^{n-2}C_{n-2}^p\frac{1}{(n+p-1)!}\int_\alpha^t\Theta(\xi)\,d\xi$$

$$\Theta(\xi) = (t-\xi)^{n+p-1}\left(\left|\frac{d}{d\xi}v_1(\xi)\right| + |v_1(\xi)|\right)$$

$$\le Mk^{n-1}\sum_{p=0}^{n-2}C_{n-2}^p\frac{1}{(n+p-1)!}\int_\alpha^t(t-\xi)^{n+p-1}d\xi$$

$$\le Mk^{n-1}\sum_{p=0}^{n-2}C_{n-2}^p\frac{1}{(n+p)!}\left[-(t-\xi)p\right]_{\xi=\alpha}^{\xi=t}$$

$$\le Mk^{n-1}\sum_{p=0}^{n-2}C_{n-2}^p\frac{1}{(n+p)!}(t-\alpha)^{n+i}$$

$$\le Mk^{n-1}\sum_{p=0}^{n-2}C_{n-2}^p\frac{1}{(n)!}(t-\alpha)^{n+p}$$

$$\le Mk^{n-1}\frac{(t-\alpha)^n}{(n)!}\sum_{p=0}^{n-2}C_{n-2}^p(t-\alpha)^p 1^{n-2-p}$$

$$\le Mk^{n-1}\frac{(t-\alpha)^n}{(n)!}(t-\alpha+1)^{n-2}.$$

Thus
$|v_n(t)| \le Mk^{n-1}\frac{(t-\alpha)^n}{(n)!}(t-\alpha+1)^{n-2} = w_n(t)$.

According to D'Alembert's rule we have
$\lim\frac{w_{n+1}(t)}{w_n(t)} = \lim\frac{k(t-\alpha)(t-\alpha+1)}{n+1} = 0 < 1$.

Then the series $\sum_{i=0}^{\infty} w_i(t)$ is convergent, therefore the series $\sum_{i=0}^{\infty} v_n(t)$ is also convergent.

We now come to prove that $(h(t))$ is a solution to the problem (10)-(12).

Indeed

Step 1 : Take the limit of the two sides of (15) when $p \longrightarrow 1$, we get $\frac{d^2h(t)}{dt^2} = A(t)\frac{dh(t)}{dt} +$

$B(t)h(t)+C(t)$, so $h(t)$ is a solution of the equation (10).

Step 2 : We have $\forall n \geq 1 : v_n(\alpha)=0$ and $\frac{dv_n}{dt}(\alpha)=0$, thus $h(\alpha)=v_0(\alpha)=a$ and $\frac{dh}{dt}(\alpha)=\frac{dv_0}{dt}(\alpha)=b$.

3.1.3 Special case : The coefficients A and B are constant

By replacing the previous data in (16), (17) and (18) we obtain

$v_0=a-b\alpha+bt$

$v_1=\int_\alpha^t\int_\alpha^\xi\begin{pmatrix}Ab+B(a-b\alpha)\\+Bbt+C(t)\end{pmatrix}dtd\xi$

$=\int_\alpha^t(t-\xi)\begin{pmatrix}Ab+B(a-b\alpha)\\+Bb\xi+C(\xi)\end{pmatrix}d\xi$

$\ldots$

$v_n=\int_\alpha^t\int_\alpha^\xi\begin{pmatrix}A\frac{d}{dt}(v_{n-1}(t))dt\\+Bv_{n-1}(t)\end{pmatrix}dtd\xi$

$=\int_\alpha^t(t-\xi)\begin{pmatrix}A\frac{d}{d\xi}(v_{n-1}(\xi))d\xi\\+Bv_{n-1}(\xi)\end{pmatrix}d\xi.$

We pose

$f(t)=Ab+B(a-b\alpha)+Bbt+C(t)$

$p^0: v_0=a-b\alpha+bt$

$p^1: v_1=\int_\alpha^t(t-\xi)f(\xi)d\xi$

$p^2: v_2=A\int_\alpha^t v_1(\xi)d\xi+B\int_\alpha^t\int_\alpha^\mu v_1(\xi)d\xi d\mu$

$= A\int_\alpha^t\int_\alpha^\mu(t-\xi)f(\xi)d\xi d\mu + B\int_\alpha^t\int_\alpha^\mu\int_\alpha^x(t-\xi)f(\xi)d\xi dx d\mu$

$= \frac{1}{2!}A\int_\alpha^t(t-\xi)^2f(\xi)d\xi + \frac{1}{3!}B\int_\alpha^t(t-\xi)^3f(\xi)d\xi$

$p^3: v_3=\frac{1}{3!}A^2\int_\alpha^t(t-\xi)^3f(\xi)d\xi$
$+2\frac{1}{4!}AB\int_\alpha^t(t-\xi)^4f(\xi)d\xi + \frac{1}{5!}B^2\int_\alpha^t(t-\xi)^5f(\xi)d\xi$

$p^4 : v_4 = \frac{1}{4!}A^3\int_\alpha^t(t-\xi)^4f(\xi)d\xi + 3\frac{1}{5!}A^2B\int_\alpha^t(t-\xi)^5f(\xi)d\xi$
$+\frac{1}{6!}3AB^2\int_\alpha^t(t-\xi)^6f(\xi)d\xi + \frac{1}{7!}B^3\int_\alpha^t(t-\xi)^7f(\xi)d\xi$

$p^5 : v_5 = \frac{1}{5!}A^4\int_\alpha^t(t-\xi)^5f(\xi)d\xi + \frac{1}{6!}4A^3B\int_\alpha^t(t-\xi)^6f(\xi)d\xi$
$+\frac{1}{7!}6A^2B^2\int_\alpha^t(t-\xi)^7f(\xi)d\xi + \frac{1}{8!}4AB^3\int_\alpha^t(t-\xi)^8f(\xi)d\xi$
$+\frac{1}{9!}B^4\int_\alpha^t(t-\xi)^9f(\xi)d\xi$

$\ldots$

$p^n: v_n=\sum_{i=0}^{n-1}\frac{1}{(n+i)!}C_{n-1}^iA^{n-i-1}B^i\int_\alpha^t\Theta(\xi)d\xi$

$\Theta(\xi)=(t-\xi)^{n+i}f(\xi).$

Where

$v_0=a+bt$

$v_n=\sum_{i=0}^{n-1}\frac{C_{n-1}^iA^{n-i-1}B^i}{(n+i)!}\int_\alpha^t\Theta(\xi)d\xi$

$\Theta(\xi)=(t-\xi)^{n+i}(Ab+Ba+Bb\xi+C(\xi)).$

Thus $h_n(t)=v_0+\sum_{j=1}^n v_j.$

We obtain

$$h_n(t)=\Theta \tag{20}$$

$$\Theta=\sum_{j=1}^n\left(\begin{array}{c}\sum_{i=0}^{j-1}\frac{C_{j-1}^iA^{j-i-1}B^i}{(j+i)!}\\ \int_\alpha^t(t-\xi)^{j+i}\begin{pmatrix}Ab+B(a-b\alpha)\\+Bb\xi+C(\xi)\end{pmatrix}d\xi\end{array}\right)$$

$+a-b\alpha+bt.$

If C is constant, then using integration by parts, we get

$$h_n(t)=\Theta \tag{21}$$

$$\Theta=\sum_{j=1}^n\left(\begin{array}{c}\sum_{i=0}^{j-1}C_{j-1}^iA^{j-i-1}B^i\\ \begin{pmatrix}\frac{Ab+Ba+C}{(j+i+1)!}(t-\alpha)^{j+i+1}\\+\frac{Bb}{(j+i+2)!}(t-\alpha)^{j+i+2}\end{pmatrix}\end{array}\right)$$

$+a-b\alpha+bt.$

3.2 Numerical example

Example 3.2 $A(t)=-e^t$

$\alpha=0$

$B(t)=-\sin t$

$C(t)=e^t(t+2)+e^{2t}(t+1)+te^t\sin t.$

So we have the following problem :

$\frac{d^2v}{dt^2}=-e^t\frac{dv}{dt}-(\sin t)v+e^t(t+2)+e^{2t}(t+1)+te^t\sin t$

$v(0)=0$

$\frac{dv}{dt}(0)=1.$

The exact solution to this problem is given by :

$v_{exa}=te^t.$

We are now looking for the solution by the method (HPM), by replacing the previous data in (16)-(18), we get

$$v_0=t. \tag{22}$$

$$v_1=\Theta \tag{23}$$

$\Theta=2\cos t-\frac{3}{4}t-e^t+\frac{1}{4}te^{2t}+te^t+t\sin t$
$+\frac{1}{2}(\cos t)e^t+\frac{1}{2}e^t\sin t-\frac{1}{2}t(\cos t)e^t-\frac{3}{2}.$

$$v_n=\int_0^t\int_0^\xi\left(e^t\frac{d}{dt}v_{n-1}+(\sin t)v_{n-1}\right)dtd\xi, \forall n\geq 2 \tag{24}$$

and

$$h_n(t)=\sum_{i=0}^n v_i(t). \tag{25}$$

For $n=3, n=6$ *and* $n=9$ *we compute* $h_n(t)$ *withe the relations* $(22)-(25)$ *and compare the Taylor development of* $h_n(t)$ *and* $v_{exa}(t)$ *in the neighborhood of* 0 $(at order 10)$.

$DL(h_3(t)) = t + t^2 + \frac{1}{2}t^3 + \frac{1}{6}t^4 + \frac{7}{120}t^5 + \frac{3}{80}t^6 + \frac{5}{144}t^7 + \frac{89}{3360}t^8 + \frac{5791}{362\,880}t^9 + \frac{9349}{1209\,600}t^{10} + \circ(t^{10})$.

$DL(h_6(t)) = t + t^2 + \frac{1}{2}t^3 + \frac{1}{6}t^4 + \frac{1}{24}t^5 + \frac{1}{120}t^6 + \frac{1}{720}t^7 + \frac{1}{6720}t^8 - \frac{1}{7560}t^9 - \frac{29}{103\,680}t^{10} + \circ(t^{10})$.

$DL(h_9(t)) = t + t^2 + \frac{1}{2}t^3 + \frac{1}{6}t^4 + \frac{1}{24}t^5 + \frac{1}{120}t^6 + \frac{1}{720}t^7 + \frac{1}{5040}t^8 + \frac{1}{40\,320}t^9 + \frac{1}{362\,880}t^{10} + \circ(t^{10})$

$DL(v_{exa}) = t + t^2 + \frac{1}{2}t^3 + \frac{1}{6}t^4 + \frac{1}{24}t^5 + \frac{1}{120}t^6 + \frac{1}{720}t^7 + \frac{1}{5040}t^8 + \frac{1}{40\,320}t^9 + \frac{1}{362\,880}t^{10} + \circ(t^{10})$

Example 3.3 $A(t) = -t$

$B(t) = t^2$

$C(t) = e^t(-t^2 + t + 1), \alpha = 1$

$a = e$

$b = e.$

So we have the following problem :

$\frac{d^2v}{dt^2} = -e^t\frac{dv}{dt} + t^2 v e^t(-t^2 + t + 1), v(1) = e$

$\frac{dv}{dt}(1) = e.$

The exact solution to this problem is given by :

$v_{exa} = e^t.$

$v_0 = e \times t$

$v_1 = \frac{43}{15}e - 7e^t - \frac{1}{6}t^3 e + \frac{1}{20}t^5 e - t^2 e^t + \frac{1}{4}te + 5te^t$

$v_n = \int_1^t \int_1^\xi \left(-t\frac{d}{dt}v_{n-1} + t^2 v_{n-1}\right) dt d\xi$

$$h_n(t) = \sum_{i=0}^{\infty} v_i(t).$$

For $n = 3, n = 6$ *and* $n = 9$ *we compute* $h_n(t)$ *and compare the Taylor development of* $h_n(t)$ *and* $v_{exa}(t)$ *in the neighborhood of* 1 $(at order 10)$.

$DLh_3(t) = e + e(t-1) + \frac{1}{2}e(t-1)^2 + \frac{1}{6}e(t-1)^3 + \frac{1}{24}e(t-1)^4 + \frac{1}{60}e(t-1)^5 + \frac{1}{90}e(t-1)^6 + \frac{1}{2520}e(t-1)^7 - \frac{17}{4032}e(t-1)^8 - \frac{317}{181\,440}e(t-1)^9 + \frac{1}{5760}e(t-1)^{10} + o\left((t-1)^{10}\right)$

$DLh_6(t) = e + e(t-1) + \frac{1}{2}e(t-1)^2 + \frac{1}{6}e(t-1)^3 + \frac{1}{24}e(t-1)^4 + \frac{1}{120}e(t-1)^5 + \frac{1}{720}e(t-1)^6 + \frac{1}{5040}e(t-1)^7 - \frac{1}{17\,280}e(t-1)^9 - \frac{19}{403\,200}e(t-1)^{10} + o\left((t-1)^{10}\right)$

$DLh_9(t) = e + e(t-1) + \frac{1}{2}e(t-1)^2 + \frac{1}{6}e(t-1)^3 + \frac{1}{24}e(t-1)^4 + \frac{1}{120}e(t-1)^5 + \frac{1}{720}e(t-1)^6 + \frac{1}{5040}e(t-1)^7 + \frac{1}{40\,320}e(t-1)^8 + \frac{1}{362\,880}e(t-1)^9 + \frac{1}{3628\,800} + o\left((t-1)^{10}\right)$

$DLv_{exa}(t) = e + e(t-1) + \frac{1}{2}e(t-1)^2 + \frac{1}{6}e(t-1)^3 + \frac{1}{24}e(t-1)^4 + \frac{1}{120}e(t-1)^5 + \frac{1}{720}e(t-1)^6 + \frac{1}{5040}e(t-1)^7 + \frac{1}{40\,320}e(t-1)^8 + \frac{1}{362\,880}e(t-1)^9 + \frac{1}{3628\,800}e(t-1)^{10} + o\left((t-1)^{10}\right)$

Example 3.4 $A(t) = t^2$

$B(t) = \cos t$

$C(t) = e^t$

$a \in R, b \in R$

$\alpha = 0.$

So we have the following problem :

$\frac{d^2v}{dt^2} = t^2\frac{dv}{dt} + v\cos t + e^t, v(0) = a$

$\frac{dv}{dt}(0) = b.$

*In this problem we do not know in advance the exact solution. We are now looking for the solution by the method (HPM), by replacing the previous data in (*16*)-(*18*), we get*

$v_0 = a + bt$

$v_1 = a - t + e^t + \frac{1}{12}bt^4 - bt - a\cos t + 2b\sin t - bt\cos t - 1$

$v_n = \int_0^t \int_0^\xi \left(t^2\frac{d}{dt}v_{n-1}(t) + v_{n-1}(t)\cos t\right) dt d\xi,$

and

$$h_n(t) = \sum_{i=0}^{n} v_i(t).$$

For $n = 4, n = 5$ *and* $n = 6$ *we compute* $h_n(t)$ *and compare the Taylor development of* $h_n(t)$ *and* $v_{exa}(t)$ *in the neighborhood of* 0 $(at order 10)$.

$DL(h_4(t)) = a + bt + \left(\frac{1}{2}a + \frac{1}{2}\right)t^2 + \left(\frac{1}{6}b + \frac{1}{6}\right)t^3 + \left(\frac{1}{12}b + \frac{1}{12}\right)t^4 + \left(\frac{1}{20}a - \frac{1}{60}b + \frac{1}{15}\right)t^5 + \left(\frac{7}{360}b - \frac{1}{144}a + \frac{1}{80}\right)t^6 + \left(\frac{1}{840}a + \frac{11}{1680}b + \frac{13}{1680}\right)t^7 + \left(\frac{3}{640}a - \frac{19}{10\,080}b + \frac{47}{8064}\right)t^8 + \left(\frac{691}{362\,880}b - \frac{11}{12\,096}a + \frac{19}{24\,192}\right)t^9 + \left(\frac{91}{518\,400}a + \frac{19}{45\,360}b + \frac{2279}{3628\,800}\right)t^{10}.$

$DL(h_5(t)) = a + bt + \left(\frac{1}{2}a + \frac{1}{2}\right)t^2 + \left(\frac{1}{6}b + \frac{1}{6}\right)t^3 + \left(\frac{1}{12}b + \frac{1}{12}\right)t^4 + \left(\frac{1}{20}a - \frac{1}{60}b + \frac{1}{15}\right)t^5 + \left(\frac{7}{360}b - \frac{1}{144}a + \frac{1}{80}\right)t^6 + \left(\frac{1}{840}a + \frac{11}{1680}b + \frac{13}{1680}\right)t^7 + \left(\frac{3}{640}a - \frac{19}{10\,080}b + \frac{47}{8064}\right)t^8 + \left(\frac{691}{362\,880}b - \frac{11}{12\,096}a + \frac{19}{24\,192}\right)t^9 + \left(\frac{319}{1814\,400}a + \frac{19}{45\,360}b + \frac{19}{30\,240}\right)t^{10}.$

$DL(h_6(t)) = DL(h_5(t)) + \circ(t^{10}).$

3.2.1 Homotopy perturbation method for a nonlinear second-order differential equation with nonconstant coefficients

We consider the following problem

$$\begin{cases} \frac{d^2v}{dt^2} &= A(t)\frac{dv}{dt} + B(t)v^m + C(t), m \in N^*/\{1\}, \\ v(\alpha) &= a, \\ \frac{dv}{dt}(\alpha) &= b. \end{cases}$$

We can build the following homotopy
$v(t,p) : I \times [0,1] \to R$

$$\frac{d^2}{dt^2}\left(\sum_{i=0}^{\infty} p^i v_i\right) = p\left(\begin{array}{c} A(t)\frac{d}{dt}\left(\sum_{i=0}^{\infty} p^i v_i\right) \\ +B(t)\left(\sum_{i=0}^{\infty} p^i v_i\right)^m + C(t) \end{array}\right).$$

By conformity, we find

$p^0 : \frac{d^2v_0}{dt^2} = 0$

$v_0(\alpha) = a$

$\frac{dv_0(\alpha)}{dt} = b$

$v_0 = a - b\alpha + bt$

$p^1 : \frac{d^2v_1}{dt^2} = (A(t)b + B(t)(a - b\alpha + bt)^m + C(t))$

$v_1(\alpha) = 0$

$\frac{dv_1(\alpha)}{dt} = 0$

$v_1 = \int_\alpha^t \int_\alpha^\xi (A(t)b + B(t)(a - b\alpha + bt)^m + C(t))\, dtd\xi$

$p^2 : \frac{d^2v_2}{dt^2} = A(t)\frac{d}{dt}v_1 + B(t)C_m^1 v_1 v_0^{m-1}$

$v_2(\alpha) = 0$

$\frac{dv_2(\alpha)}{dt} = 0$

$v_2 = \int_\alpha^t \int_\alpha^\xi \left(A(t)\frac{d}{dt}v_1 + B(t)mv_1v_0^{m-1}\right) dtd\xi$

$p^3 : \frac{d^2v_3}{dt^2} = A(t)\frac{d}{dt}v_2 + B(t)\left(C_m^1 v_2 v_0^{m-1} + C_m^2 v_1^2 v_0^{m-2}\right)$

$v_3(\alpha) = 0$

$\frac{dv_3(\alpha)}{dt} = 0$

$$v_3 = \int_\alpha^t \int_\alpha^\xi \left(\begin{array}{c} A(t)\frac{d}{dt}v_1 \\ +B(t)\left(\begin{array}{c} mv_2v_0^{m-1} \\ +\frac{m(m-1)}{2}v_1^2v_0^{m-2} \end{array}\right) \end{array}\right) dtd\xi$$

...

$p^n : \frac{d^2v_n}{dt^2} = A(t)\frac{d}{dt}(v_{n-1})$

$+B(t)\left(\sum_{k_0+k_1+\ldots\ldots+k_{m-1}=n-1}\left(\prod_{i=0}^{m-1} v_{k_i}\right)\right)$

$v_{n-1}(\alpha) = 0$

$\frac{dv_{n-1}(\alpha)}{dt} = 0$

$$v_n = \int_\alpha^t \int_\alpha^\xi \left(\begin{array}{c} A(t)\frac{d}{dt}(v_{n-1}) \\ +B(t)\left(\sum_{n-1}\left(\prod_{i=0}^{m-1} v_{k_i}\right)\right) \end{array}\right) dtd\xi$$

$k_0 + k_1 + \ldots\ldots + k_{m-1} = n-1$

Thus

$$v_0 = a - b\alpha + bt \tag{26}$$

$$v_1 = \int_\alpha^t \int_\alpha^\xi (A(t)b + B(t)(a - b\alpha + bt)^m + C(t))\, dtd\xi \tag{27}$$

$$\forall n \geq 2 : v_n = \int_\alpha^t \int_\alpha^\xi (\Theta(t,\xi))\, dtd\xi \tag{28}$$

$\Theta(t,\xi) = A(t)\frac{d}{dt}(v_{n-1})$

$+B(t)\sum_{k_0+\cdots+k_{m-1}=n-1}\left(\prod_{i=0}^{m-1} v_{k_i}\right),$

and

$$h_n(t) = \sum_{i=0}^{n} v_i(t). \tag{29}$$

Example 3.5 $m = 2, \alpha = 0, A(t) = t, B(t) = e^t, C(t) = 2 - 2t^2 - t^4e^t, a = 0$ *and* $b = 0$.

So we have the following problem :

$\frac{d^2v}{dt^2} = t\frac{dv}{dt} + e^tv^2 + 2 - 2t^2 - t^4e^t, v(0) = 0$ *and* $\frac{dv}{dt}(0) = 0$.

The exact solution to this problem is given by :

$v_{exa} = t^2$.

We are now looking for the solution by the method (HPM), by replacing the previous data in (26)-(29), we get

$v_0 = 0$

$v_1 = \int_0^t \int_0^\xi (2 - 2t^2 - t^4e^t)\, dtd\xi$

$$v_n = \int_0^t \int_0^\xi \left(\begin{array}{c} t\frac{d}{dt}(v_{n-1}) \\ +e^t\sum_{k_0+k_1=n-1}\left(\prod_{i=0}^{1} v_{k_i}\right) \end{array}\right) dtd\xi,$$

and

$h_n(t) = \sum_{i=0}^{n} v_i(t).$

Where

$v_0 = 0$

$v_1 = \int_0^t \int_0^\xi (2 - 2t^2 - t^4e^t)\, dtd\xi$

$v_n = \int_0^t \int_0^\xi \left(\frac{d}{dt}(v_{n-1}) + \sum_{k_0=0}^{n-1} v_{k_0}v_{n-k_0-1}\right) dtd\xi.$

Thus

$v_0(t) = 0$

$v_1(t) = \int_0^t \int_0^\xi (2 - 2t^2 - t^4e^t)\, dtd\xi$

$v_2(t) = \int_0^t \int_0^\xi t\left(\frac{d}{dt}(v_1) + 2e^tv_0v_1\right) dtd\xi$

$v_3(t) = \int_0^t \int_0^\xi \left(t\frac{d}{dt}v_2 + e^t(2v_0v_2 + v_1^2)\right) dtd\xi$

$v_4(t) = \int_0^t \int_0^\xi \left(t\frac{d}{dt}v_3 + 2e^t(v_0v_3 + v_1v_2)\right) dtd\xi$

$v_5(t) = \int_0^t \int_0^\xi \left(t\frac{d}{dt}v_4 + e^t(2v_0v_4 + 2v_1v_3 + v_2^2)\right) dtd\xi$

$v_6(t) = \int_0^t \int_0^\xi \left(t\frac{d}{dt}v_5 + 2e^t(v_0v_5 + v_1v_4 + v_2v_3)\right) dtd\xi.$

For $n = 4, n = 5$ *and* $n = 6$ *we compute* $h_n(t)$ *and compare the Taylor development of* $h_n(t)$ *and* $v_{exa}(t)$ *in the neighborhood of* $0\ (at order 13)$.

$DL(h_4(t)) = t^2 - \frac{59}{22\,680}t^{10} - \frac{251}{110\,880}t^{11}$

$-\frac{239}{187\,110}t^{12}-\frac{2323}{3931\,200}t^{13}+o\left(t^{13}\right)$
$DL(h_5\left(t\right))=t^2-\frac{155}{299\,376}t^{12}-\frac{2693}{4717\,440}t^{13}+o\left(t^{13}\right)$
$DL(h_6\left(t\right))=t^2+o\left(t^{13}\right)$
$DL(v_{exa})=t^2+o\left(t^{13}\right).$

Example 3.6 $m=3$
$A\left(t\right)=t^2$
$B\left(t\right)=1$
$C\left(t\right)=e^t\left(t^2+4t+2\right)-t^6e^{3t}-t^3e^t\left(t+2\right)$
$a=0$ *and* $b=0.$
So we have the following problem :
$\frac{d^2v}{dt^2}=t^2\frac{dv}{dt}+v^3+e^t\left(t^2+4t+2\right)-t^6e^{3t}-t^3e^t\left(t+2\right)$
$v\left(0\right)=0$ *and* $\frac{dv}{dt}\left(0\right)=0.$
The exact solution to this problem is given by :
$v_{exa}=t^2e^t.$
We are now looking for the solution by the method (HPM), by replacing the previous data in (26)-(28), we get
$v_0=a+bt$
$v_1=\int_0^t\int_0^\xi\left(A\left(t\right)b+B\left(t\right)\left(a+bt\right)^3+C\left(t\right)\right)dtd\xi$

$$v_n=\int_0^t\int_0^\xi\left(\begin{array}{c}A\left(t\right)\frac{d}{dt}\left(v_{n-1}\right)\\+B\left(t\right)\sum\limits_{k_0+k_1+k_2=2}\left(\prod\limits_{i=0}^{2}v_{k_i}\right)\end{array}\right)dtd\xi.$$

Where
$v_0=0$
$v_1=\int_0^t\int_0^\xi\left(e^t\left(t^2+4t+2\right)-t^6e^{3t}-t^3e^t\left(t+2\right)\right)dtd\xi$

$$v_n=\int_0^t\int_0^\xi\left(\begin{array}{c}t^2\frac{d}{dt}\left(v_{n-1}\right)\\+\left(\sum\limits_{k_1=0}^{n-1}\sum\limits_{k_0=0}^{k_1}v_{n-k_1-1}v_{k_1-k_0}v_{k_0}\right)\end{array}\right)dtd\xi,$$

and

$$h_n\left(t\right)=\sum_{i=0}^{n}v_i\left(t\right).$$

Thus
$p^0:v_0\left(t\right)=0$
$p^1:v_1\left(t\right)=\int_0^t\int_0^\xi\left(e^t\left(t^2+4t+2\right)-t^6e^{3t}-t^3e^t\left(t+2\right)\right)dtd\xi$
$p^2:v_2\left(t\right)=\int_0^t\int_0^\xi t^2\left(\frac{d}{dt}\left(v_1\right)+3v_0^2v_1\right)dtd\xi$
$p^3:v_3\left(t\right)=\int_0^t\int_0^\xi\left(t^2\frac{d}{dt}v_2+3v_2v_0^2+3v_0v_1^2\right)dtd\xi$
$p^4:v_4\left(t\right)=\int_0^t\int_0^\xi\left(t^2\frac{d}{dt}v_3+3v_3v_0^2+6v_2v_0v_1+v_1^3\right)dtd\xi$
$p^5:v_5\left(t\right)=\int_0^t\int_0^\xi\left(\begin{array}{c}t^2\frac{d}{dt}v_4+3v_4v_0^2+6v_3v_0v_1\\+3v_0v_2^2+3v_1^2v_2\end{array}\right)dtd\xi.$

For $n=4,n=5$ *and* $n=6$ *we compute* $h_n\left(t\right)$ *and compare the Taylor development of* $h_n\left(t\right)$ *and* $v_{exa}\left(t\right)$ *in the neighborhood of* $0\ (atorder13)$.
$DLh_3\left(t\right)=t^2+t^3+\frac{1}{2}t^4+\frac{1}{6}t^5+\frac{1}{24}t^6$
$+\frac{1}{120}t^7-\frac{83}{5040}t^8-\frac{209}{5040}t^9-\frac{403}{8064}t^{10}$
$-\frac{15\,551}{362\,880}t^{11}-\frac{1156\,669}{39\,916\,800}t^{12}-\frac{8522\,347}{518\,918\,400}t^{13}+o\left(t^{13}\right)$
$DLh_4\left(t\right)=t^2+t^3+\frac{1}{2}t^4+\frac{1}{6}t^5+\frac{1}{24}t^6$
$+\frac{1}{120}t^7+\frac{1}{720}t^8+\frac{1}{5040}t^9+\frac{1}{40\,320}t^{10}$
$-\frac{11\,471}{2851\,200}t^{11}-\frac{385\,549}{39\,916\,800}t^{12}-\frac{6130\,067}{518\,918\,400}t^{13}+o\left(t^{13}\right)$
$DLh_5\left(t\right)=t^2+t^3+\frac{1}{2}t^4+\frac{1}{6}t^5+\frac{1}{24}t^6$
$+\frac{1}{120}t^7+\frac{1}{720}t^8+\frac{1}{5040}t^9+\frac{1}{40\,320}t^{10}$
$+\frac{1}{362\,880}t^{11}+\frac{1}{3628\,800}t^{12}+\frac{1}{39\,916\,800}t^{13}+o\left(t^{13}\right)$
$DL(v_{exa})=t^2+t^3+\frac{1}{2}t^4+\frac{1}{6}t^5+\frac{1}{24}t^6$
$+\frac{1}{120}t^7+\frac{1}{720}t^8+\frac{1}{5040}t^9+\frac{1}{40\,320}t^{10}$
$+\frac{1}{362\,880}t^{11}+\frac{1}{3628\,800}t^{12}+\frac{1}{39\,916\,800}t^{13}+o\left(t^{13}\right).$

4 Laplace transform and Homotopy-perturbation method for solving the Telegraph equation

In the following example we expose the Homotopy-perturbation method (HPM) combined with the Laplace transformation (LT) to solve a partial differential equation of order 2.

In the rectangular domain
$\mathcal{Q}=(0,1)\times(0,T)$ or $T<\infty$.
We consider the following Telegraph equation

$$\frac{\partial^2v}{\partial t^2}-\alpha\frac{\partial^2v}{\partial x^2}+\beta\frac{\partial v}{\partial t}=f\left(x,t\right),\tag{30}$$

with the initial conditions

$$v(x,0)=\varphi(x),0<x<1,\tag{31}$$

$$v_t(x,0)=\psi(x),\ 0<x<1,\tag{32}$$

and the boundary conditions

$$v(0,t)=n(t),0<t\leq T,\tag{33}$$

$$v_x(0,t)=m(t),0<t\leq T,\tag{34}$$

where f,φ,ψ,n and m are known functions. α,β and T are positive constants, moreover the functions $\varphi(x)$ and $\psi(x)$ satisfying the following compatibility conditions

$$\varphi\left(0\right)=n(0),\ \varphi_x'\left(0\right)=m_t'(0),\tag{35}$$

and $\psi\left(0\right)=n(0),\ \psi_x'\left(0\right)=m_t'(0).$

Let us suppose that the function $v(x,t)$ is defined and of exponential order for $t\geq0$, that is to say that there exists $A,\ \gamma>0$ and $t_0>0$ such that $|v\left(t\right)|\leq A\exp\left(\gamma t\right)$ for $t\geq t_0$. Suppose also that the Laplace transformation $V(x,s)$ is exists and given by the following formula
$V\left(x,s\right)=\mathcal{L}\left\{v\left(x,t\right);t\longrightarrow s\right\}$
$=\int_0^\infty v\left(x,t\right)\exp\left(-st\right)dt,$

where s is a positive parameter. Taking the Laplace transformations on both sides of (30), we get

$$\frac{d^2V}{dx^2}(x,s) = \Theta, \qquad (36)$$

$\Theta = (\frac{s^2+\beta s}{\alpha})V(x,s) - \frac{F(x,s)+\psi(x)+(s+\beta)\varphi(x)}{\alpha}$,
where $F(x,s) = \mathcal{L}\{g(x,t); t \longrightarrow s\}$. Similarly, we have

$$V(0,s) = a(s), \qquad (37)$$

$$V_x(0,s) = b(s), \qquad (38)$$

where

$$a(s) = \mathcal{L}\{n(t); t \longrightarrow s\}, \qquad (39)$$

$$b(s) = \mathcal{L}\{m(t); t \longrightarrow s\}. \qquad (40)$$

By replacing the previous data in (20), i.e.
$t \to x$
$a \to a(s)$
$b \to b(s)$
$A \to 0$
$B \to \frac{s^2+\beta s}{\alpha}$
$C(\xi) \to -\frac{F(\xi,s)+\psi(\xi)+(s+\beta)\varphi(\xi)}{\alpha}$.
We obtain

$$H_n(x,s) = \sum_{j=1}^{n} \Theta_j \qquad (41)$$

$\Theta_j = \frac{1}{(2j-1)!}\left(\frac{s^2+\beta s}{\alpha}\right)^{j-1}\int_0^x \Theta(\xi)\,d\xi$
$+a(s) + b(s)x$.
$\Theta(\xi) = (x-\xi)^{2j-1}$
$\left(\frac{s^2+\beta s}{\alpha}(a(s)+b(s)\xi) - \frac{F(x,s)+\psi(x)+(s+\beta)\varphi(x)}{\alpha}\right)$.

The solution $h_n(x,t)$ can be recovered approximately from $H_n(x,s)$ by the analytical method or according to the Stehfest algorithm [6]. By taking the inverse $\mathcal{L}^{-1}$ on both sides of (41), we obtain the approximate solution of the problem (30)-(34).

4.1 Numerical example

By take
$\alpha = 1$
$\beta = 1$,
$f(x,t) = e^t(2x^2+2t+1)$
$\varphi(x) = x^2$
$\psi(x) = x^2+1$
$n(t) = te^t$ and $m(t) = 0$.
So we have the following problem

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - \alpha\frac{\partial^2 v}{\partial x^2} + \beta\frac{\partial v}{\partial t} &= f(x,t), \\ v(x,0) &= x^2, \\ v_t(x,0) &= x^2+1, \\ v(0,t) &= te^t, \\ v_x(0,t) &= 0. \end{cases}$$

Thus

$f(x,t) = e^t(2x^2+2t+1) \to F(x,s) = \frac{1}{(s-1)^2}(s+2sx^2-2x^2+1)$
$n(x,t) = te^t \to a(s) = \frac{1}{(s-1)^2}$
$m(x,t) = 0 \to b(s) = 0$.
The exact solution to this problem is given by
$v_{exa}(x,t) = (t+x^2)e^t$.

We are now looking for the solution by the method (LT-HPM), by replacing the previous data in (41), we obtain

$H_n(x,s) = \sum_{j=1}^{n}\left(\frac{1}{(2j-1)!}(s^2+s)^{j-1}\int_0^x \Theta(\xi)\,d\xi\right) + \frac{1}{(s-1)^2}$

$\Theta(\xi) = (x-\xi)^{2j-1}\left(\frac{-1}{s-1}(s^2\xi^2+s\xi^2-2)\right)$.

Thus

$h_n(x,t) = \mathcal{L}^{-1}\left(\sum_{j=1}^{n}\left(\begin{array}{c}\frac{1}{(2j-1)!}(s^2+s)^{j-1} \\ \int_0^x \Theta(\xi)\,d\xi\end{array}\right)\right) + te^t$

$\Theta(\xi) = (x-\xi)^{2j-1}\left(\frac{-1}{s-1}(s^2\xi^2+s\xi^2-2)\right)$

For $t \in \{0.2, 0.6, 2, 10\}$, $x \in \{0.2, 0.4, 0.6, 0.8, 1\}$, and $n \in \{3, 5\}$, we compute $h_n(x,t)$ numerically and compare this result with the exact solution in the tables Tab1, Tab2.

t/x	$x=0.2$	$x=0.4$
$t=0.2$	$v_{eax} = 0.29314$ $h_n = 0.29314$	0.4397 0.4397
$t=0.6$	$v_{eax} = 1.1662$ $h_n = 1.1662$	1.3848 1.3848
$t=2$	$v_{eax} = 15.074$ $h_n = 15.074$	15.96 15.96
$t=10$	$v_{eax} = \frac{2.2115}{10^{-5}}$ $h_n = \frac{2.2115}{10^{-5}}$	$\frac{2.2379}{10^{-5}}$ $\frac{2.2379}{10^{-5}}$

t/x	$x=0.6$	$x=0.8$	$x=1$
$t=0.2$	0.68399 0.68398	1.0260 1.0259	1.4657 1.4652
$t=0.6$	1.7492 1.7492	2.2594 2.2593	2.9154 2.9147
$t=2$	17.438 17.438	19.507 19.507	22.167 22.164
$t=10$	$\frac{2.2819}{10^{-5}}$ $\frac{2.2819}{10^{-5}}$	$\frac{2.3436}{10^{-5}}$ $\frac{2.3436}{10^{-5}}$	$\frac{2.4229}{10^{-5}}$ $\frac{2.4228}{10^{-5}}$

Tab1: $n=3$.

t/x	$x=0.2$	$x=0.4$
$t=0.2$	$v_{eax} = 0.29314$ $h_n = 0.29314$	0.4397 0.4397
$t=0.6$	$v_{eax} = 1.1662$ $h_n = 1.1662$	1.3848 1.3848
$t=2$	$v_{eax} = 15.074$ $h_n = 15.074$	15.96 15.96
$t=10$	$v_{eax} = \frac{2.2115}{10^{-5}}$ $h_n = \frac{2.2115}{10^{-5}}$	$\frac{2.2379}{10^{-5}}$ $\frac{2.2379}{10^{-5}}$

t/x	$x = 0.6$	$x = 0.8$	$x = 1$
$t = 0.2$	$0.683\,99$ $0.683\,99$	$1.\,026\,0$ $1.\,026\,0$	$1.\,465\,7$ $1.\,465\,7$
$t = 0.6$	$1.\,749\,2$ $1.\,749\,2$	$2.\,259\,4$ $2.\,259\,4$	$2.\,915\,4$ $2.\,915\,4$
$t = 2$	$17.\,438$ $17.\,438$	$19.\,507$ $19.\,507$	$22.\,167$ $22.\,167$
$t = 10$	$\frac{2.\,281\,9}{10^{-5}}$ $\frac{2.\,281\,9}{10^{-5}}$	$\frac{2.\,343\,6}{10^{-5}}$ $\frac{2.\,343\,6}{10^{-5}}$	$\frac{2.\,422\,9}{10^{-5}}$ $\frac{2.\,422\,9}{10^{-5}}$

Tab2: $n = 5$.

4.2 Conclusion

In this paper, He's homotopy perturbation method has been successfully applied to find the solution of the second order differential equation with non-constant coefficients. The method is reliable and easy to use. The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed term. Knowing that the convergence of this series is demonstrated in this paper. Then, by using this method and by introducing the Laplace transformation technique with the Stehfest algorithm, we were able to solve the telegraph problem with the Dirichlet boundary conditions.

Acknowledgements

The authors thank the referees for their careful reading and their precious comments. Their help is much appreciated.

References:

[1] A. Necib and A. Merad. Laplace transform and Homotopy perturbation methods for solving the pseudohyperbolic integro-differential problems with purely integral conditions. Kragujevac Journal of Mathematics. Volume 44(2), Pages 251–272. 2020.

[2] A. Necib and A. Merad. Problèmes aux limites avec conditions non locales. Thèse de Doctorat. Université Oum El Bouaghi. http://hdl.handle.net/123456789/6591. 2018.

[3] A. Necib and M. Bouzit. Problème mixte avec condition non locale pour une équation différenielle aux dérivées partielle d'ordre cinq de type mixte. Mémoire de Magister. Université Oum El Bouaghi. 2009.

[4] He, J.H. New Interpretation of Homotopy Perturbation Method. International Journal of Modern Physics B, 20, 2561-2568, 2006.

[5] He, J.H. Homotopy perturbation method: a new nonlinear analytical technique. Applied Mathematics and Computation 135 (1): 73–79, 2003.

[6] H. Stehfest. Numerical Inversion of the Laplace Transform, Comm. ACM 13, 47-49, 1970.

[7] J.H. He. Limit cycle and bifurcation of nonlinear problems, Chaos Soliton. Fract.26 827-833, 2005.

[8] J.H. He. The homotopy perturbation method for non-linear oscillators with discontinuities. Appl. Math. Comput. 151(1), pp. 287-292, 2004.

[9] J.H. He. A coupling method of homotopy technique and perturbation technique for nonlinear problems, Int. J. Nonlinear Mech. 35 (1)37-43, 2000.

[10] J. H. He. Homotopy perturbation technique. Computer Methods in Applied Mechanics and Engineering.178:257-262, 1999.

[11] Liu, G.L. New research directions in singular perturbation theory: artificial parameter approach and inverse-perturbation technique. Proceedings of the 7th Conference of moder, 1997.

[12] Liao, S.J. Boundary element method for general nonlinear differential operators. Engineering Analysis with Boundary Elements 20 (2): 91-99, 1997.

[13] Liao, S.J. An approximate solution technique not depending on small parameters: a special example. International Journal of Non-Linear Mechanics 30 (3): 371-380, 1995.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

We confirm that all Authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

No conflict of interest to declare