On the Monotonous Evolutions: Model and Applications

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Abstract: This article discusses and explains some phenomena that monotonously develop and presents a mathematical model that controls and describes these monotonous evolutions. Furthermore, this model is linked to some applications in several different fields of physics. Knowing that this model consists of a set of differential equations and their solutions with some mathematical properties.

Key-Words: Monotony; Linear differential equation; Evolution; Exponential increase; Radioactive decay; RC circuit; Progress of chemical reaction X

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1 Introduction

There are many phenomena that develop in one direction and this is what we call the monotony of evolution. The evolution of these phenomena is described and governed by differential equations and their solutions give us an accurate description of these developments. The most known developments are exponential. Here in this work, we try to shed light on the most important phenomena that are increasing exponentially in one direction and those that are decreasing exponentially in one direction. Such phenomena are common. We also give some of the mathematical properties of these phenomena.

A differential equation is an equation that relates one or more unknown variables (functions) and their derivatives, [1], [2]. In applications, the variables generally represent physical quantities, the derivatives represent their rates of change and the differential equation defines the evolution. Differential equations can be classified according to their order, which is determined by the highest derivative that appears in the equation. Historically, the concept of the differential equation first came into existence with the invention of calculus in the late 17th century by the mathematicians Isaac Newton and Gottfried Wilhelm Leibniz independently of each other, [3], [4], [5]. Thereafter, the development of calculus and its uses have continued to the present day. Differential equations play a prominent role in many disciplines including physics, engineering, biology, and even economics and sociology. The beauty in the study of differential equations is how solvable explicit formulas are; as well, many properties of its solutions may be determined without calculating them intricately. Differential equations play an important role in modelling virtually every biological process or physical technical, from celestial motion to an interaction between particles. In general, several techniques and approaches have been developed to solve differential equations, including undetermined coefficients and separation of variables methods, and numerical approaches such as Runge-Kutta and Euler's methods, [2]. However, in particular, differential equations such as those used to solve physical problems and describe complicated phenomena may not necessarily be directly solvable. Instead, solutions can be approximated using numerical methods.

Among the most important physicists who were credited with the application of differential equations in physics, were the following scientists: Jean le Rond d'Alembert, Leonhard Euler, Daniel Bernoulli, Joseph-Louis Lagrange, and Joseph Fourier. Now, as practical examples in physics about differential equations since their discovery, we mention: (i) the problem of a vibrating string such as that of a musical instrument, [6]. (ii) D'Alembert as well, discovered the one-dimensional wave equation in 1746, and later within ten years the 3D wave equation was discovered by [7]. (iii) The Euler–Lagrange equations, [8], [9], which are a system of second-order ordinary differential equations, were developed in 1750 by Euler and Lagrange in connection with their studies of the tautochrone problem, [10]. This later led to the development of Lagrange's method and applied it to mechanics, which yielded the formulation of

Lagrangian mechanics. (iv) The work on the heat flow in the analytic theory of heat by Fourier in 1822, [11], which was about heat equation for conductive diffusion of heat. This last partial differential equation has become an essential equation of mathematical physics that is taught to students today.

There are several monotonous developments in nature, and we will highlight some of these phenomena later. Monotonous development refers to a process or behavior that consistently increases or decreases without any significant deviation or reversal. The monotony here is merely an exponential increase or decrease in one direction. It is not an increase, then a decrease, or a combination of the two. The mathematical model that controls and describes these developments will be presented and linked to some applications in several different fields, including nuclear transformations, electricity, chemical reactions in aqueous solutions, and Newtonian mechanics.

In fact, there are many other examples, including: population growth, which is exponential growth, and light intensity in optics where the intensity of light decreases exponentially as it travels through a medium (Beer-Lambert law). Also, the decay of the magnetic field because when a magnetic field is removed or disrupted, its strength decreases over time following an exponential decay curve. Such as in MRI machines, where the decay of magnetic fields is caused by the relaxation processes, which is why MRI machines must be calibrated regularly. Heat transfer can be also an example of monotonous evolution in certain cases. Monotonic evolution refers to a continuous change in a system over time that either always increases or always decreases, without any fluctuations or reversals. Heat transfer can exhibit monotonous evolution if the transfer of heat energy is always in the same direction and does not fluctuate. For example, when a hot object is placed in contact with a cooler one, heat energy will transfer from the hotter object to the cooler one until both objects reach thermal equilibrium. This transfer of heat energy is always in the same direction, from hot to cold, and does not fluctuate or reverse. As a result, the change in temperature over time between the hot object and the cool one will follow a monotonous trend, with the temperature of the hot object decreasing and the temperature of the cool one increasing until they reach equilibrium.

This paper is outlined as follows. In Section 2, the monotonous evolution model is presented. In Section 3, some applications and illustrations are proposed and presented. In sub-section 3.1, we study radioactive decay. Then in sub-sections 3.2 & 3.3, as an illustration of the proposed model in both electricity and mechanics, we present a resistance-capacitor circuit and the motion of a real fall of a solid object in the air, respectively. Sub-section 4 is devoted to the application in chemistry, in which the proposed model is associated with the way quantities of the substance are formed or reacted during chemical reactions in an aqueous solution. We present our conclusion in Section 4.

2 Monotonous Evolution Model

The pattern of monotonous developments we present in this work is of two types: one is an exponential increase, and the other one is an exponential decrease, both in one direction. We start with the following differential equation

$$\frac{dX}{dt}+\frac{1}{\tau}X=\frac{X_0}{\tau}, \qquad (1)$$

where X is the physical variable (function in time); X_0 is the maximum value of X and τ is the time constant, which will be interpreted as the image of the crossing of the tangent of the function curve $X=f(t)$ at the zero point with the asymptote $X=X_0$. It has the dimension of time and its unit is the second (s). Note that the resolution of equation (1) yields

$$X(t)=X_0\left(1-e^{-\frac{t}{\tau}}\right). \qquad (2)$$

Figure 1 illustrates the given function in equation (2) as follows:

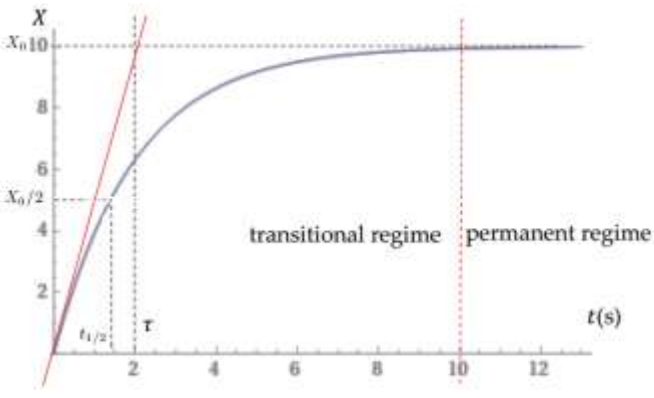

Fig. 1: Plot of $X=f(t)$ for $X_0=10$ and $\tau=2$.

We say that equations (1) and (2) constitute a system of evolution of the exponential increase in one direction. From the graph, we note that there are two phases during the evolution of the proposed system, one of them is transitional, in which the system develops exponentially, while in the second phase, the system is constant at a specific value. Now for the completely decreasing evolutions in one direction, we have the following differential equation

$$\frac{dX}{dt} + \frac{1}{\tau} X = 0, \tag{3}$$

thus the resolution of equation (3) gives

$$X(t) = X_0 e^{-\frac{t}{\tau}}. \tag{4}$$

Figure 2 illustrates the given function in equation (4) as follows:

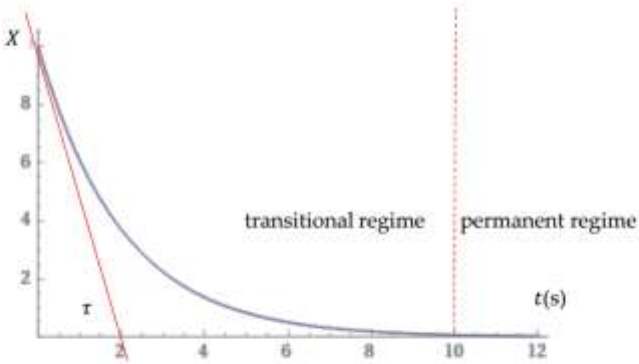

Fig. 2: Plot of $X = g(t)$ for $X_0 = 10$ and $\tau = 2$.

We may add certain characteristics such as:

1. The constant λ: it is the reciprocal of τ (in s^{-1}), sometimes referred to as simply increasing or decreasing rate. Thus, we have the expression

$$\lambda = \frac{1}{\tau}. \tag{5}$$

2. The half-life $t_{1/2}$: it is the time taken for the increasing or decreasing of a given amount of X to half of its initial value X_0, which means it is the time that correspond to $X_0/2$. So from equation (2) or equations (4), we have

$$t_{1/2} = \tau ln2. \tag{6}$$

It is easy to deduce that for $t_n = nt_{1/2}$, we have $X_n = \frac{X_0}{2^n}$.

Last but not least, it can be concluded that: (i) τ is defined as twenty percent of the time of reaching the permanent system. It can be used to indicate how rapidly an exponential function increase or decreases. Physically, in a decreasing system, τ represents the elapsed time required for the system response to decay to zero, it is $\frac{1}{e}$ of X_0. In an increasing system, τ is the time for the system's step response to reach $(1 - \frac{1}{e})$ of X_0. (ii) The systems of monotonous exponential increase (or even decrease) that we study consist of a transitional regime and a permanent one. The beginning of the permanent regime is always at $t = 5\tau$.

3 Applications

3.1 The Radioactive Decay

As known about nuclear transformations, there are two types of transformations: Stimulated (non-spontaneous) nuclear transformations, namely nuclear fission, and thermonuclear fusion; and spontaneous nuclear transformations, namely radioactivity. In this part of the article, we will focus on spontaneous nuclear reactions known as radioactivity.

Radioactivity, [12], [13], [14], is the phenomenon of the spontaneous disintegration of unstable (i.e., radioactive) atomic nuclei, so in the process of decay, one or more types of energetic ionizing radiation (particles or electromagnetic radiation) are emitted. However, in the random process of radioactive decay, a nucleus loses energy by emitting radiation, where the nature of the produced radiation depends first on if the unstable nucleus is heavy or not, in the case of heavy nuclei the produced radiation Is usually in the form of α particles (Helium nuclei ${}^{4}_{2}He$). If the emitted radiation is β^-or β^+ particles, namely electrons ${}^{\ 0}_{-1}e$ and positrons ${}^{0}_{1}e$, respectively. This is done according to the number of neutrons and protons present inside the unstable nucleus, where if the number of protons is greater, the radioactive transformation is according to the β^+ decay pattern i.e., ${}^{1}_{1}p \to {}^{1}_{0}n + \beta^+$, while if the number of neutrons is greater, the radioactive transformation pattern is according to β^- i.e., ${}^{1}_{0}n \to {}^{1}_{1}p + \beta^-$. There are also gamma rays ${}^{0}_{0}\gamma$ (high-energy photons), which automatically accompanies the previous emissions, and it occurs at the energy level, rather than at the particle level of the nucleus. Knowing that the nucleu'' energy reduces, making it more stable. In all decay processes mass, charge, and lepton number are conserved. Once α-decay occurs the radioactive nucleus changes into a different more stable one, with two fewer protons and two fewer neutrons, and α particle is emitted. On the other hand, when β^-or

β^+-decay occurs, the number of nucleons in the nucleus remains the same, so the mass number does not change, but as a neutron is converted into a proton the atomic number increases by one, and as a proton is converted into a neutron the atomic number decreases by one.

Radioactive decay is a random process, which means that it is impossible to predict when a particular radioactive nucleus will decay. It is also spontaneous; you cannot cause or influence the decay. However, with large numbers of nuclei, it is possible to predict statistically the behavior of the entire group through radioactive decay. So le''s consider the case of a nuclide A that decays into another one B through some radioactivity $A \to B$. e.g., β-decay, which is the emission of positrons e or electrons e^-or α-decay or γ-decay. Statistical study of the behavior of several unstable nuclei together yields the suggested model. The decay of an unstable nucleus is completely random in time so according to quantum theory, it is not possible to predict when a particular nucleus will decay, regardless of how long the nucleus has existed. Thus, for this reason, it appears in the equation later on the so-called probability of disintegration per second λ. Therefore, given a sample of a particular radioisotope, the number of decay events $-dN$ expected to occur in a small interval of time dt and is proportional to the number of radioactive present nuclei N, which means $-dN \propto dt$ and $-dN \propto N$. Thus, $-dN \propto Ndt$ and this yields $-dN = \lambda Ndt$ (negative because the number of nuclei decreases over time), which means that the rate of change of radioactive nuclei is proportional to the number of remaining radioactive nuclei N . Therefore, we have

$$\frac{dN}{dt} + \lambda N = 0, \tag{7}$$

the differential equation above is similar to equation (3) presented in the monotonous evolution model, and as well its solutions are

$$N(t) = N_0 e^{-\lambda t}, \tag{8}$$

which is the radioactive decay law, where $N(t)$ is the quantity of remaining radioactive (undecayed) nuclei at time t, N_0 is the initial quantity (at time $t = 0$), and the constant λ is called the decay constant or disintegration constant, or even transformation constant. So $A = \lambda N$ is the activity of radioactive nuclei, which is the number of decays per second. Its unit is the Becquerel (Bq) in SI, where $1Bq = 1$ disintegrations/Sec, also the Curie (Ci) and Rutherford (Rd) in a non-SI, where $1\ Ci = 3.7 \times 10^{10}\ Bq$ and $1Rd = 10^6\ Bq$, [15]. Since the activity is proportional to the number of radioactive atoms, it decreases exponentially with time as well

$$A(t) = A_0 e^{-\lambda t}. \tag{9}$$

The decay rate of a radioactive substance is characterized by the following time-independent parameters:

1. The half-life $t_{1/2}$ of a particular species of nuclei is the time that it would take for the number of nuclei N_0 in a given sample to decay to halve. The larger the half-life of a nuclei, the less likely it is to decay in a given time.
2. Mean lifetime τ, which is the average lifetime of a radioactive particle before decay.
3. Decay constant λ, which is the reciprocal of the mean lifetime (in s^{-1}), sometimes referred to as simply decay rate.
4. The equation that combines the aforementioned properties is
$$t_{1/2} = \tau ln2 = \frac{1}{\lambda} ln2. \tag{10}$$

Knowing that $t_{1/2}$ depends only on λ of the nuclei. It is always the same; the amount of time for the number of nuclei to decrease from 40 million to 20 million is the same amount of time as it takes the number to decrease from 4.8 to 2.4. The initial number does not have an effect.

Figure 3 illustrating the decay of remaining radioactive nuclei is as follows:

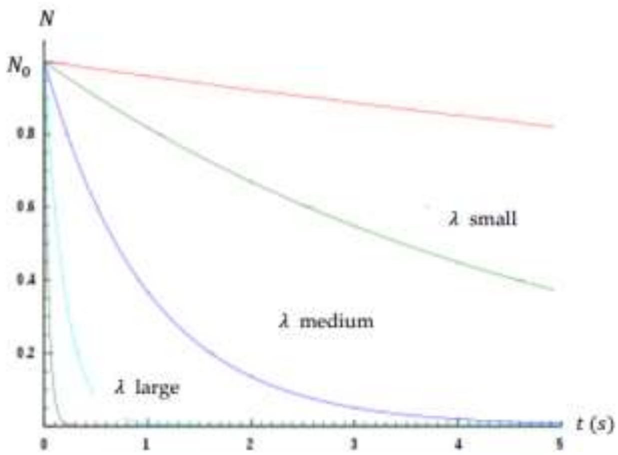

Fig. 3: The graph shows the exponential decay of a radioactive element. It shows three different exponential decays, each with a different decay constant i.e., 25, 5, 1, 1/5, and 1/25.

The remaining radioactive nuclei undergo exponential decay, where larger decay constants make the quantity N vanish much more rapidly.
Now, if we have $N_0 = N + N'$, where N' is the transformed nuclei (more stable), thus using equations (8), we have

$$N'(t) = N_0\left(1 - e^{-\lambda t}\right). \qquad (11)$$

We also note that for any quantity related to the number of remaining radioactive nuclei N, the same previous laws of a differential equation and its solutions are applied to it. For example, we derive the law of radiative decrease in terms of remaining radioactive mass $m(t)$. We have $n(t) = \frac{N(t)}{N_A} = \frac{m(t)}{M}$, where n, N_A, M are the amount of substance, Avogadro number, and the atomic molar mass, respectively. So by using equation (8), we obtain

$$m(t) = m_0 e^{-\lambda t}, \qquad (12)$$

where m_0 is the initial mass of radioactive nuclei.

3.2 Resistor–Capacitor Circuit (RC Circuit)

The RC circuit is an electric circuit composed of resistor R and capacitor C. The simplest RC circuit consists of a resistor and a charged capacitor connected in a single loop.

A capacitor is one of several kinds of devices used in electric circuits as in computers, radios, and other such equipment. The capacitors provide temporary storage of energy in circuits and the property of a capacitor that characterizes its ability to store energy is called its capacitance. When energy is stored in a capacitor, an electric field exists within the capacitor.

3.2.1 Case of Capacitor Charging Process (Presence of a Voltage Source):

We realize the electrical circuit consists of the components: a voltage source E (generator), a resistor, and a capacitor, connected to one another in a single loop. The diagram of RC circuit in the presence of a voltage source is presented in Figure 4.

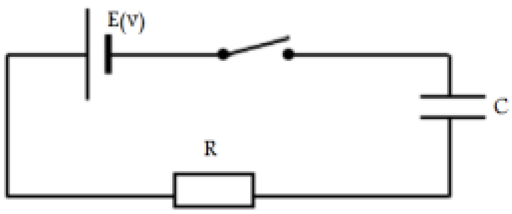

Fig. 4: Diagram of RC circuit in the presence of a voltage source.

Once the circuit is closed, the capacitor starts to charge its stored energy through E. The system we study will be described by a linear differential equation and its solution is the voltage across the capacitor U_C, which is time-dependent, and it can be found by using Kirchhoff's current law (sum of voltage law) as follows

$$U_C + U_R = E. \qquad (13)$$

By using Ohm's law

$$U_R(\mathrm{t}) = Ri(t), \qquad (14)$$

and

$$U_C(\mathrm{t}) = \frac{q(t)}{C} \ \ and \ \ i(\mathrm{t}) = \frac{dq(t)}{dt}, \qquad (15)$$

we have

$$\frac{dU_C}{dt} + \frac{U_C}{R} = \frac{E}{RC} \ \ with \ \ \tau = RC, \qquad (16)$$

which is a linear differential equation similar to that of equation (1). Knowing that U_R, U_C are resistor and capacitor voltages. τ is the time constant (in seconds), R is the electric resistance (in ohms) and C is the electric capacitance (in farads). Noting that each variable has a relationship with the capacitor voltage, the studied system can be described by a differential equation according to it such as the charge $q(t)$ on the poles of the capacitor

$$\frac{dq}{dt} + \frac{q}{\tau} = \frac{Q_{max}}{\tau} \ \ with \ \ Q_{max} = EC. \qquad (17)$$

By solving the differential equation (16) describing the studied circuit, the voltage across the capacitor is

$$U_C(t) = E(1 - e^{-\frac{1}{\tau}t}), \qquad (18)$$

where $U_{Cmax} = E$.

3.2.2 Case of Capacitor Discharging Process (Without a Voltage Source):

We use the same circuit as before, but without an electrical voltage source E. The diagram of RC circuit in the absence of a voltage source is presented in Figure 5.

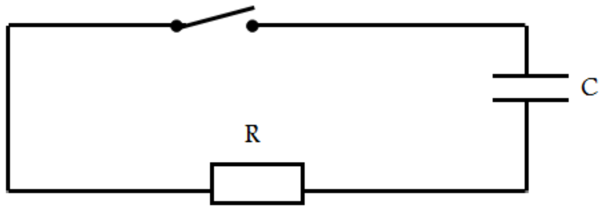

Fig. 5: Diagram of RC circuit in the absence of a voltage source.

Once the circuit is closed, the capacitor starts to discharge its stored energy through the resistor. The voltage across the capacitor as well is time-dependent. Both the linear differential equation that describes the system above and its solution (the voltage across the capacitor U_C) can be found by using the sum of voltages law as follows

$$U_C + U_R = 0. \tag{19}$$

Now using equations (14) and (15), we then obtain

$$\frac{dU_C}{dt} + \frac{U_C}{R} = 0, \tag{20}$$

which is a linear differential equation similar to that of equation (3). Each variable has a relationship with the capacitor voltage U_C can be used to study the system. Such as the following differential equation according to the charge on the poles of the capacitor

$$\frac{dq}{dt} + \frac{q}{\tau} = 0. \tag{21}$$

As well, the equation from equation (19) using the voltage across the resistor U_R (t) is

$$\frac{1}{\tau}\int U_R dt + U_R = 0, \tag{22}$$

which is a Fredholm integral equation, and once we derive it we get

$$\frac{dU_R}{dt} + \frac{U_R}{\tau} = 0. \tag{23}$$

The same regarding the equation from the current $i(t)$, so by deriving equation (21), we have

$$\frac{di}{dt} + \frac{i}{\tau} = 0. \tag{24}$$

Now the solution to equation (18) is

$$U_C(t) = e^{-\frac{1}{\tau}t}. \tag{25}$$

In this part, we can conclude that τ is the time required to charge the capacitor, through the resistor, from an initial charge voltage of zero to approximately 63.2% of E or to discharge the capacitor through the same resistor to approximately 36.8% of E.

3.3 Motion of a Real Fall of a Solid Object in the Air

An example of modelling a real problem using differential equations is the determination of the velocity v of a solid object (s) falling through the air, considering only gravity and air resistance. The ball's acceleration towards the ground is the acceleration due to gravity minus the deceleration due to air resistance. Gravity is considered constant, and air resistance may be modelled as proportional to the ball's velocity. This means that the ball's acceleration, which is a derivative of its velocity, depends on the velocity, which depends on time. Finding the velocity as a function of time involves solving a differential equation and verifying its validity. Now, we consider the system in question to be a solid object (s) falling from a certain height toward the ground (one-dimensional motion). Figure 6 presentes th scheme of the various forces to which the body is subject.

Fig. 6: Scheme of the various forces to which the body is subject.

The reference level for gravitational potential energy is the surface of the Earth. The external forces acting on the object (s) during its fall are the force from gravity $\vec{p} = M\vec{g}$ and $\vec{\pi} = m_{air}\vec{g} = \rho_{air}V\vec{g}$ denotes the buoyant force applied onto the submerged object (according to Archimedes' principle) and the force of friction $\vec{f} = kv^n \vec{e}_z$, where for low object velocities $n = 1$ and for high ones $n = 2$.

M is the mass of the body; g is Earth's gravitational acceleration; $\rho_{air} = \frac{m_{air}}{V_{air}}$ is a volumetric mass of air (fluid) and V is the volume of the solid object immersed in the fluid (equal to that of the displaced fluid i.e., $V = V_{air}$). k is the friction constant and v is the solid object velocity. By applying Newton's second law (the basic principle of motion), we have

$$\sum \vec{F}_{ext} = \vec{p} + \vec{\pi} + \vec{f} = Ma_s, \tag{26}$$

where a_s is the acceleration of the object (s). By projecting the equation (26) on the axis of motion and after simplifying, we find

$$\frac{dv}{dt} + \frac{1}{M} f = \frac{Mg - \rho_{air} V g}{M}, \tag{27}$$

with $\rho_{air} = \frac{M}{V}$ is the volumetric mass of the solid object (s).

- In the case of low velocities, we have

$$\frac{dv}{dt} + \frac{k}{M} v = g(1 - \frac{\rho_{air}}{\rho_s}), \tag{28}$$

which is a 1st-order differential equation similar to equation (1), its solution is like equation (2), which is

$$v(t) = v_l \left(1 - e^{-\frac{1}{\tau}t}\right). \tag{29}$$

Here v_l is the terminal velocity and $\tau = \frac{M}{k}$. In the permanent regime, we have $\frac{dv}{dt} = 0$, thus we obtain

$$v_l = \frac{g}{k}(m - \rho_{air} V). \tag{30}$$

The terminal velocity of the object increases with the increase in the volumetric mass of the solid object, and the following table (Table 1) gives some examples:

Table 1. Some solid objects and their terminal velocities v_l.

Solid Object	**Terminal Velocity v_l (m/s)**
Paratrooper in free vertical fall	8,5
Paratrooper with an opened parachute	6,5
Table tennis ball	7
Golf ball	30
A steel ball with a radius of 2 cm	80
Stone of radius 1 cm	30
A drop of water	10

- In the case of high velocities, we have

$$\frac{dv}{dt} + \frac{k}{M} v^2 = g(1 - \frac{\rho_{air}}{\rho_s}), \tag{31}$$

where the terminal velocity is

$$v_l = \sqrt{\frac{g}{k}(m - \rho_{air} V)}. \tag{30}$$

Now, Figure 7 presents a graph from equation (24) as follows:

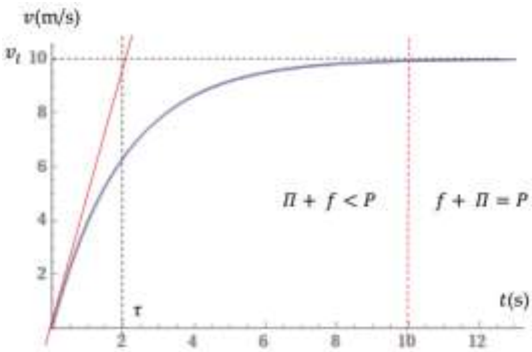

Fig. 7: Plot of $v = f(t)$ where $v_l = 10$ and $\tau = 2$.

3.4 Formed and Reacted Substance Amounts during Chemical Reactions

As another example regarding the phenomena that develop monotonously, let's consider the evolution of the formed or reacted (hidden) substance amounts and the progress of reaction (as the extent of reaction) in the chemical reactions in aqueous solutions. By considering the simple chemical equation as $A + B \rightarrow C + D$, where the chemical reactants on the left and those of the chemical products on the right. The amount of substance n_A is of the hidden reactant A and n_C is of the formed product C and the progress of chemical reaction X, which expresses the number of times the reaction occurs; it is at the macroscopic level. So, in a chemical reaction in an aqueous solution, the evolution of the amount of the formed substance will be exponentially increasing while the amount of the reacted substance will be exponentially decreasing. As a result, we present the plot of $n_A = f(t), n_C = g(t)$) where $\tau = 2$ in Figure 8.

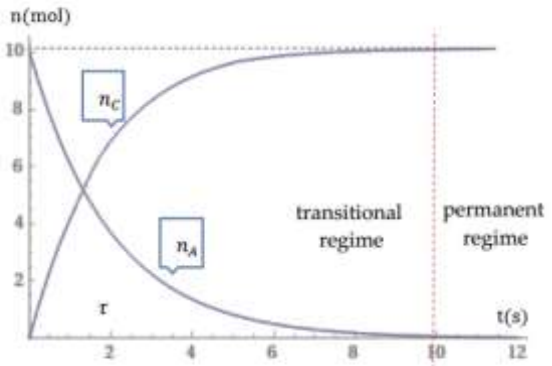

Fig. 8: Plot of $n_A = f(t); n_C = g(t)$) where $\tau = 2$.

In addition, the equation of the progress of reaction in time $X(t)$ will be

$$X(t) = X_{max}\left(1 - e^{-\frac{1}{\tau}t}\right), \tag{29}$$

where X_{max} is the maximum number of times the reaction occurs, it also corresponds to the smallest value of the progress for which the final quantity of at least one of the reactants is zero. We also have the half-life $t_{1/2}$, which is intended to compare two chemical reactions in terms of speed or to examine whether the reaction is slow or fast. So, $t_{1/2}$ of a chemical reaction is the time required for the reaction to progress half its final progression, i.e., for $t = t_{1/2}$, we have $X_{1/2} = \frac{X_\mathrm{f}}{2}$. Once the reaction is complete, then $X_\mathrm{f} = X_\mathrm{max}$.

We draw attention to that any amount related to the amount of substance, such as concentration, could be used in the study.

Finally, yet importantly, we note that with each application study, the picture becomes clearer. As well, the permanent regime (the speed constancy) is reached after a period of 5τ.

4 Conclusion

Many phenomena develop monotonously where the evolution of these phenomena is described by differential equations and their solutions give accurate descriptions of these developments. The most known developments are those that develop exponentially. In this work, we have put a mathematical model used to describe these phenomena and shed light on the most common important phenomena that are increasing exponentially in one direction and those that are decreasing exponentially in one direction. Such as the radioactive decay, charging, and discharging of a resistor–capacitor circuit and the motion of a real fall of a solid object in the air, as well, as the progress of quantities of substance when formed or reacted during chemical reactions in an aqueous solution. Besides, we explained some of the mathematical properties of these phenomena.

References:

[1] Simmons, G. F. (2016). *Differential equations with applications and historical notes*. CRC Press.

[2] Dennis G. Zill. (2012). *A first course in differential equations with modeling applications*. Cengage Learning. 10th Edition.

[3] Newton, Isaac. (1671). *Methodus fluxionum et serierum infinitarum*. Opuscula mathematica. 1, 66. Published in 1744.

[4] Leibniz, Gottfried Wilhelm: Nova methodus pro maximis et minimis, *Acta eruditorum*, 1684.

[5] Leibniz, Gottfried Wilhelm Freiherr. La naissance du calcul différentiel: 26 articles des. *Acta eruditorum*. Vrin, 1989.

[6] Wheeler, G. F., & Crummett, W. P. (1987). The vibrating string controversy. *American Journal of Physics*, 55(1), 33-37. https://doi.org/10.1119/1.15311

[7] Speiser, David. *Discovering the principles of mechanics 1600-1800*: essays by David Speiser. Vol. 1. Springer Science & Business Media, 2008.

[8] Fox, Charles. *An introduction to the calculus of variations*. Courier Corporation, 1987.

[9] Pierce, Allan D. *Acoustics: an introduction to its physical principles and applications*. Springer, 2019.

[10] Blackwell, Richard J. (1986). Christiaan Huygens' *The Pendulum Clock*. Ames, Iowa: Iowa State University Press. Part II, Proposition XXV, p. 69. ISBN 0-8138-0933-9

[11] Fourier, Jean Baptiste Joseph, and Gaston Darboux. *Théorie analytique de la chaleur*. Vol. 504. Paris: Didot, 1822.
[12] Patel, S.B. (2000). *Nuclear physics: an introduction*. New Delhi: New Age International. pp. 62–72. ISBN 978-81-224-0125-7.
[13] Glascock, M. D. 15.15-*Nuclear Spectroscopy*. Treatise on Geochemistry (2nd Edition), Elsevier, (2014) 273-290. https://doi.org/10.1016/B978-0-08-095975-7.01419-4
[14] L'Annunziata, Michael F. (2007). *Radioactivity: Introduction and History*. Amsterdam, Netherlands: Elsevier Science. p. 2. ISBN 9780080548883.
[15] Vincent P. Guinn, *in Encyclopedia of Physical Science and Technology* (3rd Edition), 2003. https://www.sciencedirect.com/topics/physics-and-astronomy/radioactivity

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The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

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