Stability Analysis of a Single Species Model with Allee Effect and Density Dependent Birth Rate

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Abstract: Abstract: - A single species model with Allee effect and density-dependent birth rate

$$\frac{dx}{dt}=x(\frac{a}{b+cx}-d-ex)\frac{x}{\beta+x}$$

is proposed and studied in this paper, where a, b, c, d, e and β are all positive constants. Sufficient conditions which ensure the system admits a unique globally stable positive equilibrium are obtained. Numeric simulations show that with the increasing Alee effect, the system takes a much longer time to reach its stable steady-state the solution, however, Allee effect has no influence on the final density of the species.

Key-Words: Single species model; Allee effect; Global stability
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1 Introduction

The aim of this paper is to investigate the dynamic behaviors of the following single-species model with Allee effect and nonlinear birth rate

$$\frac{dx}{dt}=x(\frac{a}{b+cx}-d-ex)\frac{x}{\beta+x} \quad (1.1)$$

where a,b,c,d,e and β are all positive constants. $x(t)$ is the densities of the species at time t, here we make the following assumptions:

(a) $\frac{a}{b+cx}$ is the birth rate of the species, which is density-dependent, the birth rate of the species is declining as the density of the species is increasing;
(b) d is the death rate of the species, e is the density dependent coefficients;
(c) We incorporate the Allee effect $\alpha(x)=\frac{x}{\beta+x}$ on the species, such an Allee effect describes the fact of limitations in finding mates, which is also called the weak Allee effect function. $\alpha(x)$ is the probability of finding a mate where β is the individual searching efficiency ([4], [8], [11],[12], [13], [14],[15], [16], [18]). The bigger β is the stronger Allee effect.

Allee effect, which was first time observed by Allee ([1]), describes a negative density dependence of the species, become one of the main topics on the ecosystem, many scholars have done works in this direction, see [14], [16] and the references cited therein.

Huseÿın Merdan [15] proposed the following predator-prey system with Allee effect on prey species

$$\begin{aligned}\frac{dx}{dt}&=rx(1-x)\frac{x}{\beta+x}-axy,\\ \frac{dy}{dt}&=ay(x-y).\end{aligned} \quad (1.2)$$

where β is a positive constant, which describes the intense of the Allee effect. *Huseÿın* Merdan showed that the system subject to an Allee effect takes a longer time to reach its steady-state solution and the Allee effect reduces the population densities of both predator and prey at the steady-state.

Stimulated by the work of Merdan, Guan, Liu and Xie [8] argued that the higher the hierarchy in the food chain, the more likely it is to become extinct. Hence, they proposed the following predator-prey model with predator species subject to Allee effect:

$$\begin{aligned}\frac{dx}{dt}&=rx(1-x)-axy,\\ \frac{dy}{dt}&=\frac{ay^2}{\beta+y}(x-y),\end{aligned} \quad (1.3)$$

where r,a are positive constants. They obtained a set of sufficient conditions which ensure the existence of a unique globally asymptotically stable positive equilibrium. Their numeric simulations showed that the system subject to an Allee effect takes a much longer time to reach its stable steady-state solution, however, Allee effect has no influence on the final density of the predator and prey species. Such a finding is very different from that of *Huseÿın* Merdan [15].

During the past decade, the study of dynamic behaviors of mutualism or commensalism model becomes one of the main topics in the study of population dynamics ([4]-[23]), among those works, some of them ([4], [13], [18]) studied the influence of Allee effect to the commensalism model.

Wu, Li, and Lin [18] proposed the following two species commensal symbiosis model with Holling type functional response and Allee effect on the second species

$$\begin{aligned} \frac{dx}{dt} &= x(a_1 - b_1 x + \frac{c_1 y^p}{1+y^p}), \\ \frac{dy}{dt} &= y(a_2 - b_2 y)\frac{y}{\beta + y}, \end{aligned} \qquad (1.4)$$

where $a_i, b_i, i = 1,2\, p, \beta$ and c_1 are all positive constants, $p \geq 1$. They showed by numeric simulations that with the increasing Allee effect, the system takes much more time to reach its stable steady-state solution.

Chen [4] proposed the following two species commensal symbiosis model involving Allee effect and one party can not survive independently, which takes the form:

$$\begin{aligned} \frac{dx}{dt} &= x(-a_1 - b_1 x + \frac{c_1 y}{x+y}), \\ \frac{dy}{dt} &= y(a_2 - b_2 y)\frac{y}{u + y}, \end{aligned} \qquad (1.5)$$

where a_1, b_1, c_1, a_2, b_2 and u are all positive constants, $x(t)$ and $y(t)$ are the densities of the first and second species at time t. The author investigated the local and global stable properties of the boundary equilibrium and the positive equilibrium.

Lin [13] investigated the dynamic behaviors of the following two species commensal symbiosis model incorporating Allee effect to the first species:

$$\begin{aligned} \frac{dx}{dt} &= rx(1-x)\frac{x}{\beta + x} - axu, \\ \frac{du}{dt} &= -bu + cx, \end{aligned} \qquad (1.6)$$

where $b_i, a_{ii}, i = 1,2$ β and a_{12} are all positive constants. He found that with the increase of Allee effect, the final density of the species subject to Allee effect is also increased. Such a phenomenon is the first time observed, which is quite different from the known results ([4], [15]).

Lin [14] proposed a single species Logistic model with Allee effect and feedback control

$$\begin{aligned} \frac{dx}{dt} &= rx(1-x)\frac{x}{\beta + x} - axu, \\ \frac{du}{dt} &= -bu + cx, \end{aligned} \qquad (1.7)$$

where β, r, a, b and c are all positive constants. He showed that for the system without Allee effect, the system admits a unique positive equilibrium which is globally attractive, however, for the system with Allee effect, depending on the intense of the Allee effect, the system could admit a unique positive equilibrium which is locally asymptotically stable or the species may be driven to extinction. Allee effect reduces the population density of the species.

It brings to our attention that in the system (1.2) - (1.7), without the influence of the other species or other factors, the species subject to Allee effect takes the form

$$\frac{dx}{dt} = x(a_1 - ex)\frac{x}{\beta + x}. \qquad (1.8)$$

That is, all the works of [8], [13]-[15] are based on the traditional single species Logistic equation

$$\frac{dx}{dt} = x(a_1 - ex). \qquad (1.9)$$

System (1.9) is very famous and is the cornerstone of population biology. Noting that system (1.9) could be revised as

$$\frac{dx}{dt} = x(a - d - ex). \qquad (1.10)$$

where a is the birth rate of the species and d is the death rate of the species. Already, Brauer and Castillo-Chavez [3], Tang and Chen [17] and Berezansky, Braverman, Idels [2] had shown that in some cases, the density dependent birth rate of the species is more suitable. If we take the famous Beverton Holt function ([2]) as the birth rate, then the system (1.10) should be revised to

$$\frac{dx}{dt} = x(\frac{a}{b + cx} - d - ex). \qquad (1.11)$$

If we further consider the influence of Allee effect to the system (1.11), by adding the term $\frac{x}{\beta + x}$ to the right hand side of the above system, this will lead to the system (1.1).

The paper is arranged as follows. In section 2, we investigated the dynamic behaviors of the system (1.1); Section 3 presents some numerical simulations to show the feasibility of the main results. We end this paper with a brief discussion.

2 Dynamic behaviors of the system (1.1)

The equilibrium of system (1.1) is determined by the equation

$$x(\frac{a}{b+cx}-d-ex)\frac{x}{\beta+x}=0. \quad (2.1)$$

Assume that

$$\frac{a}{b}>d \quad (2.2)$$

holds, then system (1.1) admits a boundary equilibrium $x_1=0$ and a positive equilibrium x^* where

$$x^*=\frac{-(dc+eb)+\sqrt{(dc+eb)^2-4ec(db-a)}}{2ec}. \quad (2.3)$$

For the biology meaning, we will focus our attention on the stability of the positive equilibrium, we have the following result.

Theorem 2.1. *Assume that (2.2) holds, then the system (2.1) admits a unique positive equilibrium* x^* *which is globally stable.*

Proof. Obviously, x^* satisfies the equation

$$\frac{a}{b+cx^*}-d-ex^*=0. \quad (2.4)$$

Now let's consider the Lyapunov function

$$V(x)=x-x^*-x^*\ln\frac{x}{x^*}. \quad (2.5)$$

One could easily see that the function V is zero at the positive equilibrium x^* and is positive for all other positive values of x. By applying (2.4), the time derivative of V along the trajectories of (1.1) is

$$\begin{aligned}
&D^+V(t)\\
&=(x-x^*)(\frac{a}{b+cx}-d-ex)\frac{x}{\beta+x}\\
&=(x-x^*)(\frac{a}{b+cx}-\frac{a}{b+cx^*}\\
&\quad+ex^*-ex)\frac{x}{\beta+x}\\
&=(x-x^*)(\frac{ac(x^*-x)}{(b+cx)(b+cx^*)} \quad (2.6)\\
&\quad+e(x^*-x))\frac{x}{\beta+x}\\
&=-(\frac{ac}{(b+cx)(b+cx^*)}+e)\times\\
&\quad\frac{x}{\beta+x}(x-x^*)^2
\end{aligned}$$

It then follows from (2.6) that $D^+V(t)<0$ strictly for all $x>0$ except the positive equilibrium x^*, where $D^+(t)=0$. Thus $V(x)$ satisfies Lyapunov's asymptotic stability theorem ([6]), and the positive equilibrium x^* of system (1.1) is globally asymptotically stable. This ends the proof of Theorem 2.1.

3 Numeric simulations

Now let's consider the following example.

Example 3.1

$$\frac{dx}{dt}=x(\frac{2}{1+x}-1-x)\frac{x}{\beta+x}. \quad (3.1)$$

In this system, corresponding to system (1.1), we take $a=2, b=1, c=1, d=1, e=1,$ since $a=2>1=db,$ it follows from Theorem 2.1 that for all β, system (3.1) always admits a unique positive equilibrium $x^*\approx 0.4142$, which is globally asymptotically stable. Fig.1 is the case $\beta=1$. Now let's take $\beta=0, 0.2$ and 0.5 and 1, respectively, together with the initial condition $x(0)=0.1$, Fig. 2 shows that with the increasing of the β (i. e., the increasing of the Allee effect), the solution takes much time to reach its steady state.

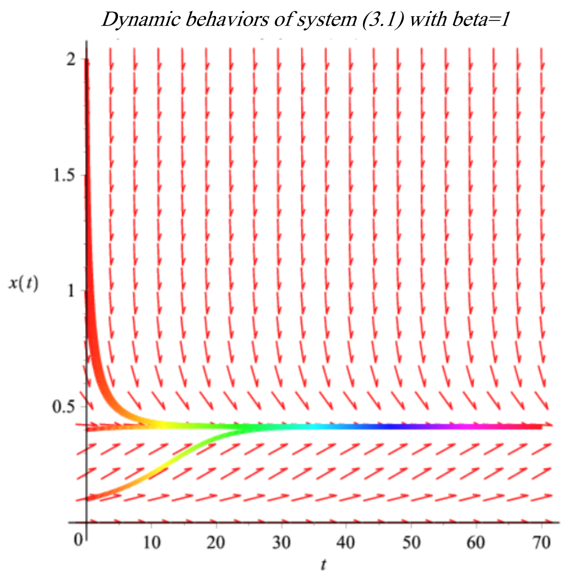

Figure 1: Dynamic behaviors of the system (3.1) with $\beta = 1$, the initial condition $x(0) = 0.1, 0.4, 1, 1.5$ and 2, respectively.

4 Conclusion

Recently, many scholars investigated the dynamic behaviors of the ecosystem subject to Allee effect, see [4], [8], [11],[12], [13], [14],[15], [16], [18] and the references cited therein. By carefully checking the models considered in [14], [15], we found that all of them are based on the traditional Logistic model. However, a more suitable model should consider the influence of density on the birth rate. Thus, we propose a single species model with Allee effect and density-dependent birth rate, i.e., system (1.1).

Our study shows that the conditions which ensure

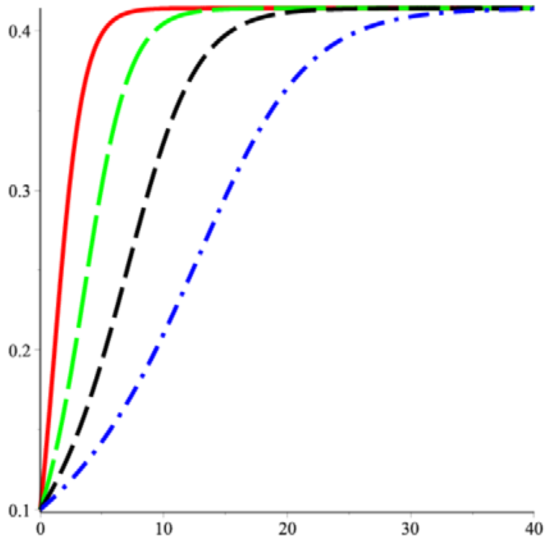

Figure 2: Dynamic behaviors of the system (3.1) with the initial condition $x(0) = 0.1$, where a red curve is the solution of $\beta = 0$, green curve is the solution of $\beta = 0.2$, the black curve is the solution of $\beta = 0.5$ and the blue curve is the solution of $\beta = 1$, respectively.

the existence of the positive equilibrium is enough to ensure its global asymptotically stability, that is, once the positive equilibrium exists, it is globally asymptotically stable. Numeric simulations show that the Allee effect has no influence on the final density of the species, however, with the increasing of the Allee effect, the system takes more time to reach its steady state. Such kind of property is similar to that of the commensalism model ([4], [18]).

It seems interesting to consider the multi-species system with both the nonlinear birth rate and Allee effect, we leave this for future discussion.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Qun Zhu wrote the draft.
Qianqian Li carried out the simulation.
Fengde Chen proposed the issue and revise the paper.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.