Perturbation of multiagent linear systems

M. I. GARCÍA-PLANAS
Universitat Politècnica de Catalunya
Departament de Matemàtica Aplicada
Mineria 1, 08038 Barcelona
SPAIN

Abstract: The main objective of this note is to explore if, making a small perturbation of an uncontrollable multi-agent linear system with a previously interrelationship topology established, a controllable multi-agent system with the same topology can be obtained. Arnold geometric techniques will be used for the objective, and versal deformations will be constructed in the set of equivalent systems.

Key–Words: Multiagent linear systems, deformation, controllability.

Received: May 13, 2022. Revised: January 23, 2023. Accepted: February 20, 2023. Published: March 7, 2023.

1 Introduction

Recently, the study of multiagent systems has attracted the attention of many researchers because this class of systems appears in various areas of knowledge, such as the cooperative control of unmanned aerial vehicles, the consensus problem of communication networks, the training control of mobile robots, neural networks modeling the brain structure, Etc., [5], [9], [10].

An interesting technique for analyzing perturbations and investigating complicated objects such as singularities and bifurcations in multiparametric dynamical multi-systems is the construction of versal deformations since they provide a particular parametrization of families of multi-systems.

Versal deformation permits speaking of generic families relative to a generic property of interest as, for example, controllability. In this case, generic families with controllable members have the property that such members remain even when the family is perturbed. The generic property permits us that a small perturbation may eliminate all the non-controllable cases. In order to construct versal deformation, one defines an equivalence relation in the space of multi-agent systems preserving controllability character.

The knowledge of a versal deformation provides a particular parametrization of the space of multi-agent systems in a neighborhood of a fixed point, which can be effectively applied to the perturbation analysis of this point.

(For more information about versal deformations, see [1], [2], [3]).

2 Preliminaries

Let us consider a group of k agents. The following linear dynamical systems give the dynamic of each agent

$$\left.\begin{array}{ll}\dot{x}^1(t) & = A_1x^1(t)+B_1u^1(t)+C_1v^1(t)\\ \quad\vdots & \\ \dot{x}^k(t) & = A_kx^k(t)+B_ku^k(t)+C_1v^k(t)\end{array}\right\}\quad(1)$$

$A_i\in M_n(\mathbb{C})$, $B_i\in M_{n\times m}(\mathbb{C})$, $C_i\in M_{n\times p}(\mathbb{C})$, $x^i(t)\in\mathbb{C}^n$, $u^i(t)=f^i(x^1(t),\dots,x^k(t)\in\mathbb{C}^m$, $v^i(t)\in\mathbb{C}^p$, $1\leq i\leq k$.

Sometimes, the considered internal controls u^i are given by means a communication topology defined by an undirected graph with

i) Vertex set: $V=\{1,\dots,k\}$

ii) Edge set: $E=\{i,j)\mid i,j\in\subset V\times V$

iii) Neighbor of i: $\mathcal{N}_i=\{j\in\mid (i,j)\in E\}$. (In the case where $j=i$ the edge is called self-loop).

defining the communication topology among agents:

$$u^i(t)=\sum_{j\in\mathcal{N}_i}(x^i(t)-x^j(t)),\quad 1\leq i\leq k$$

for some $K_i\in M_{m\times n}(\mathbb{C})$, $1\leq i\leq k$

Writing $\mathbb{X}=(x^i,\dots,x^k)^t$, $\mathbb{U}=(u^i,\dots,u^k)^t$ and $\mathbb{V}=(v^i,\dots,v^k)^t$, and considering the Laplacian matrix associated with the graph that is defined in the following manner

$$\mathcal{L}=(l_{ij})=\begin{cases}|\mathcal{N}_i| & \text{if } i=j\\ -1 & \text{if } j\in\mathcal{N}_i\\ 0 & \text{otherwise}\end{cases}\quad(2)$$

To simplify notations, we will write the systems in matrix language:

$$\begin{array}{ll}\mathbb{A} & = \operatorname{diag}(A_1,\dots,A_k)\\ & \in M_n(\mathbb{C})\times \overset{k}{\dots}\times M_n(\mathbb{C}) = \mathfrak{M}_1\\ \mathbb{B} & = \operatorname{diag}(B_1,\dots,B_k)\\ & \in M_{n\times m}(\mathbb{C})\times \overset{k}{\dots}\times M_{n\times m}(\mathbb{C}) = \mathfrak{M}_2\\ \mathbb{C} & = \operatorname{diag}(C_1,\dots,C_k)\\ & \in M_{n\times p}(\mathbb{C})\times \overset{k}{\dots}\times M_{n\times p}(\mathbb{C}) = \mathfrak{M}_3\end{array}$$

and $\mathbb{X} = (x^i,\dots,x^k)^t$, $\mathbb{U} = (\mathcal{L}\otimes I_n)\mathbb{X} = (u^i,\dots,u^k)^t$ and $\mathbb{V} = (v^i,\dots,v^k)^t$, and in the case where the communication topology for internal control is considered, the control is written as $\mathbb{U} = (\mathcal{L}\otimes I_n)\mathbb{X}$,

Finally. we will call by $\mathfrak{M}$ the set $\mathfrak{M} = \mathfrak{M}_1\times\mathfrak{M}_2\times\mathfrak{M}_3$. This set is clearly, a differentiable manifold.

3 Equivalence relation for differentiable families of multi-systems

In the set $\mathfrak{M}$ we consider the following equivalence relation

Definition 1 *Two systems* $(\mathbb{A}_1,\mathbb{B}_1,\mathbb{C}_1)$ *and* $(\mathbb{A}_2,\mathbb{B}_2,\mathbb{C}_2)$ *in* $\mathfrak{M}$ *are said equivalent if and only if, there exist* $\mathbb{P} = \operatorname{diag}(P_1,\dots,P_k)\in Gl(n;\mathbb{C})\times\overset{k}{\dots}\times Gl(n;\mathbb{C})$, $\mathbb{Q} = \operatorname{diag}(Q_1,\dots,Q_k)\in Gl(m;\mathbb{C})\times\overset{k}{\dots}\times Gl(m;\mathbb{C})$, $\mathbb{R} = \operatorname{diag}(R_1,\dots,R_k)\in Gl(p;\mathbb{C})\times\overset{k}{\dots}\times Gl(p;\mathbb{C})$, $\mathbb{K} = \operatorname{diag}(K_1,\dots,K_k)\in M_{m\times n}(\mathbb{C})\times\overset{k}{\dots}\times M_{m\times n}(\mathbb{C})$ *and* $\mathbb{F} = \operatorname{diag}(F_1,\dots,F_k)\in M_{p\times n}(\mathbb{C})\times\overset{k}{\dots}\times M_{p\times n}(\mathbb{C})$*, such that*

$$\begin{array}{l}(\mathbb{A}_2,\mathbb{B}_2,\mathbb{C}_2) =\\ (\mathbb{P}^{-1}\mathbb{A}_1\mathbb{P}+\mathbb{P}^{-1}\mathbb{B}_1\mathbb{K}+\mathbb{P}^{-1}\mathbb{C}_1\mathbb{F},\mathbb{P}^{-1}\mathbb{B}\mathbb{Q},\mathbb{P}^{-1}\mathbb{C}\mathbb{R})\end{array}$$

This equivalence relation can be seen as the action by a Lie group in the following manner:

Let $\mathcal{G} = \{(\mathbb{P},\mathbb{Q},\mathbb{R},\mathbb{K},\mathbb{F}) \in Gl(n,\mathbb{C})\times Gl(m;\mathbb{C})\times Gl(p;\mathbb{C})\times M_{m\times n}(\mathbb{C})\times M_{p\times n}(\mathbb{C})$. It is a Lie group to the usual product of matrices in the form:

$$\begin{pmatrix}\mathbb{P}_1 & 0 & 0\\ \mathbb{K}_1 & \mathbb{Q}_1 & 0\\ \mathbb{F}_1 & 0 & \mathbb{R}_1\end{pmatrix}\cdot\begin{pmatrix}\mathbb{P}_2 & 0 & 0\\ \mathbb{K}_2 & \mathbb{Q}_2 & 0\\ \mathbb{F}_2 & 0 & \mathbb{R}\end{pmatrix} = \begin{pmatrix}\mathbb{P}_1\mathbb{P}_2 & 0 & 0\\ \mathbb{K}_1\mathbb{P}_2+\mathbb{Q}_1\mathbb{K}_2 & \mathbb{Q}_1\mathbb{Q}_2 & 0\\ \mathbb{F}_1\mathbb{P}_2+\mathbb{R}_1\mathbf{F}_2 & 0 & \mathbb{R}_1\mathbb{R}_2\end{pmatrix}$$

and

$$\begin{pmatrix}\mathbb{P}_1 & 0 & 0\\ \mathbb{K}_1 & \mathbb{Q}_1 & 0\\ \mathbb{F}_1 & 0 & \mathbb{R}_1\end{pmatrix}^{-1} = \begin{pmatrix}\mathbb{P}_1^{-1} & 0 & 0\\ -\mathbb{Q}_1^{-1}\mathbb{K}_1P_1^{-1} & \mathbb{Q}_1^{-1} & 0\\ -\mathbb{R}_1^{-1}\mathbb{F}_1\mathbb{P}_1^{-1} & 0 & \mathbb{R}_1^{-1}\end{pmatrix}$$

$\mathcal{G}$ acts over $\mathfrak{M}$ in the following manner

Calling $G = (P,Q,R,K,F)$ and $M = (\mathbb{A},\mathbb{B},\mathbb{C})\in\mathfrak{M}$,

$$\begin{array}{ll}\varphi:\mathcal{G}\times\mathfrak{M} & \longrightarrow\mathfrak{M}\\ (G,M) & \longrightarrow\varphi(G,M) = \bar{M}\end{array}\tag{3}$$

Where $\bar{M} = (\mathbb{P}^{-1}\mathbb{A}\mathbb{P}+\mathbb{P}^{-1}\mathbb{B}\mathbb{K}+\mathbb{P}^{-1}\mathbb{C}\mathbb{F},\mathbb{P}^{-1}\mathbb{B}\mathbb{Q},\mathbb{P}^{-1}\mathbb{C}\mathbb{R})$

φ is differentiable and surjective.

Fixing $M^0 = (\mathbb{A}^0,\mathbb{B}^0,\mathbb{C}^0)\in\mathfrak{M}$ we have the differentiable map

$$\begin{array}{ll}\varphi_{M^0}:\mathcal{G} & \longrightarrow\mathfrak{M}\\ G & \longrightarrow\varphi(G,M^0)\end{array}\tag{4}$$

The image of this map $\operatorname{Im}\varphi_{M^0}$ is the set of equivalent systems to M^0 and it is called orbit of M^0 and it is denoted by $\mathcal{O}(X_0)$. On the other hand, the subset of $\mathcal{G}$ leaving invariant the system M^0, $\{G\in\mathcal{G}\mid\varphi_{M^0}(G) = M^0\}$ is called the stabilizer of M^0.

The differentiability of the action allows a local study of the orbit and the stabilizer computing the differential of φ_{M^0}.

Lemma 2 *The differential* $d\varphi_{M^0,\mathbb{I}}: T_{\mathbb{I}}\mathcal{G}\longrightarrow\mathfrak{M}$ *at the identity point* $\mathbb{I}\in\mathcal{G}$*, on any element* $G\in T_{\mathbb{I}}\mathcal{G}$*, is given by*

$$\begin{array}{l}d\varphi_{M^0,\mathbb{I}}(G)) =\\ ([\mathbb{A},\mathbb{P}]+\mathbb{B}\mathbb{K}+\mathbb{C}\mathbb{F},\mathbb{B}\mathbb{Q}-\mathbb{P}\mathbb{B},\mathbb{C}\mathbb{R}-\mathbb{P}\mathbb{C})\end{array}$$

Remark 3 $T_{\mathbb{I}}\mathcal{G} = \{(\mathbb{P},\mathbb{Q},\mathbb{R},\mathbb{K},\mathbb{F})\in M_n(\mathbb{C})\times M_m(\mathbb{C})\times M_p(\mathbb{C})\times M_{m\times n}(\mathbb{C})\times M_{p\times n}(\mathbb{C})$. *and* $T_{M^0}\mathfrak{M} = \mathfrak{M}$.

Proof:

It suffices to compute the linear approximation of the map on the identity.

$$\begin{array}{l}\varphi_{M^0}(\mathbb{I}+\varepsilon G) =\\ M^0+\varepsilon(([\mathbb{A},\mathbb{P}]+\mathbb{B}\mathbb{K}+\mathbb{C}\mathbb{F},\mathbb{B}\mathbb{Q}-\mathbb{P}\mathbb{B},\mathbb{C}\mathbb{R}-\mathbb{P}\mathbb{C}))\\ \qquad +\varepsilon^2\dots\end{array}$$

$\square$

4 Versal deformations

There is quite a lot of literature in which it can find the definition of deformation and versal deformation; in this case, we take the one found in [6]

Definition 4 *A deformation of an element $M^0 \in \mathfrak{M}$ is a family of elements of $\mathfrak{M}$ indexed by $\lambda \in \Lambda$ $\varphi : \Lambda \longrightarrow \mathfrak{M}$ where $\Lambda \subset \mathbb{C}^m$ is a neighborhood of 0, and where $\varphi(0) = M^0$ and φ depends smoothly on the parameters.*

Definition 5 *A deformation $\varphi(\lambda) = \varphi(\lambda_1, \dots, \lambda_m)$ of M^0 is versal if and only if for any deformation $\varphi'(\mu_1, \dots, \varphi_k) \in \mathfrak{M}$ of M^0, $\varphi'(\mu)$ is induced by $\varphi(\lambda)$, that is to say, there exists a neighborhood V of 0 in $\mathbb{C}^k$, a map $\psi : V \longrightarrow \mathbb{C}^m$ with $\psi(0) = 0$, and a map $g : V \longrightarrow \mathcal{G}$ with $g(0) = I$ such that $\forall \mu \in V$, $\varphi'(\mu) = g(\mu)\varphi(\psi(\mu))g^{-1}(\mu)$ with ψ and g holomorphic (smooth).*

It is obvious that if we have a versal deformation of an element, automatically we have a versal deformation of any element that is equivalent to it, since if $M = \varphi(G, M^0)$ is an equivalent element of M^0 and $\varphi(\lambda)$ is a versal deformation of M^0 then $\varphi(G^{-1}, \varphi(\lambda))$ is a versal deformation of M'.

A versal deformation having a minimal number of parameters is called *miniversal*.

4.1 Transversality

The versatility condition admits a useful geometric characterization in terms of transversality. We begin, then, by recalling the notion of transversality.

Definition 6 *Let $S \subset W$ be a differentiable submanifold of a manifold W. Consider a differentiable map $\psi\Lambda \longrightarrow W$, of another manifold S on W. Let $\lambda \in \Lambda$ such that $\psi(\lambda) \in S$*

It is said that the map ψ is transversal to S in λ, if the tangent space to W, in $\psi(\lambda)$ decomposes in the way:

$$T_{\psi(\lambda)}W = \operatorname{Im} d\psi_\lambda + T_{\psi(\lambda)}S.$$

It is called mini transversal if said sum is direct.

Transversality allows obtaining local trivializations along the orbits:

Proposition 7 *Let $\psi : \Lambda \longrightarrow \mathfrak{M}$ be a deformation of M^0 minitransversal to the orbit $O(M^0)$ in 0, and $G_1 \subset G$ a submanifold minitransversal to the stabilizer of M^0 in $\mathbb{I}$. then the application*

$$\beta : \Lambda \times G_1 \longrightarrow \mathfrak{M}$$

defined by $\beta(\lambda, G) = \varphi(G, M^0(\lambda)$ is a local diffeomorphism at $(0, \mathbb{I})$.

Proof: It suffices to apply the inverse function theorem and to prove that $d\beta$ is exhaustive at $(0, \mathbb{I})$. □

The following result was proved by Arnold [1], in the case where $\mathrm{Gl}(n;\mathbb{C})$ acts on $M_n(\mathbb{C})$, remarking that what was important was the action of the Lie group on the variety, thus generalizing the result and providing the relationship between a versal deformation of M^0 and the local structure of the orbit.

Theorem 8 ([1]) *1. A deformation $\varphi(\lambda)$ of (M^0) is versal if and only if it is transversal to the orbit $\mathcal{O}(M^0)$ at (M^0).*

2. The minimal number of parameters of a versal deformation is equal to the codimension of the orbit of M^0 in $\mathfrak{M}$, $\ell = \operatorname{codim} \mathcal{O}(M^0)$.

Corollary 9 *Then $\varphi(\lambda) = M^0 + (T_{M^0}\mathcal{O}(M^0))^\perp$ for some scalar product is a miniversal deformation.*

Let $\{v_1, \dots, v_\ell\}$ be a basis of any arbitrary complementary subspace $(T_{M^0}\mathcal{O}(M^0)^c$ to $T_{M^0}\mathcal{O}(M^0)$.

Corollary 10 *The deformation*

$$\varphi : \Lambda \subset \mathbb{C}^\ell \longrightarrow M, \quad \varphi(\lambda) = M^0 + \sum_{i=1}^{\ell} \lambda_i v_i$$

is a miniversal deformation.

The versatility condition admits a useful geometric characterization in terms of transversality. We begin, defining scalar products in $\mathfrak{M}$ and $T_{\mathbb{I}}\mathcal{G}$, we can consider the adjoint application of $d\varphi_{M^0}$.

The Euclidean scalar product considered in this paper is defined as follows:

For all $M^i = (\mathbb{A}_i, \mathbb{B}_i, \mathbb{C}_i) \in \mathfrak{M}$

$$\langle M^1, M^2 \rangle_1 = \operatorname{trace}(\mathbb{A}_1\mathbb{A}_2^*) + \operatorname{trace}(\mathbb{B}_1\mathbb{B}_2^*) + \operatorname{trace}(\mathbb{C}_1\mathbb{C}_2^*), \tag{5}$$

where $\mathbb{A}^*$ denotes the conjugate transpose of a matrix $\mathbb{A}$.

Theorem 11 *The normal complementary subspace to tangent space to the orbit of the system M^0 is defined by the set of elements $(\mathbb{X}, \mathbb{Y}, \mathbb{Z}) \in \mathfrak{M}$ verifying*

$$\left.\begin{aligned} [\mathbb{X}^*, \mathbb{A}] - \mathbb{B}\mathbb{Y}^* - \mathbb{C}\mathbb{Z}^* &= 0 \\ \mathbb{X}^*\mathbb{B} &= 0 \\ \mathbb{X}^*\mathbb{C} &= 0 \\ \mathbb{Y}^*\mathbb{B} &= 0 \\ \mathbb{Z}^*\mathbb{C} &= 0 \end{aligned}\right\}$$

Proof:

It suffices to observe that:

$$\langle((([\mathbb{A},\mathbb{P}]+\mathbb{B}\mathbb{K}+\mathbb{C}\mathbb{F},\mathbb{B}\mathbb{Q}-\mathbb{P}\mathbb{B},\mathbb{C}\mathbb{R}-\mathbb{P}\mathbb{C}),(\mathbb{X},\mathbb{Y},\mathbb{Z}))\rangle=0$$

for any $(\mathbb{P},\mathbb{Q},\mathbb{R},\mathbb{K},\mathbb{F})\in T_{\mathbb{I}}\mathcal{G}$, if and only if

$$\text{trace}\begin{pmatrix}[\mathbb{X}^*,\mathbb{A}]-\mathbb{B}\mathbb{Y}^*-\mathbb{C}Z^* & \mathbb{X}^*\mathbb{B} & \mathbb{X}^*\mathbb{C}\\ 0 & \mathbb{Y}^*\mathbb{B} & 0\\ 0 & 0 & \mathbb{Z}^*\mathbb{C}\end{pmatrix}\cdot\begin{pmatrix}\mathbb{P} & \star & \star\\ \mathbb{K} & \mathbb{Q} & \star\\ \mathbb{F} & \star & \mathbb{R}\end{pmatrix}=0$$

□

After this theorem, it is easy to compute these spaces.

5 Controllability

The importance of the qualitative property of dynamic systems in the control theory, known as controllability, is well known.

The controllability concept involves taking the system from any initial state to any final state in finite time, regardless of the path or input. Let us consider the multi-agent system 1

It is important to emphasize that various definitions of controllability are derived, depending to a large extent on the class of dynamic systems and the form of allowable controls, [7].

In our particular setup, the controllability character can be described as

$$\text{rank}\begin{pmatrix}\mathbb{A}-\lambda I_{n\times k} & \mathbb{B} & \mathbb{C}\end{pmatrix}=nk$$

Proposition 12 *The controllability character is invariant under the equivalence relation considered.*

Proof:

$$\text{rank}\begin{pmatrix}\mathbb{A}-\lambda I_{n\times k} & \mathbb{B} & \mathbb{C}\end{pmatrix}=$$
$$\text{rank}\,\mathbb{P}^{-1}\begin{pmatrix}\mathbb{A}-\lambda I_{n\times k} & \mathbb{B} & \mathbb{C}\end{pmatrix}\begin{pmatrix}\mathbb{P} & & \\ \mathbb{K} & \mathbb{Q} & \\ \mathbb{F} & & \mathbb{R}\end{pmatrix}$$

□

For a fixed $\mathbb{B}$-feedback $\mathbb{K}$ and the fixed topology comunication, the system 1 can be written as

$$\dot{\mathbb{X}}(t)=(\mathbb{A}+\mathbb{B}\mathbb{K}(\mathcal{L}\otimes I_n))\mathbb{X}(t)+\mathbb{C}\mathbb{V}(t). \qquad (6)$$

(See [8] for Kronecker product properties).

The controllability of the system can be analyzed by computing the rank of the controllability matrix:

$$\begin{matrix}(\mathbb{C} & (\mathbb{A}+\mathbb{B}\mathbb{K})(\mathcal{L}\otimes I_n))\mathbb{C})\\ \dots & (\mathbb{A}+\mathbb{B}\mathbb{K})(\mathcal{L}\otimes I_n))^{nk-1}\mathbb{C})\end{matrix}$$

The rank of this matrix is invariant under feedback, that is to say

Proposition 13 *The matrix controllability of the system 1 is invariant under external feedback*

Proof:

$$\begin{aligned}&\text{rank}\,(\mathbb{C}\ (\mathbb{A}+\mathbb{B}\mathbb{K})(\mathcal{L}\otimes I_n)+\mathbb{C}\mathbb{F})\mathbb{C})\quad\dots\\ &(\mathbb{A}+\mathbb{B}\mathbb{K})(\mathcal{L}\otimes I_n)+\mathbb{C}\mathbb{F})^{nk-1}\mathbb{C})\\ &=\text{rank}\,(\mathbb{C}\ (\mathbb{A}+\mathbb{B}\mathbb{K})(\mathcal{L}\otimes I_n))\mathbb{C})\quad\dots\\ &(\mathbb{A}+\mathbb{B}\mathbb{K})(\mathcal{L}\otimes I_n))^{nk-1}\mathbb{C})\\ &\cdot\begin{pmatrix}I & \mathbb{F}\mathbb{B} & & \dots\\ & I & \mathbb{F}\mathbb{B} & \dots\\ & \ddots & \ddots & \end{pmatrix}\end{aligned}$$

□

We are going to carry out the study for a particular case in which all the systems have the same dynamics, that is, $A_i=A$, $B_i=B$, $C_i=C$ and $K_i=K$ for all $1\leq i\leq k$; and the graph defining the topology relating to the systems is undirected and connected. For being un undirected graph the matrix $\mathcal{L}$ is symmetric, then there exist an orthogonal matrix P such that $P\mathcal{L}P^t=\mathcal{D}$, and the connection ensures that 0 is a simple eigenvalue of $\mathcal{L}$.

Proposition 14 *Under these conditions, the system can be described as*

$$\dot{\mathbb{X}}(t)=\\((I_k\otimes A)+(I_k\otimes BK)(\mathcal{L}\otimes I_n))\mathbb{X}(t)+(I_n\otimes C)\mathbb{V}(t) \qquad (7)$$

In our particular setup, we have that there exists an orthogonal matrix $P\in Gl(k,\mathbb{R})$ such that $P\mathcal{L}P^t=\mathcal{D}=\text{diag}\,(\lambda_1,\dots,\lambda_k)$, $(\lambda_1\geq\dots\geq\lambda_{k-1}>\lambda_k=0)$.

Corollary 15 *The system can be described in terms of the matrices A, B, C the feedback K and the eigenvalues of $\mathcal{L}$.*

Proof:

Following the properties of Kronecker product, we have that.

$$\begin{array}{l}(P\otimes I_n)(I_k\otimes A)(P^t\otimes I_n)=(I_k\otimes A)\\ (P\otimes I_n)(I_k\otimes BK)(P^t\otimes I_n)=\\ (I_k\otimes BK)\\ (P\otimes I_n)(I_k\otimes C)(P^t\otimes I_k)=(I_k\otimes C)\\ (P\otimes I_n)(\mathcal{L}\otimes I_n)(P^t\otimes I_n)=(\mathcal{D}\otimes I_n)\end{array}$$

and calling $\widehat{\mathbb{X}}=(P\otimes I_n)\mathbb{X}$, and $\widehat{\mathbb{V}}=(P\otimes I_k)\mathbb{V}$ we have

$$\dot{\widehat{\mathbb{X}}}=((I_k\otimes A)+(I_k\otimes BK)(\mathcal{D}\otimes I_n))\widehat{\mathbb{X}}+(I_k\otimes C)\widehat{\mathbb{V}}.$$

Using this description, the analysis of controllability is easier.

Proposition 16 *The system* (7) *is controllable if and only if the systems* $(A+\lambda_i BK, C)$ *are controllable for each* $1\leq i\leq k$.

5.1 Perturbation

The controllability character is generic in multi-agent systems' space $\mathfrak{M}$.

Proposition 17 *The subset* $\mathfrak{C}\subset\mathfrak{M}$ *of the controllable multi-agent systems is an open and dense set in the space* $\mathfrak{M}$ *of multi-agent systems.*

Proof:

Taking into account that if a nk-order minor is non-zero, it is in the neighborhood, we conclude that all small perturbations of this minor are non-zero, and in particular for all perturbed minors corresponding to a perturbed system, so the set $\mathfrak{C}$ is open in $\mathfrak{M}$, and for density it suffices to take into account the fact that the function rank : $\mathbb{C}^{m\times n}\longrightarrow\mathbb{R}$ is lower semicontinuous in the space of rectangular matrices of size $r\times s$ for all pair of non-zero positive numbers r and s. □

Then, the set of controllable systems is the union of orbits of controllable systems. Each non-controllable system is located on the frontier of one of these orbits.

Proposition 18 *For each non-controllable system, there exists a neighborhood of this system containing controllable systems. These controllable systems can be described using the miniversal deformation of the given system.*

For the case of the non-controllable systems 6, and in order to preserve the fixed feedback $\mathbb{K}$, the equivalence relation is reduced to external feedback in the following sense

Definition 19 *Two systems* $\dot{\mathbb{X}}(t)=(\mathbb{A}+\mathbb{B}\mathbb{K}(\mathcal{L}\otimes I_n))\mathbb{X}(t)+\mathbb{C}\mathbb{V}(t)$ *and* $\dot{\mathbb{X}}(t)=(\mathbb{A}_1+\mathbb{B}\mathbb{K}(\mathcal{L}\otimes I_n))\mathbb{X}(t)+\mathbb{C}_1\mathbb{V}(t)$ *are equivalent if and only if*

$$\mathbb{A}_1=\mathbb{A}+\mathbb{C}\mathbb{F},\ \mathbb{C}_1=\mathbb{C}\mathbb{R}$$

for some $\mathbb{F}\in\prod_k M_{p\times n}(\mathbb{C})$ *and* $\mathbb{R}\in\prod_k Gl(p,\mathbb{C})$

This relation can be given in a matritial expression:

$$\begin{pmatrix}\mathbb{A}_1 & 0 & \mathbb{C}_1\end{pmatrix}=\begin{pmatrix}\mathbb{A} & 0 & \mathbb{C}\end{pmatrix}\begin{pmatrix}I & & \\ & I & \\ \mathbb{F} & & \mathbb{R}\end{pmatrix}$$

where $\begin{pmatrix}I & & \\ & I & \\ \mathbb{F} & & \mathbb{R}\end{pmatrix}\in\mathcal{G}_1\subset\mathcal{G}$

$\mathcal{G}_1$ has the structure group with the same operation as $\mathcal{G}$.

Proposition 20 *For a non-controllable system of type 6, If in a neighborhood of it, there exists a controllable one it can be found in* $(\mathbb{A}+\mathbb{X},0,\mathbb{C}+\mathbb{Z})$ *with* $(\mathbb{X},0,\mathbb{Z})$ *in a neighborhood of* $0\in\mathfrak{M}$ *verifying that* $\mathbb{X}^*\mathbb{C}=0$ *and* $\mathbb{Z}^*\mathbb{C}=0$.

Proof: It suffices to consider the miniversal deformation restricted to this case.

□

6 Conclusion

In this work, we have explored whether, with a small perturbation of a non-controllable multi-agent linear system with a previously established interrelationship topology, we can obtain a controllable multi-agent system with the same topology. For this, we have used geometric techniques defining transversal families to the set of equivalent systems under a previously defined equivalence relation that preserves controllability.

References:

[1] V.I. Arnold. *On matrices depending on parameters*, Russian Math. Surveys, **26**, (2), pp.29-43, (1971).

[2] V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko. *Versal deformations*. In: Singularities of Differentiable Maps. Monographs in Mathematics, **82**, pp.145-156, (1985). Birkhäuser Boston.

[3] García Planas, M. I. Sensivity and stability of singular systems under Sensivity and stability of singular systems under proportional and derivative feedback. WSEAS transactions on mathematics, **8**, (11), pp. 635-644, (2009).

[4] M.I. Garcia-Planas. *Control properties of multi-agent dynamical systems modelling brain neural networks.* In 2020 International Conference on Mathematics and Computers in Science and Engineering (MACISE) (pp. 106-113). IEEE. (2020, January).

[5] M.I. Garcia-Planas, M.V. Garcia-Camba. *Controllability of brain neural networks in learning disorders—a geometric approach*. Mathematics. **10**, (3), pp. 331:1-331:13, (2022). doi.org/10.3390/math10030331

[6] M.I. Garcia-Planas, T. Klymchuk. *Differentiable families of traceless matrix triples*. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas. **114**, (11), pp. 1 - 8. (2019). doi.org/10.1007/s13398-019-00754-w

[7] J. Klamka. *Controllability of dynamical systems. A survey*. Bulletin of the Polish Academy of Sciences: Technical Sciences, **61**, (2), pp. 335–342, (2013).

[8] P. Lancaster, M. Tismenetsky, *The Thoery of Matrices*. Academic–Press. San Diego, 1985.

[9] J. Wang, D. Cheng, X. Hu. (2008). *Consensus of multi-agent linear dynamic systems*. Asian Journal of Control, **10**, (2), pp. 144-155, (2008).

[10] G. Xie, L. Wang. *Consensus control for a class of networks of dynamic agents: switching topology*, Proc.2006 Amer.Contro.Conf.,pp.13821387,(2007).

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.