The New Way to Solve Physical Problems Described by ODE of the Second Order with the Special Structure

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Abstract- In the last decade, many researchers have studied extensively theoretical and practical problems of natural sciences using ODEs as a means to analyze and understand them. Specifically, second-order ODEs with special complex structures provide the necessary tools to construct mathematical models for several physical - and other- processes such as the Schturm-Liouville, Schrölinger, Population, etc. As a result, it is of great importance to construct special stable methods of a higher order as a means to solve differential equations. One of the most important efficiency methods for solving these problems is the Stërmer-Verlet method which consists of hybrid methods with constant coefficients. In this paper, we expand on recent studies that prove that the hybrid methods are more precise than the Stërmer-Verlet method while investigating the convergence variable. This paper aims to prove the existence of a new, stable hybrid method using a special structure of degree(p)=3k+2, where k is the order of the multistep methods. Lastly, we also provide a detailed mathematical explanation of how to construct stable methods on the intersection of multistep and hybrid methods having a degree(p)≤3k+3.

Key-Words: Hybrid method of Stërmer type, multistep second derivative method, stability and degree, relationship between order and degree.

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1 Introduction

As is known, some theoretical and applied problems are reduced to solve the initial value problem for ODE of the second order, which can be presented as:

$$y''(x)=F(x,y,y'),\quad y'(x_0)=y'_0,\ y(x_0)=y_0,\quad x_0\le x\le X. \qquad (1)$$

There are wide classes of numerical methods for solving this problem. For solving problem (1), here proposed to use the multistep multiderivative methods with constant coefficients of the hybrid type. Let us assume that problem (1) has a unique solution for a continuous total arguments function that has defined or some closed area. Some known scientists who investigated the solution using ODE participated in the problem(1). For the illustration of this, let us consider the following generalization of the known Schrödinger and Sehturm-Liouville equations (see [1], [2], [3). Note that the Schrölinger, and Schturm-Liouville problems are usually formulated using the boundary-value problem for the abovementioned equation (see for example, which can be reduced to the solution of the initial value problem for the ODE of the second order (see for example [4], [5], [6]). The problem (1) has been investigated by many authors by using the one-step or multistep methods (see for example [7], [8], [9]). For solving the problem (1), here proposed to use some generalization of the Stërmer-Verlet methods, the construction of which has used the hybrid method with the special structure. To determine the maximum value of the order of accuracy for the proposed methods, have used the way of unknown coefficients and the theory of nonlinear systems of algebraic equations (see for example [10], [11], [12], [13], [14], [15]).

As is known, the Stërmer-Verlet method can be applied to solving the initial-value problem for the above-mentioned second order with the special structure. For the illustration of this, consider the construction of an effective method for solving named problems.

Suppose that the right-hand side of the ODE in the problem (1) has been presented as $F(x,y,y')=f(x)$. In this case, the problem (1) can be written as the following:

$$y''=f(x), y'(x_0)=y'_0, y(x_0)=y_0, x\in[x_0,X]. \qquad (2)$$

The solution to this problem can be presented as:

$$y(x) = y_0 + y'_0(x - x_0) + \int_{x_0}^{x} (x - s) f(s) ds. \quad (3)$$

By using this equality one can write the following:

$$y(x+h) = y(x_0) + y'(x_0)(x - x_0) + hy'(x_0) + \int_{x_0}^{x+h} (x-s) f(s) ds + h \int_{x_0}^{x+h} f(s) ds,$$
$$y(x-h) = y(x_0) + y'(x_0)(x - x_0) - hy'(x_0) + \int_{x_0}^{x+h} (x-s) f(s) ds + h \int_{x_0}^{x+h} f(s) ds. \quad (4)$$

Here, the solution of the understudy is constructed in such a form that allows one to construct the Stërmer-Verlet method using a minimum number of operations.

Applying any quadrature method to the calculation of definite integrals involved in the above equality and comparing them, we obtain the following numerical method for solving the problem, which can be presented in the following form:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i f_{n+i}. \quad (5)$$

The results of the investigation of method (5) received some connection (see for example, [7], [8], [9], [10], [11]).

For using method (4) it is necessary to construct some algorithm for the calculation of the values $y'_m = (m, 1, 2, \ldots)$. For this aim, let us consider the following:

$$y'(x+h) = y'(x) + \int_{x}^{x+h} f(s) ds;$$
$$y'(x-h) = y'(x) \int_{x-h}^{x} f(s) ds. \quad (6)$$

By using above mentioned, for the calculation of the value y'_m one can be recommended the following:

$$\sum_{i=0}^{k} \alpha'_i y'_{m+i} = h \sum_{i=0}^{k} \beta^j_i y''_{n+i}. \quad (7)$$

This method can be received from the known multistep method, which is applied to solve ODE of the first order, by the change $y(x)$ with the $y'(x)$. Noted that for solving the problem (2), let us use the points $x_0 - h,\ x_0,\ x_0 + h,$ or the points $x - h,\ x,\ x + h$. For the construction methods similar to the Stërmer method let us consider the following equalities:

$$y(x+h) - y(x-h) = 2hy'x + \int_{x-h}^{x+h} (x-s) f(s) ds + h \int_{x-h}^{x+h} f(s) ds, \quad (8)$$
$$y(x+h) - y(x) = hy'x + \int_{x}^{x+h} (x-s) f(s) ds + h \int_{x_0}^{x+h} f(s) ds.$$

By using (3) and (4) or the equalities (8), receive:

$$y(x+h) - 2y(x) + y(x-h) = \int_{x}^{x+h} (x-s) f(s) ds - \int_{x-h}^{x} (x-s) f(s) ds + h \int_{x-h}^{x+h} f(s) ds - h \int_{x-h}^{x} f(s) ds. \quad (9)$$

If put $x = x_n + (k-1)h$ in the equality of (9), then receive:

$$y_{n+k} - 2y_{n+k-1} + y_{n+k-2} = \int_{x_{n+k-2}}^{x_{n+k}} (x_{n+k} - s) f(s) ds - \int_{x_{n+k}}^{x_{n+k-1}} (x_{n+k-1} - s) f(s) ds + h \int_{x_{n+k-2}}^{x_{n+k}} f(s) ds. \quad (10)$$

To use some quadrature formula for the calculation of definite integrals participated in equation (10), receive

$$y_{n+k} - 2y_{n+k-1} + y_{n+k-2} = h^2 \sum_{i=0}^{k} \gamma_i f_{n+i} \quad (11)$$

Here the coefficients of $\gamma_i (i = 0, 1, \ldots, k)$ are calculated by the values of the coefficients of the quadrature formula which has been applied to the calculation of the definite integrals participating in the equality of (10). By the generalization of the linear part of the method (11), one can receive the following:

$$\sum_{i=0}^{k} \bar{\alpha}_i y_{n+i} = h^2 \sum_{i=0}^{k} \bar{\beta}_i f_{n+i}. \quad (12)$$

By the comparison of methods (5) and (12), method (12) can be obtained from method (5) as the partial case. As is known, one of the basic conceptions for comparison of numerical methods is in regards to the degree and its stability, which can be defined in the following way (see for example [7], [8], [9], [10]).

Definition 1. The integer value is called the degree for the method of (5) if the following asymptotic equality takes place:

$$\sum_{i=0}^{k} (\alpha_i y(x+ih) - h\beta_i y'(x+ih) - h^2 \gamma_i y''(x+ih)) = O(h^{p+1}), h \to 0. \quad (13)$$

If $\beta_i = 0\ (i = 0, 1, 2, ..k)$, then the asymptotic equality presented as:

$$\sum_{i=0}^{k}(\bar{\alpha}_i y(x+ih)-h^2\bar{\gamma}_i y''(x+ih)=O(h^{p+2})\quad h\to 0. \tag{14}$$

Definition 2. Method of (12) is called as stable if the roots of the polynomial $\rho(\lambda)=\alpha_k^{\lambda^k}+\alpha_{k-1}\lambda^{k-1}+...+\alpha_1\lambda+\alpha_0$ located in the united circle, on the boundary of which there are no multiple roots.

From here we receive that method (12) is an independent object for the investigation. Note that method (12) was investigated by many authors (see for example, [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]). And have defined the conditions which must satisfy its coefficients for its convergence. The relation between the exact and numerical solution of problem (1) is investigated in the next paragraph.

2 Application of Hybrid Methods to Solve the problem (1).

Usually, finding numerical solutions to problem (1) has used methods which are called multistep methods with constant coefficients or the finite difference method. For construction, more exact methods here have proposed to use the forward-jumping and hybrid methods, so these methods have some advantages. For the construction methods of type (12), for solving the following problem:

$$y(x)=f(x,y(x)),\quad y'(x)=y_0',\ y(x_0)=y_0. \tag{15}$$

Let us consider the finding of the coefficients in the method (12). For this aim, consider the approximation of definite integrals participating in the equality of (8). As is known there are some classes of methods for the calculation of definite integrals. Here proposed to use the following method, which can compare with the equality of (11):

$$\int_{x_{n+1}}^{x_{n+k}}(x_{n+k-1}-s)f(s)ds-\int_{x_n}^{x_{n+k-1}}(x_{n+k-2}-s)f(s)ds+h\int_{x_n}^{x_{n+k}}f(s)ds=$$
$$h^2\sum_{i=0}^{k}\bar{\beta}_i f_{n+i}+h^2\sum_{i=0}^{k}\bar{\gamma}_i f_{n+i+\nu_i},\ (|\nu_i|<1,\quad i=0,1,...,k). \tag{16}$$

By using these equalities in the asymptotic equality of (12) one can receive:

$$\sum_{i=0}^{k}\bar{\alpha}_i y_{n+i}=h^2\sum_{i=0}^{k}\bar{\beta}_i f_{n+i}+h^2\sum_{i=0}^{k}\bar{\gamma}_i f_{n+i+\nu_i},\ (|\nu_i|<1;\ i=0,1,...,k). \tag{17}$$

Note that for the calculation of definite integrals, one can use the method (17). In this case $\nu_i=0(i=0,1,\dots,k)$ follows from the formula (17) the known quadrature methods. In other cases, the method (17) receives the new method, the properties of which are depending on the values of coefficients of the formula (14) and from the values of. $\nu_i=0(i=0,1,\dots,k)$.

If applied the method of unknown coefficients is to determine the values of the coefficients which are participated in the formula (12), then receive methods which are usually called the finite-difference method. To determine the values of the coefficients, a nonlinear system of algebraic equations is obtained. For the construction of the named system, let us use the following Taylor series:

$$y(x+ih)=y(x)+ihy'(x)+\frac{(ih)^2}{2!}y''(x)+\cdots+\frac{(ih)^{p+1}}{(p+1)!}y^{p+1}(x)+O(h^{p+2}).$$

$$y''(x+ih)=y''(x)+ihy'''(x)+\frac{(ih)^2}{2!}y^{IV}(x)+..+\frac{(ih)^{p-1}}{(p-1)!}y^{p-1}(x)+O(h^p).$$

By using these equalities in the asymptotic equality of (12) one can receive:

$$\sum_{i=0}^{k}\bar{\alpha}_i y(x)+h\sum_{i=0}^{k}i\bar{\alpha}_i y'(x)+h^2\sum_{i=0}^{k}\left(\frac{i^2}{2!}\bar{\alpha}_\iota-\bar{\beta}_\iota\right)y''(x)+..+h^{p+1}\sum_{i=0}^{k}\left(\frac{i^{p+1}}{(p+1)!}\bar{\alpha}_\iota-\frac{i^{p-1}}{(p-1)!}\bar{\beta}_\iota\right)y^{p+1}(x)+O(h^{p+2})=O(h^{p+2}),h\to 0. \tag{18}$$

It follows from here that, if the method of (12) has the degree of p, then by the comparison of the asymptotic equalities and receive that the following must satisfy (see for example, [16], [17], [18], [19], [20], [21]):

$$\sum_{i=0}^{k}\left(\bar{\alpha}_\iota y(x)+i\mathrm{h}\,\bar{\alpha}_\iota y'(x)+h^2\left(\frac{i^2}{2!}\bar{\alpha}_1-\bar{\beta}_\iota\right)y''(x)+\cdots+h^{p+1}\left(\frac{i^{p+1}}{(p+1)!}\bar{\alpha}_\iota-\frac{i^{p-1}}{(p-1)}\bar{\beta}_\iota\right)y^{(p+1)}(x)\right)=0. \tag{19}$$

By taking into account that the systems $1,x,x^2,\dots,x^{p+1}$ or $y(x),y'(x),\dots,$

$y^{(p+1)}(x)\left(y^j(x)\not\equiv 0, j=0,1,\dots,p+1\right)$ are independent, receive that for satisfying the equality of (19) the following system must have the solution (see for example, [13], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30]) :

$$\sum_{i=0}^{k}\bar{\alpha}_i=0;\ \sum_{i=0}^{k} i\bar{\alpha}_i=0;\ \sum_{i=0}^{k}\left(\frac{i^j}{j!}\bar{\alpha}_i-\frac{i^{j-2}}{(j-2)!}\bar{\beta}_i\right)=0;\ j=2,3,\dots,p+1. \tag{20}$$

By finding the coefficients of method (12) we receive the linear system of algebraic equations. The error for method (12) can be estimated by the error of the quadrature methods. For the sake of objectivity, let us note that in the application of the method (12) to solving initial-value problems some errors.

From the stability of method (12), it follows that all the errors arising in using method (19) will be bounded. Therefore, let us investigate the system of (20). In this system, the amount of the unknowns is equal to $2k+2$, but the amount of the equations is equal to $p+2$. It is not difficult to prove that the linear system has a unique solution for the case $p=2k$. But this equality for the stable methods of type (11) may be written as: $p\le[2/k/2]+2$. And also, the constant k must satisfy the condition $k\ge 2$. This condition follows from equality (11). As a result, it is proved the following lemma:

Lemma. If method (12) has the degree of p, then its coefficients must satisfy the system of (20) and vice versa. If the coefficients of method (12) satisfy condition(20), then method (12) will have the degrees of p, which satisfies the condition $p\le[k/2]+2$ for the stable and the condition $p\le 2$k for other methods.

And now let us consider the investigation of the method (17). For the investigation method (17), here offer one way for finding the values of the coefficients $\bar{\alpha}_i,\bar{\beta}_i,\bar{\gamma}_i,\nu_i (i=0,1,2,\dots,k)$. For this aim, one can be used the above-presented Taylor series with the following:

$$y''(x+l_ih)=y''(x)+l_ihy'''(x)+\frac{l_i^2}{2!}h^2y^{IV}(x)+\dots+\frac{(l_ih)^{p-1}}{(p-1)!}y^{p+1}(x)+O(h^p),$$

here $l_i=i+\nu_i (i=0,1,\dots,k)$.

By repeating the above using description for the finding of the coefficients α_i, $\beta_i,\gamma_i,\nu_i (i=0,1,2,\dots,k)$, receive the following nonlinear system of algebraic equations:

$$\sum_{i=0}^{k}\bar{\alpha}_i=0;\ \sum_{i=0}^{k} i\bar{\alpha}_i=0; \tag{21}$$

$$\sum_{i=0}^{k}\left(\frac{i^j}{j!}\bar{\alpha}_i-\frac{i^{j-2}}{(j-2)!}\bar{\beta}_i-\frac{l_i^{j-2}}{(j-2)!}\bar{\gamma}_i\right)=0; j=2,3,\dots p+1.$$

And now consider the explanation of the condition $k\ge 2$. Note that by using the first two equations of system (20) or (21) receive that $\lambda=1$ is the double root of the polynomial$\rho(\lambda)$. Note that this condition is necessary for convergence of the method (17), therefore $\rho(\lambda)$ can be written as:

$$\rho(\lambda)=(\lambda-1)^2\left(\bar{\alpha}_{k-2}\lambda^{k-2}+\bar{\alpha}_{k-3}\lambda^{k-3}+\bar{\alpha}_1\lambda+\bar{\alpha}_0\right).$$

By taking this into account in the system of (21) receive that the amount of the unknowns equal to $4k+4$, but the amount of the equations equal to $p+2$ and the received system will be linear nonhomogeneous which will have a unique solution in case $p=2k$ and $\gamma_i=0 (i=0,1,\dots,k)$. From here it follows that $p\le 2k$, if $\gamma_i=0 (i=0,1,2,\dots,k)$.

System (21) is different from the system (20) as system (21) is nonlinear. Moreover, system (21) follows the system (20) in the case of $\gamma_i=0 (i=0,1,\dots.k)$.

Here, also by using the properties of the first two equations receive that the amount of the equation can be taken as $p+1$ and in this case, the homogeneous system (21) becomes the non-homogeneous system. From here it follows receive that the system (21) can have the solution by which will construct methods with the degree$p\le 4k+2$. And by the construction of the concrete stable methods with the degree $p=3k+3$, reserve that method (17) can have the degree $p=3k+3$. From here we obtain the following theorem.

Theorem: Method (17) has the degree of p, satisfying its coefficients $\bar{\alpha}_i\bar{\beta}_i,\bar{\gamma}_i,\bar{\nu}_i (i=0,1,2,\dots,k)$ the nonlinear system of (21) is necessary and sufficient. If method (17) is stable, then there are methods with the degree $p\le 3k+3$.

It is not difficult to prove that if there exist methods of type (5) with the degree $p\le 3k+1$. But if method (5) in this case has the degree p and is stable then the degree for method (5) satisfies the condition:: $p\le 2k+2$.

Method (17) coefficients the nonlinear system of (21) which is necessary and sufficient. If method (17) is stable, then there are methods with this degree.

By comparison of all the above-described advantages and disadvantages properties of the suggested methods, receive that the methods of type (17) have some advantages. Therefore, they can be taken as the perspective. Note that, [31], [32], [33], [34], [35], [36] obtained very interesting results to solve some practical problems. We would like to note that, [37], [38] obtained very investing results, which are related to the construction of linear and nonlinear models. Similar studies are found in recent bibliography, [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54]. In the following sections, we compare the results received here, with the existing results derived from the recent literature.

3 Numerical Results

To compare the results obtained here with the recent academic literature, let us consider defining the numerical solution of the following simple examples:

$$y'' = \lambda^2(1 + a - ay(x)) + (1 + a)\lambda^3 \int_0^x y(s)ds, \quad y(0) = 1, \quad y'(0) = \lambda, \quad 0 \le x \le 1, \tag{22}$$

the exact solution can be represented: $y(x) = \exp(\lambda x)$. By using that the right-hand side of the example does not depend on $y'(x)$, to find the solution of this example one can use the following method:

$$y_{n+2} = 2y_{n+1} - y_n + h^2(4y''_{n+1-\beta} + y''_{n+1} + 4y''_{n+1+\beta})/9, \ \ \beta = \sqrt{3}/4, \ p = 4, \tag{23}$$

Which was applied to solving our example in the case, when $a = 1$ and $a = -1$. To compare the results obtained, here have used the following method of Numerov:

$$y_{n+2} = 2y_{n+1} - y_n + h^2(y''_{n+1} + 10y''_{n+1} + y''_{n+2})/12, \ \ p = 4. \tag{24}$$

The results received by method (23) are presented in table 1. Similarly, results received by method (2 are presented in table 2.

Table 1. Results received by the method (23) for $a = 1; \ h = 0.01; \ \lambda = \pm 1; \pm 5; \pm 10$

Variable x	$\lambda = 1$	$\lambda = -1$	$\lambda = 5$	$\lambda = -5$	$\lambda = 10$	$\lambda = -10$
0.20	2.04 E-12	2.16 E-12	2.53 E-08	3.26 E-08	8.86 E-07	1.34 E-06
0.60	1.7E-11	2.05 E-11	3.17 E-08	1.17 E-08	1.79 E-04	6.22 E-06
1.00	4.1E-11	5.53 E-11	2.86 E-06	1.97 E-07	2.2E-02	1.84 E-05

Table 2. Results received by the method (24) for $a = -1; \ h = 0.01; \ \lambda = \pm 1; \pm 5; \pm 10$

Variable x	$\lambda = 1$	$\lambda = -1$	$\lambda = 5$	$\lambda = -5$	$\lambda = 10$	$\lambda = -10$
0.20	5.66 E-13	4.92 E-13	1.28 E-08	6.5E-09	1.49 E-06	4.16 E-07
0.60	6.23 E-12	4.17 E-12	4.31 E-07	8.13 E-08	3.04 E-04	2.52 E-05
1.00	2.12 E-11	1.10 E-11	5.76 E-06	6.11 E-07	2.88 E-02	1.37 E-03

Here we have used several methods for solving problems (1)-(2). The methods (23)-(24) have the same degree and are stable. Comparing the above results, we find that the hybrid methods are promising.

4 Remark

Method (17) can be taken as more effective than the others so the stable method of type (17) has the degree $p \le 3k + 3$. But in this case, $\overline{\beta}_i = 0$, the maximum value for the stable method of type (17) satisfies the condition $p \le 2k + 2$, which has the same as the degree for the stable method of type (5). This means that the stable method of type (5) will be implicit, but the stable method with the maximum degree of type (17) will be explicit. Consequently, methods of type (17) have some advantages. To further explain this statement, let us consider the following methods:

$$y_{n+2} = 2y_{n+1} - y_n + h^2(5y''_{n+1-\gamma} + 14y''_{n+1} + 5y''_{n+1+\gamma})/24, \ \gamma = \sqrt{10}/5, \tag{25}$$

$$y_{n+2} = 2y_{n+1} - y_n + h^2(19y''_{n+2} + 87y''_{n+1} + 19y''_n)/1740 + h^2(1323y''_{n+1-\alpha} + 58y''_{n+1} + 1323y_{n+1+\alpha})/5655, \ \ \alpha = \sqrt{13}/42. \tag{26}$$

By the simple comparison of these methods with the known, receive that, the hybrid methods can be compared with the Gauss method because they have some similarities. For example, they can be taken as symmetries. Note that hybrid methods are more exact than the Gauss method. Methods (25) and (26) are stable and have the $p = 6$ and $p = 8$, consecutively.

5 Conclusion

In this research article, we have investigated some of the generalizations of the Stërmer -Verlet method and considered the application of these to solve the initial-value problem for ODE of the second order with the special structure. From our research one can receive many known equations such as Schrōdinger, Sehturm-Liouville, and others. Moreover, it is a fact there are several classes of methods constructed for solving the initial value problem for ODEs of the second order. Analytically, one of the effective methods for solving the problem (1) is method (5), as it is more accurate than the others. Based on our findings, we have proven that one of the most efficient methods for solving problems (2) and (15) is the hybrid method of Stërmer -Verlet type, which can be received from (17) as the partial case.

By using the Dahlquist laws, method (5) is more exact than the others. But here, we have shown that the hybrid method is more exact than the method of (5). By taking into account this, here for solving these problems with a high order of accuracy, we suggested investigating hybrid methods, which are constructed by using the formula (17). In our research, we have proved that if the method defined by the formula (17) is stable and has a specific degree of p, then the degree will satisfy the necessary condition. Specifically, it was shown that the hybrid methods are more exact than the others, therefore these methods are perspective.

Furthermore, it comes naturally that one can prove that the stable hybrid methods with the maximum degree can be considered as optimal. Moreover, since these methods are stable, they have maximum accuracy and a simple structure. It is worth noticing that in the application of these methods, researchers usually use some of the existing methods for the calculation of the values of the solution at the hybrid points. In addition, we note that depending on the properties of the solution of the investigated problem, the methods with fractional step-size can give results, which are worse than the results obtained by the non-fractional methods. Consenquately, it must be considered that it is not difficult to obtain methods with the fractional step size that are also a part of the class of hybrid methods. This means that the results of this paper have a significant advantage in solving many applied physical problems including the above-mentioned problems of Schrölinger, Sturm-Liouville, and others.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.