# Laboratory Student Workshop: Experience with Darboux Mechanism

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*Abstract:* - One of the mechanisms from the museum of History of Saint-Petersburg State University is being analysed. This mechanism is named after the French mathematician and mechanician, supposedly the creator of this design, Jean Gaston Darboux. There is little description of such a mechanism in the literature on the theory of machines and mechanisms. Its properties need to be studied. At the laboratory workshop, simple formulas are found for the implementation of the animation of the movement of this mechanism. This model for the mechanism of the 19 century is created during the student workshop.

*Key-Words:* - Laboratory Student Workshop**,** Darboux mechanism, system Maple, creation of model for the mechanism of 19 century, bifurcation points, theory of machines and mechanisms.

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## 1 Introduction

From the second half of the 19th century, in Russian universities appeared places called “cabinets” where the textbooks and different types of models, instruments and mechanisms were stored. At the St. Petersburg Imperial University there was a Cabinet of Practical Mechanics (or Applied Mechanics), later it was called a Mechanical Cabinet. This collection has been forgotten. Articles about this cabinet [1-3] and descriptions in books have now begun to be published. In 2018, St. Petersburg University published a richly illustrated book-album [4], which contains modern photographs of preserved mechanisms. Similar cabinets existed in other countries. In 1894 - 95, on the instructions of the Cabinet of Practical Mechanics of St. Petersburg University, Russian scientist I.V. Meshchersky visited some higher educational institutions in Europe (Italy, France, Switzerland and Germany). During the trip, he also visited auxiliary rooms with instruments, mechanisms and various manuals in mathematics and mechanics and reported after the trip [5]. These devices, mechanisms and visual aids have survived to our times in different countries, there are catalogs on the websites of some universities. One of the rich historical collections on the Internet in various fields of mathematics and mechanics is the collection of the University of Göttingen (Gottingen collection of mathematical models and instruments: http://modellsammlung.uni-goettingen.de/). Other collections are listed, for example, on the website of the Kharkiv National University V.N. Karazin http://geometry.karazin.ua/ru/geometric-models-collection.html or in the article [1].

Since about 2012, the author has been conducting an educational laboratory-computational workshop “Kinematic Analysis of Mechanisms” [1], in which the objects of study are historical models. Some mechanisms of the 19th century are convenient to use for computer implementations in laboratory workshops. The author showed some of these implementations at a seminar on the history of mathematics, for example, at the end of 2017 the implementation of the Darboux mechanism for rectilinear motion by student T. Shugailo (portal http://www.mathnet.ru/php/seminars.phtml?option_lang=rus&presentid=18589). In this article, the authors will present their work.

## 2 Problem Formulation

Recently in the archives of the Museum of History of St. Petersburg State University, a model of the device in figure 1 was found. It registered as one of Somov's posters [6]. For reference, Osip (Joseph) Ivanovich Somov (1815 -1876) is Russian mathematician and mechanics, academician of the Imperial St. Petersburg Academy of Sciences.

The system in figure 1 is named after the French mathematician and mechanics, supposedly the creator of this design, Jean Gaston Darboux (1842 – 1917).

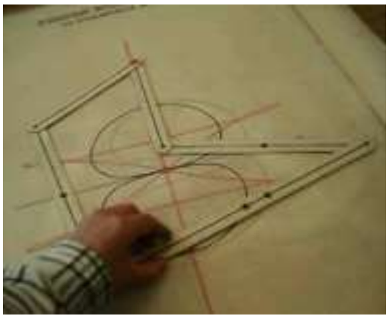

Fig. 1: Model of the Darboux mechanism from the Museum of History of Saint-Petersburg State University

The task for the 5th year student T. Shugailo was to describe the mathematical model of this mechanism and build an animation of its movement. In this paper some details of the mechanism kinematics will be done from his work.

## 2.1 Description of the Mechanism

A mechanical system is considered, consisting of 5 successive, cyclically, pivotally connected links, and two links have an additional one attached hinge. There are 5 links, let's designate them GE, EC, CD, DF and FG (Fig. 2).

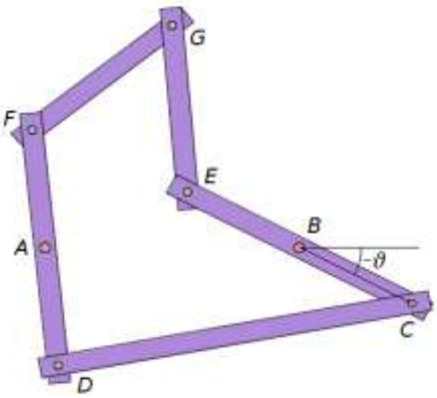


Fig. 2: Scheme of the mechanism

Points A and B correspond to fixed hinges. So, in total it will be m=5 movable links, n=5 movable hinges and v=2 fixed hinges. According to Chebyshev's formula 3m-2(n+v) we get that this mechanism has one degree of freedom. Let us take as a generalized coordinate ϑ - the angle of deviation of the rib BC from the horizontal axis.

Let us now determine the lengths of all edges. First of all, studying the model presented in the museum, we note that AD=AF=BC=BE=1/2 AB and GE=FG (Fig. 3, 4 and 5). The lengths of the links were measured by us.

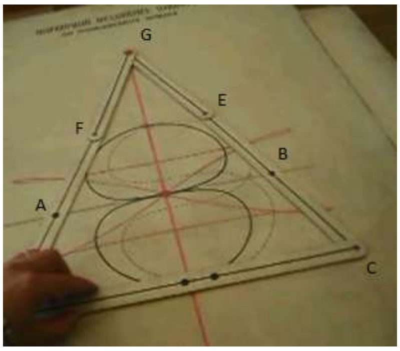


Fig. 3: One of the position for Darboux mechanism: all links form an isosceles triangle

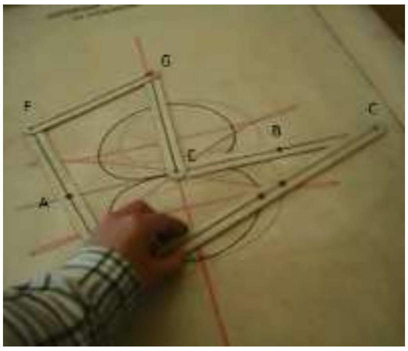


Fig. 4: One of the position for Darboux mechanism: points E, B, C is in one horizontal line.


Fig. 5: One of the position for Darboux mechanism: the links FE and DC are crossed (ϑ=π/2).

In addition, in the position where ϑ=π/2 (Fig. 5) it is that CD=EG+GF. Summarizing the above, the values of all edges depending on one parameter are following: $AD = AF = BC = BE = \frac{1}{2}AB = 2a$, $EG = GF = \frac{1}{2}CD = 2\sqrt{2}a$.

# 3 Building an Animation of Mechanism

To build animation, we need to be able to find the positions of points C, D, E, F, G depending on ϑ - (the angle of deviation of the rib BC from the horizontal axis). Like the whole mechanism, it has one degree of freedom, therefore, the position of the four-link ABCD determines the position of the entire mechanism as a whole. Denote the coordinates of points B and C as $\bar{B}$ and $\bar{C}$ respectively. The position of point C is found by the formula:

$$\bar{C} = \bar{B} + a\bar{e}, \bar{e} = \begin{pmatrix} cos\vartheta \\ sin\vartheta \end{pmatrix}.$$

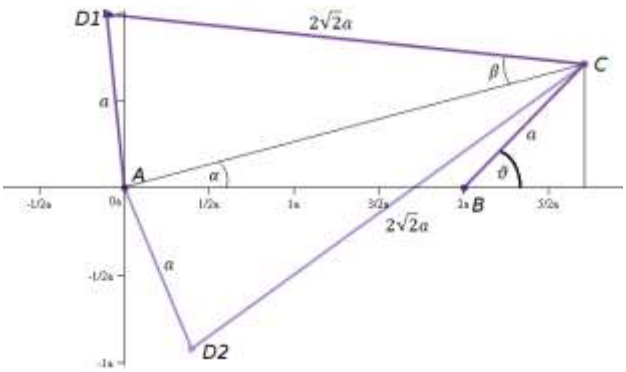


Fig. 6: Possible positions for BC and CD.

There are two possible positions for point D ($D_1$ is above the point A and $D_2$ is below A).

We turned out that for the point D there is not one possible position but there are two possible positions ($D_1$, $D_2$) for one value of the angle ϑ (fig. 6), but only if ∆ACD exists. In order to find the position of the point D the triangle inequality is used in the form:

$$(AC - CD)^2 - AD^2 < 0. \qquad (1)$$

This inequality will be called the criterion for the existence of ∆ACD.

Let's build a graph of the value of the right side of this inequality (1) depending on ϑ. The graph in figure 7 is constructed with a set of admissible ϑ corresponding to the function $(AC - CD)^2 - AD^2$ is below zero. We know that the definition of phase space in mathematics and physics is a space, each point of which corresponds to one and only one state of many possible states of the system. We can consider a set of admissible ϑ and the corresponding values of the function $(AC - CD)^2 - AD^2$ as a two-dimensional space where we see the possible states of the mechanism.

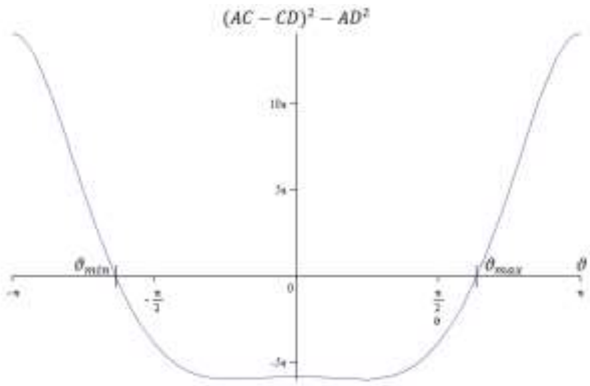


Fig. 7: Set of allowable ϑ
(attention to the graph below zero)

Here some commands from our coding using Maple mathematical package are done. To build the graph (1), our procedure *Value* in Maple mathematical package) is used. The result of the procedure *Value* is drawn by the command with *display* (below):

Now the exact values of the angles

```
Value := proc(CD, A, AD, B, BC, α)
  # A function that calculates the value that determines the existence of triangle ACD
local C, AC;
C := B + BC·⟨cos(α), sin(α)⟩ :
#Determine the coordinates of point C depending on the angle of inclination of the link BC
AC := Norm(A − C, 2) :            #Compute the length of side AC
evalf((CD − AC)^2 − AD^2)
    #If this value is greater than zero, then triangle ACD exists, follows from the triangle
    inequalities
end:
```

$\vartheta_{min}, \vartheta_{max}$ can be found by analyzing the image of the four-link ABCD at the maximum deviation of the link BC from the horizontal axis (Figure 8). The

```
display(plot(Value(CD, A, AD, B, BC, α), α = 0 ..2·Pi, thickness = 1), axes = normal,
    tickmarks = [spacing(π/2), default])
# Graph of the value determining the existence of the ACD triangle

    # defines the area of existence of the ACD polygon depending on the angle of inclination of
    the BC link
```

cosine theorem for the triangle ACB gives us

$$(2\sqrt{2} - 1)^2 a^2 = 4a^2 + a^2 + 4a^2 \cos \vartheta_{max}.$$

Whence $\vartheta_{max} = \arccos(1 - \sqrt{2})$ and $\vartheta_{min} = -\arccos(1 - \sqrt{2})$ due to symmetry. Thus

$$\vartheta \in [\arccos(1 - \sqrt{2}), -\arccos(1 - \sqrt{2})].$$

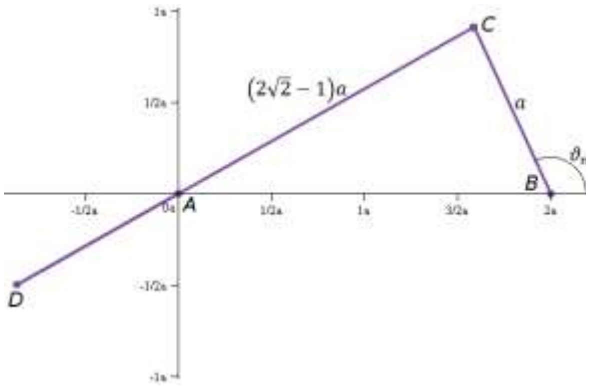


Fig. 8: System position at maximum deflection BC

Knowing the provisions of points A and C, as well as the lengths of the ribs AD and CD, we get $D_1, D_2$ by the following procedure (*FindCross*):

So, the position of ABCD depending on the

```
FindCross := proc(A, AC, B, BC)
#A function that returns the coordinate of the third vertex of a triangle according to the two
   vertices and the lengths of its three sides
local AB, sina, cosa, sinb, cosb;
AB := Norm(A - B, 2) :
cosa := (B[1] - A[1]) / AB :      # Cosine of the angle of inclination of the side AB
sina := (B[2] - A[2]) / AB :      # Sine of the angle of inclination of the AB side
cosb := (BC^2 + AB^2 - AC^2) / (2 BC AB) :   #Cosine of the angle ABC by the cosine theorem
sinb := sqrt(1 - cosb^2) :        # Sine of the angle ABC

   #The angle of inclination of the BC side will be pi-(b-a) or b-(pi-a) from where two
   possible points C are obtained

evalf(B + BC·⟨-sina·sinb-cosa·cosb, sinb·cosa - sina·cosb⟩), evalf(B + BC·⟨sina·sinb
   - cosa·cosb, -sinb·cosa - sina·cosb⟩)
end:
```

angle ϑ is built.

Now let's deal with the rest of the mechanism, first of all, the positions of points E and F will be found. To do this the lines DA and CB by their length beyond points A and B are extended (Fig. 9). Two possible points $F_i$ (i=1,2) for $D_1$ and for $D_2$ are obtained. The coordinates of points are written

$$\overline{F_1} = \bar{A} + \overline{D_1}\bar{A}, \qquad \overline{F_2} = \bar{A} + \overline{D_2}\bar{A},$$
$$\bar{E} = \bar{B} + \bar{C}\bar{B}.$$

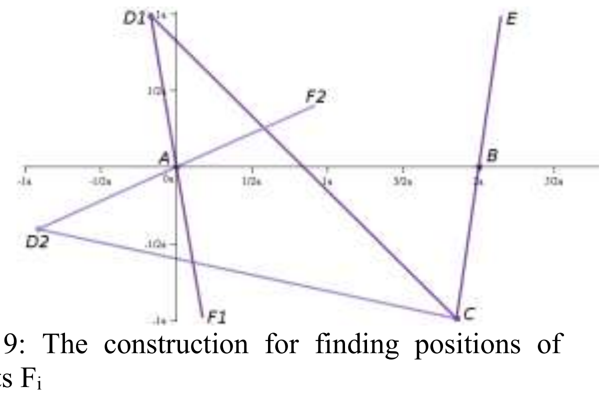


Fig. 9: The construction for finding positions of points $F_i$

To complete the construction, it remains to find the position of the point G. To do this, you can use the previously described algorithm based on the cosine theorem, only pass the positions of the points F, E and the lengths of the links FG, EG into the procedure as parameters. As before, for each such set we will get two positions of the point G (Fig. 10):

$$F_1, E, FG, EG \rightarrow G_1, G_2;$$
$$F_2, E, FG, EG \rightarrow G_3, G_4.$$

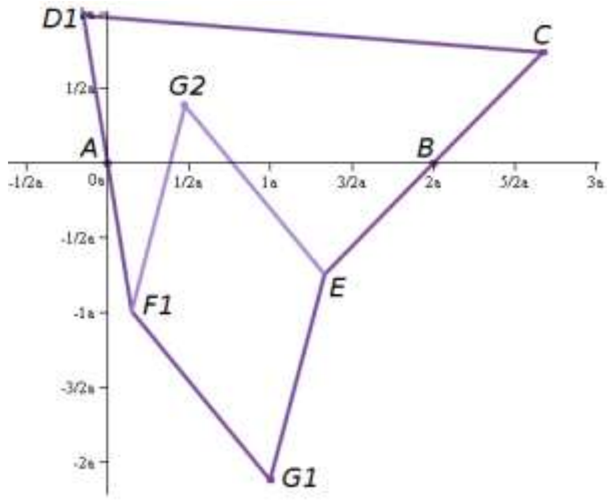


Fig. 10: Position of mechanism showing possible points $G_1$ and $G_2$

Finally for each allowable angle ϑ, there are 4 different possible positions for the Darboux mechanism.

A schematic representation of the phase space (Fig. 11) brings some clarity to the nature of the movement and the relationship between each of the four possible states of the mechanism

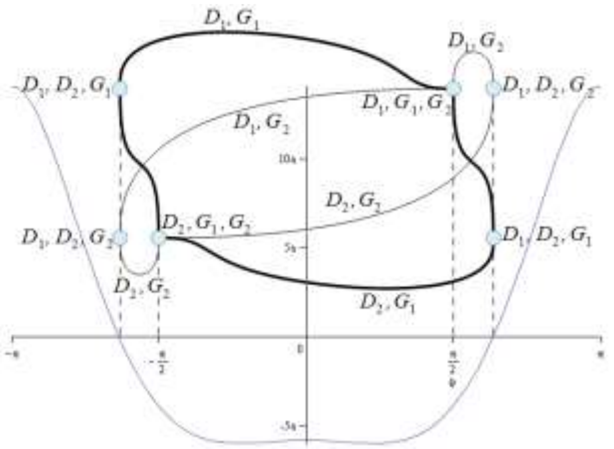


Fig. 11: Phase space of the Darboux mechanism

The phase space scheme helps a lot in building a continuous animation of the movement of the mechanism. For example, we need to highlight the part of the movement when point G moves only along a vertical straight line. If, from the two possible positions of this point $G_1$ and $G_2$, we assume that it is $G_1$ that will always be on the vertical line, and $G_2$, respectively, on the "quasi-eight" (Fig. 12), then on the phase space diagram the mechanism will go through a cycle marked with a thick line in the figure 11, in any direction we choose. At the same time, for a full cycle of motion, the angle $\vartheta$ will completely pass its set of allowable values twice, in the forward and backward directions, and the system will be at the bifurcation points (branching of the phase state) four times.

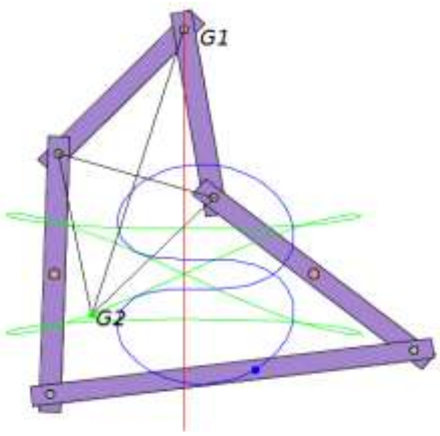


Fig. 12: Frame from the mechanism animation

# 4 A Little about the Methodology

At the laboratory practice, we usually give tasks on building computer code in different systems (such as Maple mathematical package) for animating the movement of rectilinear-guiding mechanisms (for example, Lipkin-Posselier, Watt and others).

An unusual exhibit from the museum was chosen for student T. Shugailo. Unlike other mechanisms we have previously chosen, such as the Watt mechanism or the Posselier mechanism, the Darboux mechanism has not been described anywhere in the literature. We had the exhibit itself at our disposal, we could move it, observe its movement, dimensions, properties. With a simple observation, we knew that when moving there are branching points (for example, point G from Figure 12 can move on different curves). We did not have theoretical formulas, but we had an exhibit and we could carry out measuring experiments. The first thing we did was to measure all the links, then we entered the main parameter - the angle $\vartheta$ - the angle of deviation from the horizontal of the BC rod. Then we began to connect the rod CD to the rod BC, and it turned out that there are two positions of this rod relative to the fixed-point A (Figure 6). It was expected. We looked for geometric properties for the angle theta and found them from the triangle inequality (1). Using geometric formulas for angles, the positions of other links were found depending on the angle theta. This is how the positions of all ABCD points were constructed. Our further study was based on the study of properties when adding the DF bar - in Figure 9, the FG bar in Figure 10. It was taken into account that there may be several positions of points for both point F and point G.

Further developments were made by our colleagues from the Department of Geometry [7], existence theorems and proofs of the rectilinear motion of the point G were considered.

Future developments consist in studying the properties of bifurcation points, studying the properties of the mechanism when an additional force
(or forces and moments) is introduced, for example, along a straight line where the G point moves. It will be studied as a kinematic, static and dynamic analysis of the mechanism with singular points.

# 5 Conclusion

Kinematic analysis of Darboux mechanism is done. It was managed to compile a mathematical description of the device and compile an algorithm for determining its full state depending on the angle of inclination of one of the ribs. Based on the obtained algorithm, a program was written in the Maple 13 mathematical package that allows you to get images of any possible state of the device, as well as allows you to get an animation of the movement of the mechanism. In addition, based on the analysis of the obtained program and the results obtained with the help of it in the form of images, it was possible to depict schematically the phase space of states of the mechanism under consideration.

Independently of our laboratory workshop at the Department of Geometry in SPbGU, this Darboux mechanism was also being studied. PhD dissertation [7] (by the author S.Burian) presents geometric theorems on the rectilinear motion of a point G, talks about a smooth parametrization of motion, and more. Our study directly concerns the construction of a computer model, all calculations and graphs were made to understand this. We communicated with S. Burian. Our work is an addition to his dissertation.

*References:*

[1] Kuteeva G.A. (2020) Laboratornyy praktikum “Kinematicheskiy analiz mekhanizmov” na primere istoricheskikh modeley iz kollektsiy SPbGU // V sbornike: Vtoraya mezhregional'naya nauchno-prakticheskaya konferentsiya prepodavateley matematiki i fiziki pod devizom "Matematika - eto prosto!" Materialy konferentsii. S. 122-137 (In Russian)

[2] Kuteeva G. A., Sinilshchikova G. A., Trifonenko B. V. (2019) The exhibits of the historical collection at the Faculty of Mathematics and Mechanics of St. Petersburg State University. Vestnik of Saint Petersburg University. Mathematics. Mechanics. Astronomy, vol. 6 (64), issue 3, pp. 493–504. https://doi.org/10.21638/11701/spbu01.2019.313 (In Russian)

[3] Tovstik P.E., Yushkov M.P., Kuteeva G.A., Sinilshchikova G.A., Trifonenko B.V. (2021) The collection of mechanisms by P.L. Chebyshev at the department of theoretical and applied mechanics of St. Petersburg University, Journal of Physics: Conference Series. Cep. "International Scientific Conference on Mechanics "The Ninth Polyakhov's Reading", ISCM 2021" C. 012050.

[4] Collection of knowledge. Museums and collections of St. Petersburg State University, St. Petersburg State University (2018) / St. Petersburg State University: [ed.-ed.: G.F. Anastasenko and others; editorial board: E.G. Chernova and others; per. in English. lang.: V. Yu. Golubev, Walker W. Trimble]. Publishing house of St. Petersburg State University. 287 p.

[5] Meshcherskiy I.V. (1895) Prepodavaniye mekhaniki i mekhanicheskiye kollektsii v nekotorykh vysshikh uchebnykh zavedeniyakh Italii, Frantsii, Shveytsarii i Germanii. SPb, 72s. (In Russian)

[6] Matematicheskiy Peterburg. Istoriya, nauka, dostoprimechatel'nosti. Spravochnik-putevoditel,' (2018) Redaktor-sostavitel' G.I. Sinkevich, nauchnyy redaktor A.I. Nazarov. Sankt-Peterburg: Obrazovatel'nyye proyekty. 336 s. (In Russian)

[7] Burian S. (2021) Mechanics of holonomic systems with geometric singular points. SPbGU. 283s. https://disser.spbu.ru/files/2021/disser_burian.pdf (In Russian and in English)

**Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)**

Galina Kuteeva gave an idea for the exercise of practicum, involved this student in a research project, gave recommendations on how to carry out the calculations, and revised the results.

Timofei Shugailo is a talent student, has done the calculations, implemented the algorithms.

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**Conflict of Interest**

The authors have no conflicts of interest to declare that are relevant to the content of this article.