On Fractional φ- and biφ-calculi

AMER H. DARWEESH [1], ABDELAZIZ M.D. MAGHRABI [2]
[1]Department of Mathemtics, Jordan University of Science and Technology, Irbid 22110, JORDAN
[2]Department of Mathematics, Eastern Mediterranean University, Gazimagusa, Mersin 10, TURKEY

Abstract: - In this paper we introduce fractional φ- and biφ-calculi using Riemann-Liouville approach and Caputo approach as well. An effort is put into explaining the basic principles of these calculi since they are not as common as classical calculus. This was also done for tanh-, bi-tanh- multiplicative, and bigeometric calculi and in the general case as well. Generalizations are also investigated where the homeomorphisms φ, η are arbitrary.

Key-Words: - Derivative, integral, non-Newtonian calculus, fractional derivative, homeomorphism.

Received: April 19, 2022. Revised: January 14, 2023. Accepted: February 9, 2023. Published: March 7, 2023.

1 Introduction

In the seventeenth century, Isaac Newton and Gottfried Leibniz laid the foundations for the classical -or sometimes called- Newtonian calculus. That particular calculus has proved its mathematical strength. Indeed, it is the most applicable theory used in sciences. Fractional calculus, even though it is usually thought that it is a relatively new subject, it has dated back to 1695 when L'Hoptial wrote to Leibnitz asking about the interpretation of $\frac{d^n f}{dx^n}$ when $n = \frac{1}{2}$, see [1]. In the previous century, many mathematicians have given different perspectives and approaches in an attempt to answer this question. The same question arises when one considers $\frac{d^{*(n)} f}{dx^n}$ or $\frac{d^{\pi(n)}}{dx^n}$. These are the multiplicative and bigeometric derivatives respectively. In the period 1967 to 1970, Michael Grossman and Robert Katz initiated many calculi considering different operations and viewing classical calculus as an additive type that depend on addition and subtraction as their foundation [2]. Using that view, they came up with what we call multiplicative and bigeometric calculi [1-6], that which depends on multiplication and division. More precisely, defining $\varphi\text{-}arithmetic$ to represent the main algebraic operations performed on $\mathbb{R}$. The function φ is a bijection from $\mathbb{R}$ onto an interval $\mathbb{I}$ that induced the field and metric structures from $\mathbb{R}$ onto $\mathbb{I}$. Letting $\varphi(x) = e^x$, we see that on $\mathbb{I} = (0, \infty)$, the exponential-operations give rise to two pairs of calculi on functions $f : \mathbb{I} \to \mathbb{I}$. This will be further explained later on in the second section. This paper is organized in the following way. In Section 2, we explain briefly the principles of φ- and biφ- calculi, and give examples regarding multiplicative and bigeometric calculi. Moreover, we mention the Newtonian versions of Caputo and Riemann-Liouville approaches to this subject. Then, we introduce some theorems for φ- and biφ-calculi and we also mention theorems from [6] as well which are the stepping stones used in this paper. In Section 3, we define φ- and biφ- fractional calculi, and based on them we also define it with respect to non-Newtonian calculi, which are the bigeometric, tanh-, and bi-tanh- fractional calculi. The multiplicative case is discussed in [6]. Moreover, we mention some results which are the relations between φ- and biφ- fractional calculi and the Newtonian fractional calculi considering the mentioned approaches. The notation is rather different than the one that was introduced in [2], which denotes the bijection as α instead of φ. This was done because the letter φ is more convenient when discussing fractional calculi since α is commonly used for denoting the order.

2 Elements of φ- and biφ-Differentiation

Consider an increasing homeomorphism $\varphi: \mathbb{R} \to \mathbb{I}$. For $x, y \in \mathbb{I}$, we define the following operations:

1. $x \oplus_\varphi y = \varphi(\varphi^{-1}(x) + \varphi^{-1}(y))$,

2. $x \ominus_\varphi y = \varphi(\varphi^{-1}(x) - \varphi^{-1}(y))$,

3. $x \otimes_\varphi y = \varphi(\varphi^{-1}(x) \times \varphi^{-1}(y))$,

4. $x \oslash_\varphi y = \varphi(\varphi^{-1}(x) / \varphi^{-1}(y))$,

5. $x \leq y$ if and only if $\varphi^{-1}(x) \leq \varphi^{-1}(y)$.

It is easy to check that $\mathbb{I}$ under the above operations becomes an ordered field. We call this field the φ-non-Newtonian interval. Moreover, the following real-valued function φ defines a metric on $\mathbb{I}$:

6. $d_\varphi(x, y) = |\varphi^{-1}(x) - \varphi^{-1}(y)|$.

Moreover, for any $\alpha \in \mathbb{R}$, we define the φ-$alpha$ power of $x \in \mathbb{I}$ by

$$x^{\otimes\alpha} = \varphi([\varphi^{-1}(x)]^\alpha).$$

It is easy to check that the following properties are true:

1. For $\alpha, \beta \in \mathbb{R}$, we have $x^{\otimes\alpha} \otimes x^{\otimes\beta} = x^{\otimes\beta} \otimes x^{\otimes\alpha} = x^{\otimes(\alpha+\beta)}$.

2. For $\alpha, \beta \in \mathbb{R}$, we have $x^{\otimes\alpha} \oslash x^{\otimes\beta} = x^{\otimes(\alpha-\beta)}$.

3. For $\alpha, \beta \in \mathbb{R}$, we have $\left(x^{\otimes\alpha}\right)^{\otimes\beta} = \left(x^{\otimes\beta}\right)^{\otimes\alpha} = x^{\otimes(\alpha\beta)}$.

Remark 1 *This metric is compatible with the operations on the field $\mathbb{I}$, that is, the above operations are continuous.*

With this metric we can define, as usual, the limit of a function that is defined on a φ-non-Newtonian interval $\mathbb{I}$.

Definition 2 *Let $\mathbb{I}$ be φ-interval, and $f: \mathbb{I} \to \mathbb{I}$. For $x_0 \in \mathbb{I}$, we define*

$$\text{bi}\varphi - \lim_{x \to x_0} f(x) = L \in \mathbb{I}$$

to be the limit from the metric $(\mathbb{I}, d_\varphi)$ to itself. That is, if $d_\varphi(f(x), L) \to 0$ as $d_\varphi(x, x_0) \to 0$.

In the next proposition, we see the relation between the usual limit and the biφ-limit.

Proposition 3 *Let $\mathbb{I}$ be a φ-interval, and $f: \mathbb{I} \to \mathbb{I}$. Then,*

$$\text{bi}\varphi - \lim_{x \to x_0} f(x) = \varphi\left(\lim_{t \to \varphi^{-1}(x_0)} \varphi^{-1} \circ f \circ \varphi(t)\right).$$

Proof. Let $\text{bi}\varphi - \lim_{x \to x_0} f(x) = L$. By the definition of biφ-limit, we have

$$\left|\varphi^{-1}\big(f(x)\big) - \varphi^{-1}(L)\right| \to 0 \quad \text{as } |\varphi^{-1}(x) - \varphi^{-1}(x_0)| \to 0.$$

In other words,

$$|\varphi^{-1} \circ f \circ \varphi(\varphi^{-1}(x)) - \varphi^{-1}(L)| \to 0 \quad \text{as } |\varphi^{-1}(x) - \varphi^{-1}(x_0)| \to 0.$$

Hence,

$$|\varphi^{-1} \circ f \circ \varphi(t) - \varphi^{-1}(L)| \to 0 \quad \text{as } |t - \varphi^{-1}(x_0)| \to 0.$$

That is,

$$\lim_{t \to \varphi^{-1}(x_0)} \varphi^{-1} \circ f \circ \varphi(t) = \varphi^{-1}(L).$$

Therefore,

$$\varphi\left(\lim_{t\to\varphi^{-1}(x_0)}\varphi^{-1}\circ f\circ\varphi(t)\right)=L.$$

Definition 4 *Let $\mathbb{I}$ be a φ-interval, and $f:\mathbb{R}\to\mathbb{I}$. For $x_0\in\mathbb{I}$, we define*

$$\varphi-\lim_{x\to x_0}f(x)=L\in\mathbb{I}$$

to be the limit from the usual metric on $\mathbb{R}$ to the metric $(\mathbb{I},d_\varphi)$. That is, $d_\varphi(f(x),L)\to 0$ as $|x-x_0|\to 0$.

Proposition 5 *Let $\mathbb{I}$ be a φ-interval, and $f:\mathbb{R}\to\mathbb{I}$. Then,*

$$\varphi-\lim_{x\to x_0}f(x)=\varphi\left(\lim_{x\to x_0}\varphi^{-1}\circ f(x)\right).$$

Proof. Let $\varphi-\lim_{x\to x_0}f(x)=L$. By the definition of φ-limit, we have

$$|\varphi^{-1}(f(x))-\varphi^{-1}(L)|\to 0 \quad \text{as } |x-x_0|\to 0.$$

Therefore,

$$\lim_{x\to x_0}\varphi^{-1}\circ f(x)=\varphi^{-1}(L).$$

Or equivalently,

$$\varphi\left(\lim_{x\to x_0}\varphi^{-1}\circ f(x)\right)=L.$$

Based on these types of limits, we can develop φ- and biφ-calculi, that is to define a derivative and an integral with respect to φ-operations.

Remark 6 *From now on, for the sake of convenience and brevity, we will use the operations $\oplus$, $\ominus$, $\otimes$, and $\oslash$ instead of $\oplus_\varphi$, $\ominus_\varphi$, $\otimes_\varphi$, and $\oslash_\varphi$.*

Definition 7 *The biφ-derivative of a function $f:\mathbb{I}\to\mathbb{I}$, where $\mathbb{I}\subseteq\mathbb{R}$ with $\varphi:\mathbb{R}\to\mathbb{I}$ is denoted and given by*

$$f^{\mathrm{bi}\varphi}(x)=$$

$$\mathrm{bi}\varphi-\lim_{y\to x}[f(y)\ominus f(x)]\oslash[y\ominus x]. \quad (1)$$

Consider Equation 1. Using Proposition 3, one has

$$f^{\mathrm{bi}\varphi}(x)=$$

$$\mathrm{bi}\varphi-\lim_{y\to x}[f(y)\ominus f(x)]\oslash[y\ominus x]$$

$$=\varphi\left(\lim_{t\to\varphi^{-1}(x)}\varphi^{-1}([f(\varphi(t))\ominus f(x)]\oslash [\varphi(t)\ominus x])\right)$$

$$=\varphi\left(\lim_{t\to\varphi^{-1}(x)}\frac{\varphi^{-1}(f(\varphi(t))-\varphi^{-1}(f(x))}{t-\varphi^{-1}(x)}\right)$$

$$=\varphi\left(\frac{d}{dt}(\varphi^{-1}\circ f\circ\varphi)(t)|_{t=\varphi^{-1}(x)}\right)$$

$$=\varphi\circ(\varphi^{-1}\circ f\circ\varphi)'\circ\varphi^{-1}(x).$$

This yields the following results.

Proposition 8 *The biφ-derivative of a function $f:\mathbb{I}\to\mathbb{I}$, where $\mathbb{I}$ is a φ-interval,is given by*

$$f^{\mathrm{bi}\varphi}(x)=$$

$$\varphi\left(\lim_{t\to\varphi^{-1}(x)}\frac{\varphi^{-1}(f(\varphi(t))-\varphi^{-1}(f(x))}{t-\varphi^{-1}(x)}\right)$$

$$=\varphi\circ(\varphi^{-1}\circ f\circ\varphi)'\circ\varphi^{-1}(x),$$

or in the other notations,

$$\frac{d^{\mathrm{bi}\varphi}}{dx}f(x)=\varphi\left(\frac{d}{dt}[\varphi^{-1}\circ f\circ\varphi](t)|_{t=\varphi^{-1}(x)}\right).$$

If we denote the n^{th} biφ-derivative of $f(x)$ by $\frac{d^{\mathrm{bi}\varphi(n)}}{dx^n} f(x) = f^{\mathrm{bi}\varphi(n)}(x)$, we can easily obtain the following result.

Proposition 9 *Let* $\mathbb{I}$ *be a* φ*-interval, and* $f: \mathbb{I} \to \mathbb{I}$. *Then,*

$$f^{bi\varphi(n)}(x) = \varphi \circ (\varphi^{-1} \circ f \circ \varphi)^{(n)} \circ \varphi^{-1}(x), \tag{2}$$

or in the other notations,

$$\frac{d^{\mathrm{bi}\varphi(n)}}{dx^n} f(x) = \varphi\left(\frac{d^n}{dt^n}[\varphi^{-1} \circ f \circ \varphi](t)|_{t=\varphi^{-1}(x)}\right).$$

Conversely, we can write the ordinary derivative in terms of biφ-derivative as follows.

Proposition 10 *Let* $g: \mathbb{R} \to \mathbb{R}$, $\mathbb{I}$ *be a* φ*-interval. Then,* $\varphi \circ g \circ \varphi^{-1}: \mathbb{I} \to \mathbb{I}$ *and*

$$g^{(n)}(t) = \varphi \circ \frac{d^{\mathrm{bi}\varphi(n)}}{dx^n}[\varphi \circ g \circ \varphi^{-1}] \circ \varphi(t),$$

or in the other notations,

$$\frac{d^n}{dt^n} g(t) = \varphi\left(\frac{d^{\mathrm{bi}\varphi(n)}}{dx^n}[\varphi \circ g \circ \varphi^{-1}](x)|_{x=\varphi(t)}\right).$$

Example 11 *Take* $\varphi(x) = e^x$, *with* $\mathbb{I} = (0, \infty)$, *then for functions* $f : \mathbb{I} \to \mathbb{I}$, *we can use Proposition 8 to define the bigeometric derivative as follows*

$$\frac{d^{\pi} f}{dx}(x) = \varphi\left(\lim_{t \to \varphi^{-1}(x)} \frac{\varphi^{-1} \circ f \circ \varphi(t) - \varphi^{-1}(f(x))}{t - \varphi^{-1}(x)}\right)$$

$$= \varphi\left(\lim_{t \to \ln x} \frac{\ln \circ f \circ \exp(t) - \ln(f(x))}{t - \ln x}\right)$$

$$= \varphi\left(\lim_{y \to x} \frac{\ln(f(y)) - \ln(f(x))}{\ln y - \ln x}\right)$$

$$(\text{here}, x \in \mathbb{I} \text{ as a subset of } \mathbb{R})$$

$$= e^{x(\ln f(x))'},$$

as it is expected in the bigeometric calculus.

If we define f from the Newtonian field $\mathbb{R}$ into the φ-non-Newtonian interval $\mathbb{I}$, we can introduce a weaker version of differentiablity and integrability.

Definition 12 *The* φ*-derivative of a function* $f: \mathbb{R} \to \mathbb{I}$, *where* $\mathbb{I} \subseteq \mathbb{R}$ *with* $\varphi: \mathbb{R} \to \mathbb{I}$ *is denoted and given by*

$$f^{\varphi}(x) = \varphi - \lim_{y \to x}[f(y) \ominus_{\varphi} f(x)] \oslash_{\varphi} [\varphi(y) \ominus_{\varphi} \varphi(x)].$$

Proposition 13 *The* φ*-derivative of a function* $f: \mathbb{R} \to \mathbb{I}$, *where* $\mathbb{I}$ *is a* φ*-interval, is given by*

$$f^{\varphi}(x) = \varphi\left(\lim_{y \to x} \frac{\varphi^{-1}(f(x)) - \varphi^{-1}(f(y))}{x - y}\right)$$

$$= \varphi\left(\frac{d}{dx}[\varphi^{-1} \circ f](x)\right).$$

Proposition 14 *Let* $f: \mathbb{R} \to \mathbb{I}$. *Then,* f *is* φ*-differentiable at* $x \in \mathbb{R}$ *if and only if* $f \circ \varphi^{-1}$ *is bi*φ*-differentiable at* $\varphi(x)$. *In this case,* $f^{\varphi}(x) = (f \circ \varphi^{-1})^{bi\varphi}(\varphi(x))$.

Proof. The proof follows from Propositions 8 and 13.

Theorem 15 *Let $\varphi: \mathbb{R} \to \mathbb{I}$ be differentiable with $\varphi'(x) > 0$ on $\mathbb{R}$, and let $f: \mathbb{R} \to \mathbb{I}$. Then f is differentiable if and only if f is φ-differentiable.*

Proof. Let f be differentiable at x. By the inverse function theorem, the function $\varphi^{-1}(x)$ is differentiable on $\mathbb{I}$. Hence, the function $\varphi^{-1} \circ f$ is differentiable at x. That is,

$$\frac{d}{dx}(\varphi^{-1} \circ f)(x) = \lim_{y \to x} \frac{\varphi^{-1}(f(x)) - \varphi^{-1}(f(y))}{x - y} = L$$

exists. Therefore,

$$\varphi\left(\lim_{y \to x} \frac{\varphi^{-1}(f(x)) - \varphi^{-1}(f(y))}{x - y}\right) = \varphi(L).$$

By Proposition 13, we have $f^{\varphi}(x) = \varphi(L)$. Hence, f is φ-differentiable at x. Now, suppose that f is φ-differentiable. Then,

$$\varphi\left(\lim_{y \to x} \frac{\varphi^{-1}(f(x)) - \varphi^{-1}(f(y))}{x - y}\right) = L,$$

and hence

$$\lim_{y \to x} \frac{\varphi^{-1}(f(x)) - \varphi^{-1}(f(y))}{x - y} = \varphi^{-1}(L).$$

It follows that the function $\varphi^{-1} \circ f(x)$ is differentiable at x. By the chain rule and the fact that $\varphi(x)$ is differentiable everywhere, we conclude that $f(x) = \varphi \circ \varphi^{-1} \circ f(x)$ is differentiable at x.

Theorem 16 *Let $\varphi: \mathbb{R} \to \mathbb{I}$ be differentiable with $\varphi'(x) > 0$ on $\mathbb{R}$, and let $f: \mathbb{I} \to \mathbb{I}$. Then f is differentiable if and only if f is biφ-differentiable. Moreover,*

$$f^{bi\varphi}(x) = \varphi\left(\frac{\frac{d}{dx}(\varphi^{-1} \circ f)(x)}{\frac{d}{dx}\varphi^{-1}(x)}\right).$$

Proof. Let f be differentiable at x_0. By the inverse function theorem, the function $\varphi^{-1}(x)$ is differentiable on $\mathbb{I}$. Hence, the function $\varphi^{-1} \circ f$ is differentiable at x_0. Let $t_0 = \varphi^{-1}(x_0)$, then $\varphi(t)$ is differentiable at t_0. By the chain rule, $\varphi^{-1} \circ f \circ \varphi(t)$ is differentiable at t_0, and $\frac{d}{dx}[\varphi^{-1} \circ f \circ \varphi](t_0) =$

$$\frac{d}{dx}[\varphi^{-1} \circ f](\varphi(t_0)) \frac{d}{dx}\varphi(t_0).$$

Therefore,

$$\frac{d}{dx}[\varphi^{-1} \circ f \circ \varphi](\varphi^{-1}(x_0)) = \frac{\frac{d}{dx}[\varphi^{-1} \circ f](x_0)}{\frac{d}{dx}\varphi^{-1}(x_0)}.$$

By Proposition 8, $f^{bi\varphi}(x_0)$ exists and

$$f^{bi\varphi}(x) = \varphi\left(\frac{\frac{d}{dx}(\varphi^{-1} \circ f)(x)}{\frac{d}{dx}\varphi^{-1}(x)}\right).$$

Now, suppose that f is biφ-differentiable. Then,

$$f^{bi\varphi}(x) = \varphi\left(\lim_{t \to \varphi^{-1}(x)} \frac{\varphi^{-1}(f(\varphi(t)) - \varphi^{-1}(f(x))}{t - \varphi^{-1}(x)}\right) = \varphi\left(\lim_{y \to x} \frac{\varphi^{-1}(f(x)) - \varphi^{-1}(f(y))}{\varphi^{-1}(x) - \varphi^{-1}(y)}\right)$$

Therefore,

$$\lim_{y \to x} \frac{\varphi^{-1}(f(x)) - \varphi^{-1}(f(y))}{\varphi^{-1}(x) - \varphi^{-1}(y)} = \varphi^{-1}\left(f^{bi\varphi}(x)\right).$$

Since

$$\lim_{y\to x}\frac{\varphi^{-1}(x)-\varphi^{-1}(y)}{x-y}=\frac{d}{dx}\varphi^{-1}(x)\neq 0,$$

we have

$$\lim_{y\to x}\frac{\varphi^{-1}(f(x))-\varphi^{-1}(f(y))}{x-y}=\varphi^{-1}\left(f^{bi\varphi}(x)\right)\frac{d}{dx}\varphi^{-1}(x).$$

It follows that the function $\varphi^{-1}\circ f(x)$ is differentiable at x. By the chain rule, we conclude that $f(x)=\varphi\circ\varphi^{-1}\circ f(x)$ is differentiable at x.

We denote the n^{th} φ-derivative by $f^{\varphi(n)}(x)$. With this notation, one can obtain the following result.

Theorem 17 *Let* $f:\mathbb{R}\to\mathbb{I}$, *then* $f^{\varphi(n)}(x)=\varphi\left(\frac{d^n}{dx^n}(\varphi^{-1}\circ f)(x)\right)$.

Proof. The proof will be done using mathematical induction and Proposition 13.

Example 18 *For* $\mathbb{I}=\mathbb{R}_+$ *and* $\varphi(x)=exp(x)$, *we obtain the geometric derivative (which is also called ∗derivative or multiplicative derivative) of* $f(x)$.

$$\frac{d^*f}{dx}(x)=\lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}=e^{\frac{f(\prime x)}{f(x)}}=e^{(\ln\circ f)\prime(x)} \tag{3}$$

By Theorem 17, we have

$$\frac{d^{*(n)}f}{dx}(x)=e^{(\ln\circ f)^{(n)}(x)} \tag{4}$$

Moreover, one immediately realizes the relation

$$f^{\pi}(x)=(f^*(x))^x. \tag{5}$$

The multiplicative derivative and the additive derivative can be used to express each other. Indeed, we have the following equation

$$f^{(n)}(x)=\sum_{k=0}^{n-1}\binom{n-1}{k}f^{(k)}(x)(\ln\circ f^{*(n-k)})(x) \tag{6}$$

Using Faà di Bruno formula on equation (4), one also arrives at the following

$$f^{*(n)}=\exp\Big(\sum_{\substack{k_1+2k_2+\cdots+nk_n=n\\ k:=k_1+\cdots+k_n}}\frac{-(k-1)!\,n!}{k_1!\cdot\ldots\cdot k_n!}(-f(x))^{-k}\prod_{i=1,\ldots,n}\left(\frac{f^{(i)}(x)}{i!}\right)^{k_i}\Big) \tag{7}$$

For a simpler variant of Faà di Bruno formula, refer to [7]. This gives a brief overview of multiplicative and bigeometric calculi.

Example 19 *The tanh-derivative for* $x\in\mathbb{I}=(-1,1)$, *is denoted and given by*

$$f^{\star}(x)=\frac{d^{\star}f(x)}{dx}=\frac{e^{\frac{2f(\prime x)}{1-f(x)^2}}-1}{e^{\frac{2f(\prime x)}{1-f(x)^2}}+1}=\frac{e^{2\frac{d\tanh^{-1}f(x)}{dx}}-1}{e^{2\frac{d\tanh^{-1}f(x)}{dx}}+1}. \tag{8}$$

Moreover, the n^{th} order tanh-derivative is given by

$$f^{\star(n)}(x)=\frac{e^{(\ln\frac{1+f(x)}{1-f(x)})^{(n)}}-1}{e^{(\ln\frac{1+f(x)}{1-f(x)})^{(n)}}+1},n=0,1,2,\ldots \tag{9}$$

Theorem 20 *Let* $f(x)$ *be n-times* ${}^{bi\varphi}$*differentiable, then*

$$\frac{d^{\varphi(n)}f(x)}{dx^n}=\frac{d^{\mathrm{bi}\varphi(n)}f(\varphi^{-1}(t))}{dt^n},x=\varphi^{-1}(t), \tag{10}$$

Thus, the biφ- derivative is a Gauss vector of the φ-derivative. Equivalently,

$$\frac{d^{\varphi(n)}f(x)}{dx^n} = \frac{d^{\mathrm{bi}\varphi(n)}f(x)}{d\varphi(x)^n} \quad (11)$$

$$\frac{d^{\varphi(n)}f(x)}{d\varphi^{-1}(x)^n} = \frac{d^{\mathrm{bi}\varphi(n)}f(x)}{dx^n} \quad (12)$$

Proof. This proof will be done using mathematical induction. Let $x = \varphi^{-1}(t)$, then we have $dx = \varphi^{-1}(t)'dt$. This implies,

$$\frac{d^{\varphi}f(x)}{dx} = \varphi(\frac{d\varphi^{-1}(f(x))}{dx})$$

$$= \varphi(\frac{1}{\varphi^{-1}(t)'}\frac{d\varphi^{-1}(f(\varphi^{-1}(t)))}{dt})$$

$$= \frac{d^{\mathrm{bi}\varphi}f(\varphi^{-1}(t))}{dt}.$$

Hence, the theorem is true at $n = 1$. Assume that it is true for $n = k - 1$, by the induction hypothesis we have,

$$\frac{d^{\varphi(k)}f(x)}{dx^n} = \varphi\left(\frac{d^k\varphi^{-1}\big(f(x)\big)}{dx^k}\right)$$

$$= \varphi\left(\frac{d}{dx}\frac{d^{k-1}\varphi^{-1}\big(f(x)\big)}{dx^{k-1}}\right)$$

$$= \varphi\left(\frac{d}{dx}\varphi^{-1}\left(f^{\varphi(k-1)}(x)\right)\right)$$

$$= \varphi\left(\frac{1}{\varphi^{-1}(t)'}\frac{d}{dt}\varphi^{-1}\left(\frac{d^{\mathrm{bi}\varphi(k-1)}f\big(\varphi^{-1}(t)\big)}{dt^{k-1}}\right)\right)$$

$$= \frac{d^{\mathrm{bi}\varphi(k)}f\big(\varphi^{-1}(t)\big)}{dt^k}.$$

This concludes the proof. The other forms are obtained by manipulating the substitution $x = \varphi^{-1}(t)$.

Remark 21 *The first form which includes the variables x and t were introduced to obtain a simple proof.*

Example 22 *Let f be n-times $^{\pi}$differentiable, then*

$$\frac{d^{*(n)}f(x)}{dx^n} = \frac{d^{\pi(n)}f(\ln t)}{dt^n}, x = \ln t. \quad (13)$$

Which has equivalent forms,

$$\frac{d^{*(n)}f(x)}{dx^n} = \frac{d^{\pi(n)}f(x)}{d\exp x^n} \quad (14)$$

$$\frac{d^{*(n)}f(x)}{d\ln x^n} = \frac{d^{\pi(n)}f(x)}{dx^n} \quad (15)$$

Example 23 *Let $f(x)$ be n-times $^{bi\star}differentiable$. Then,*

$$\frac{d^{\star(n)}f(x)}{dx^n} = \frac{d^{\mathrm{bi}\star(n)}f(\tanh^{-1}x)}{dt^n}, x = \tanh^{-1}t. \quad (16)$$

Which are equivalent to the forms,

$$\frac{d^{\star(n)}f(x)}{dx^n} = \frac{d^{\mathrm{bi}\star(n)}f(x)}{d\tanh x^n} \quad (17)$$

$$\frac{d^{\star(n)}f(x)}{d\tanh^{-1}x^n} = \frac{d^{\mathrm{bi}\star(n)}f(x)}{dx^n}. \quad (18)$$

Remark 24 *By Theorem (20), we can comprehend the relation between φ- and biφ-calculi. Indeed, φ-calculus is not only a weakened version of biφ-calculus, rather biφ-calculus is the change in φ-calculus with respect to $\varphi^{-1}(x)$, which is equivalent to stating that φ-calculus is the change in biφ-calculus with respect to $\varphi(x)$.*

3 Elements of φ- and biφ-Riemann integration

Using the structure of the metric $\mathbb{I}$, one can define the boundedness of $f: A \to \mathbb{I}$. Precisely, f is φ-bounded if $d_{\varphi}(f(x), \varphi(0)) \leq M$, for all $x \in A$. That is, if $|\varphi^{-1}(f(x))| \leq M$, for all $x \in A$.

Definition 25 *Let $f: \mathbb{I} \to \mathbb{I}$ be φ-bounded. Let $a < b$ in $\mathbb{I}$, and $P = \{x_0, x_1, \dots, x_n\}$ be a partition on $[a, b]$. The function f is called biφ-Riemann integrable if there is $L \in \mathbb{I}$ such that* *for any $\epsilon > 0$ and any choice of $x_{i-1} \le c_i \le x_i$, there is a $\delta > 0$ satisfies:*

$$d_\varphi(\oplus_{i=1}^n f(c_i) \otimes (x_i \ominus x_{i-1}), L) < \epsilon$$

whenever $sup_i d_\varphi(x_i, x_{i-1}) < \delta$. In this case we write,

$$\int_a^b f(x) d^{\mathrm{bi}\varphi} x = \lim_{n\to\infty} \bigoplus_{i=1}^{n} (f(c_i) \otimes (x_i \ominus x_{i-1})) = L.$$

Remark 26 *The definition above is independent of the choice of $c_i \in [x_{i-1}, x_i]$. That is, if the above limit exists, then for any choice of $\{c_i\}$, the limit is the same.*

It is worth mentioning that if $f(x)$ is Riemann integrable, and $\varphi^{-1}(x)$ is piecewise continuously differentiable on $[a, b]$, then

$$\int_a^b f(x) d^{\mathrm{bi}\varphi} x = \varphi\left(\int_a^b \varphi^{-1}(f(x)) \frac{d}{dx}\varphi^{-1}(x) dx\right). \tag{19}$$

Example 27 *The bigeometric integral, denoted $\int_a^b f\ (x)^{\mathbf{d}x}$ is given by*

$$\int_a^b f(x)^{\mathbf{d}x} = \lim_{n\to\infty} \bigoplus_{i=1}^{n} (f(c_i) \otimes (x_i \ominus x_{i-1}))$$

$$= \lim_{n\to\infty} \prod_{i=1}^{n} f(c_i)^{\ln(x_i/x_{i-1})}$$

$$= \exp\left(\int_a^b \frac{\ln f(x)}{x} dx\right).$$

Definition 28 *Let $f: \mathbb{R} \to \mathbb{I}$ be φ-bounded. Let $a < b$ in $\mathbb{I}$, and $P = \{x_0, x_1, \dots, x_n\}$ be partition on $[a, b]$. The function f is called φ-Riemann integrable if there is $L \in \mathbb{I}$ such that for any $\epsilon > 0$ and any choice of $x_{i-1} \le c_i \le x_i$, there is a $\delta > 0$ satisfies: $d_\varphi(\oplus_{i=1}^n f(c_i) \otimes (\varphi(x_i) \ominus \varphi(x_{i-1})), L) < \epsilon$ whenever $sup_i |x_i - x_{i-1}| < \delta$. In this case we write*

$$\int_a^b f(x) d^\varphi x = \lim_{n\to\infty} \bigoplus_{i=1}^{n}{}_\varphi \left(f(c_i) \otimes_\varphi (\varphi(x_i) \ominus_\varphi \varphi(x_{i-1}))\right)$$

$$= \lim_{n\to\infty} \varphi\left(\sum_{i=1}^{n} \varphi^{-1} \circ f(c_i)(x_i - x_{i-1})\right) = L.$$

It is clear that from the definition above if f is Riemann integrable, and hence $\varphi^{-1} \circ f$ is Riemann integrable, then f is φ-Riemann integrable and

$$\int_a^b f(x) d^\varphi x = \varphi(\int_a^b \varphi^{-1}(f(x)) dx).$$

Example 29 *The geometric integral, denoted $\int_a^b f\ (x)^{dx}$ is given by*

$$\int_a^b f(x)^{dx} = \lim_{n\to\infty} \exp\left[\sum_{i=1}^{n} \ln(f(c_i))(x_i - x_{i-1})\right]$$

$$= \exp\left(\int_a^b \ln f(x) dx\right).$$

Theorem 30 *Let $\int_a^b f(x) d^\varphi x$ be the φ-integral of $f(x)$. Then,*

$$\int_a^b f(x) d^{\varphi(n)} x = \varphi\left(\int_a^b \varphi^{-1}\big(f(x)\big) d^{(n)} x\right), n \in N \cup \{0\}. \tag{20}$$

Proof. The proof will be done using mathematical induction. For $n = 0,1$, it is clear. Assume that it holds true for $n = k - 1$, then we get

$$\int_a^b f(x) d^{\varphi(k)} x = \int_a^b \left(\int_a^b f(x) d^{\varphi(k-1)} x\right) d^\varphi x$$

$$= \varphi\left(\int_a^b \varphi^{-1}\left(\int_a^b f(x)d^{\varphi(k-1)}x\right)dx\right)$$

$$= \varphi\left(\int_a^b \varphi^{-1}\left(\varphi\left(\int_a^b \varphi^{-1}\big(f(x)\big)d^{(k-1)}x\right)\right)dx\right)$$

$$= \varphi\left(\int_a^b \left[\int_a^b \varphi^{-1}\big(f(x)\big)d^{(k-1)}x\right]dx\right)$$

$$= \varphi\left(\int_a^b \varphi^{-1}\big(f(x)\big)d^{(k)}x\right).$$

Example 31 *It is clear from the definition of the multiplicative integral that*

$$\left[I^{*(n)}f\right](x) = e^{\left[I^{(n)}(\ln\circ f)\right](x)} \quad (21)$$

Additionally, we have the tanh-integral

$$[I^{\star(n)}f](x) = \frac{e^{2\int_a^b \ln\frac{1+f(x)}{1-f(x)}d^{(n)}x}-1}{e^{2\int_a^b \ln\frac{1+f(x)}{1-f(x)}d^{(n)}x}+1}, n \in \mathbb{N}. \quad (22)$$

4 Definitions of fractional φ- and biφ-calculi

For $\Re(\alpha) > 0$, the Riemann-Liouville integral is given by,

$$[I^{(\alpha)}f](x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt.$$

Moreover, for $n-1 \leq \Re(\alpha) < n$, $n := \lceil\alpha\rceil$, by analytic continuation of the RL-integral to $\Re(\alpha) \leq 0$, the Riemann-Liouville fractional derivative is given by

$$[D^{(\alpha)}f](x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}f(t)dt.$$

Whereas the Caputo derivative is given by,

$$[{}^C D^{(\alpha)}f](x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt.$$

Definition 32 *For* $\Re(\alpha) > 0$, *define the* φ-*Gamma function by*

$$\Gamma_\varphi(\alpha) = \int_0^{\lim_{t\to\infty}\varphi(t)} \varphi(x^{\alpha-1}e^{-x})d^\varphi x = \varphi\left(\int_0^\infty x^{\alpha-1}e^{-x}dx\right).$$

The definitions that will be discussed in this section are dealt with in a similar fashion to that logic used in the definitions of Riemann-Liouville and Caputo. Indeed, we have the following definitions:

Definition 33 *Let* $f: \mathbb{R} \to \mathbb{I}$, *where* $\mathbb{I}$ *is a* φ-*interval. The* φ- *fractional Riemann-Liouville integral of order* $\Re(\alpha) > 0$ *is denoted and given by*

$$[I^{\varphi(\alpha)}f](x) = \int_a^x [\varphi(x) \ominus \varphi(t)]^{\otimes(\alpha-1)} \otimes f(t) \oslash \Gamma_\varphi(\alpha)d^\varphi t.$$

It is easy to see that

$$[I^{\varphi(\alpha)}f](x) = \varphi\left[\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi^{-1}\circ f(t)dt\right]$$

$$= \varphi\big[I^{(\alpha)}(\varphi^{-1}\circ f)(x)\big].$$

As an example, the fractional multiplicative Riemann-Liouville integral of order $\Re(\alpha) > 0$ is defined as follows:

$$[I^{*(\alpha)}f](x) = \exp\left[\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}(\ln \circ f)(t)dt\right].$$

(Check) Moreover, the tanh- fractional integral is denoted and given by

$$[I^{\star(\alpha)}f](x) = \frac{e^{2\int_a^x \ln\frac{1+f(x)}{1-f(x)}d^{(\alpha)}x} - 1}{e^{2\int_a^x \ln\frac{1+f(x)}{1-f(x)}d^{(\alpha)}x} + 1}.$$

Proposition 34 *The φ- fractional Riemann-Liouville integral operator satisfies the property*

$$I^{\varphi(\alpha)} \otimes I^{\varphi(\beta)}f = I^{\varphi(\alpha+\beta)}f = I^{\varphi(\beta)} \otimes I^{\varphi(\alpha)}f.$$

Proof. For $\Re(\alpha), \Re(\beta) > 0$

$$I^{\varphi(\alpha)} \otimes I^{\varphi(\beta)}f(x)$$

$$= \varphi\left[I^{(\alpha)}(\varphi^{-1}\circ f)(x)\right] \otimes \varphi\left[I^{(\beta)}(\varphi^{-1}\circ f)(x)\right]$$

$$= \varphi\left[I^{(\alpha)}(\varphi^{-1}\circ f)(x) I^{(\beta)}(\varphi^{-1}\circ f)(x)\right]$$

$$= \varphi\left[I^{(\alpha)}I^{(\beta)}(\varphi^{-1}\circ f)(x)\right]$$

$$= \varphi\left[I^{(\alpha+\beta)}(\varphi^{-1}\circ f)(x)\right]$$

$$= I^{\varphi(\alpha+\beta)}f(x).$$

Similarly,

$$I^{\varphi(\alpha)} \otimes I^{\varphi(\beta)}f(x) = I^{\varphi(\beta+\alpha)}f(x) = I^{\varphi(\alpha+\beta)}f(x).$$

This proves the assertion.

Definition 35 *Let $f: \mathbb{I} \to \mathbb{I}$, where $\mathbb{I}$ is a φ-interval. The biφ- fractional Riemann-Liouville integral of order $\Re(\alpha) > 0$ is denoted and given by*

$$[I_a^{bi\varphi(\alpha)}f](x) = \int_a^x [x \ominus t]^{\otimes(\alpha-1)} \otimes f(t) \oslash \Gamma_\varphi(\alpha) d^{bi\varphi}t.$$

It is easy to see that

$$[I_a^{bi\varphi(\alpha)}f](x) = \varphi\left[\frac{1}{\Gamma(\alpha)}\int_{\varphi^{-1}(a)}^{\varphi^{-1}(x)} (\varphi^{-1}(x) - s)^{\alpha-1}(\varphi^{-1}\circ f\circ\varphi)(s)ds\right]$$

$$= \varphi\left[\frac{1}{\Gamma(\alpha)}\int_a^x (\varphi^{-1}(x) - \varphi^{-1}(t))^{\alpha-1}(\varphi^{-1}\circ f)(t)\frac{d}{dt}\varphi^{-1}(t)dt\right]$$

$$= \varphi\left[I_{\varphi^{-1}(a)}^{(\alpha)}(\varphi^{-1}\circ f\circ\varphi)(\varphi^{-1}(x))\right].$$

Proposition 36 *The biφ- fractional Riemann-Liouville integral operator satisfies the property*

$$I_a^{bi\varphi(\alpha)} \otimes I_a^{bi\varphi(\beta)}f = I_a^{bi\varphi(\alpha+\beta)}f = I_a^{bi\varphi(\beta)} \otimes I_a^{\varphi(\alpha)}f.$$

Proof. For $\Re(\alpha), \Re(\beta) > 0$, one has

$$I_a^{bi\varphi(\alpha)} \otimes I_a^{bi\varphi(\beta)}f(x)$$

$$= \varphi\left[I_{\varphi^{-1}(a)}^{(\alpha)}(\varphi^{-1}\circ f\circ\varphi)(\varphi^{-1}(x))\right] \otimes \varphi\left[I_{\varphi^{-1}(a)}^{(\beta)}(\varphi^{-1}\circ f\circ\varphi)(\varphi^{-1}(x))\right]$$

$$= \varphi\left[I_{\varphi^{-1}(a)}^{(\alpha)}(\varphi^{-1}\circ f\circ\varphi)(\varphi^{-1}(x)) I_{\varphi^{-1}(a)}^{(\beta)}(\varphi^{-1}\circ f\circ\varphi)(\varphi^{-1}(x))\right]$$

$$= \varphi\left[I_{\varphi^{-1}(a)}^{(\alpha+\beta)}(\varphi^{-1}\circ f\circ\varphi)(\varphi^{-1}(x))\right]$$

$$= I_a^{bi\varphi(\alpha+\beta)} f(x).$$

The other part is similar.

Definition 37 *Let $n-1<\alpha<n$ and $f:\mathbb{R}\to\mathbb{I}$, where $\mathbb{I}$ is a φ-interval. The φ- fractional Riemann-Liouville derivative of order α is denoted and given by*

$$[D^{\varphi(\alpha)}f](x)=\frac{d^{\varphi(n)}}{dx^n}\int_a^x[\varphi(x)\ominus\varphi(t)]^{\otimes(n-1-\alpha)}\otimes\frac{f(t)}{\Gamma_\varphi(\alpha)d^\varphi t}.$$

It is easy to see that

$$[D^{\varphi(\alpha)}f](x)=\frac{d^{\varphi(n)}}{dx^n}\varphi\left(\frac{1}{\Gamma(n-\alpha)}\int_a^x(x-t)^{n-\alpha-1}\varphi^{-1}\circ f(t)dt\right)$$

$$=\varphi\left(\frac{d^n}{dx^n}\frac{1}{\Gamma(n-\alpha)}\int_a^x(x-t)^{n-\alpha-1}\varphi^{-1}\circ f(t)dt\right)$$

$$=\varphi[D^{(\alpha)}(\varphi^{-1}\circ f)(x)]. \quad (25)$$

As examples we have the multiplicative and tanh- versions, respectively, which are denoted and given by

$$[D^{*(\alpha)}f](x)=\frac{d^{*(n)}}{dx^n}\int_a^x(f(t)^{\frac{(x-t)^{n-\alpha-1}}{\Gamma(n-\alpha)}})^{dt},\Re(n-\alpha)>0, \quad (26)$$

$$[D^{\star(\alpha)}f](x)=\frac{d^{\star(n)}}{dx^n}\frac{e^{2\int_a^b\ln\frac{1+f(x)}{1-f(x)}d^{(n-\alpha)}x}-1}{e^{2\int_a^b\ln\frac{1+f(x)}{1-f(x)}d^{(n-\alpha)}x}+1},\Re(n-\alpha)>0. \quad (27)$$

Definition 38 *Let $n-1<\alpha<n$ and $f:\mathbb{I}\to\mathbb{I}$, where $\mathbb{I}$ is a φ-interval. The biφ- fractional Riemann-Liouville derivative of order α is denoted and given by*

$$[D_a^{bi\varphi(\alpha)}f](x)=\frac{d^{bi\varphi(n)}}{dx^n}\int_a^x[x\ominus t]^{\otimes(n-1-\alpha)}\otimes f(t)\oslash\Gamma_\varphi(\alpha)d^{bi\varphi}t.$$

It is easy to see that

$$[D_a^{bi\varphi(\alpha)}f](x)=\frac{d^{bi\varphi(n)}}{dx^n}\varphi\left(\frac{1}{\Gamma(n-\alpha)}\int_{\varphi^{-1}(a)}^{\varphi^{-1}(x)}(\varphi^{-1}(x)-\varphi^{-1}(t))^{n-\alpha-1}\varphi^{-1}\circ f(t)dt\right)$$

$$=\varphi\left(\frac{d^n}{dx^n}\frac{1}{\Gamma(n-\alpha)}\int_a^x(\varphi^{-1}(x)-\varphi^{-1}(t))^{n-\alpha-1}\varphi^{-1}\circ f(t)dt\right)$$

$$=\varphi\left[D_{\varphi^{-1}(a)}^{(\alpha)}(\varphi^{-1}\circ f)\left(\varphi^{-1}(x)\right)\right]. \quad (28)$$

Definition 39 *The φ- fractional Caputo derivative of order α is denoted and given by*

$$[{}^CD^{\varphi(\alpha)}f](x)=\varphi(\frac{1}{\Gamma(n-\alpha)}\int_a^x(x-t)^{\alpha-1}\varphi^{-1}(f^{\varphi(n)}(t))dt),$$

$$\Re(n-\alpha)>0.$$

$$=\varphi([{}^CD^{(\alpha)}\varphi^{-1}\circ f](x)) \quad (29)$$

As examples,

$[{}^{C}D^{*(\alpha)}f](x) =$
$\int_a^x (f^{*(n)}(t)^{\frac{(x-t)^{n-\alpha-1}}{\Gamma(n-\alpha)}})^{dt}, \Re(n-\alpha) > 0.$ (30)

$$[{}^{C}D^{\star(\alpha)}f](x) = \frac{e^{2\int_a^x \ln\frac{1+\left(f^{\star(n)}(x)\right)}{1-\left(f^{\star(n)}(x)\right)} d^{(n-\alpha)}x} - 1}{e^{2\int_a^x \ln\frac{1+\left(f^{\star(n)}(x)\right)}{1-\left(f^{\star(n)}(x)\right)} d^{(n-\alpha)}x} + 1},$$

$\Re(n-\alpha) > 0.$ (31)

Theorem (20) paves the way for the biφ-fractional calculi without the use of heavy machinery. It is an immediate out-growth of it in some sense. In the following definitions, the subscript $\varphi^{-1}(t)$ is to clarify that the operations are carried out with respect to $\varphi^{-1}(t)$.

Definition 40 *The biφ- fractional integral of order α is denoted and given by*

$$[I^{\mathrm{bi}\varphi(\alpha)}f](x)$$
$$= \varphi(\frac{1}{\Gamma(\alpha)}\int_a^x (\varphi^{-1}(x) - \varphi^{-1}(t))^{\alpha-1}\varphi^{-1}(f(\varphi^{-1}(t)))d\varphi^{-1}(t)),$$
$$Re(\alpha) > 0.$$
$$= \varphi([I^{(\alpha)}_{\varphi^{-1}(t)}f](x))$$

As examples,

$[I^{\pi(\alpha)}f](x) =$
$\int_a^x (f(\ln t)^{\frac{(\ln\frac{x}{t})^{\alpha-1}}{\Gamma(\alpha)}})^{d\ln t}, \Re(\alpha) > 0.$ (33)

$[I^{\mathrm{bi}\star(\alpha)}f](x) =$
$\int_a^x f(\tanh^{-1}t)d^{\star(\alpha)}\tanh^{-1}t, \Re(\alpha) > 0.$ (34)

This notation in (34) is just an abbreviation, since equation (32) is rather tedious. It means that all the arguments would change from $t, x \to \tanh^{-1}(t), \tanh^{-1}(x)$ everywhere except at the boundaries of integration.

Definition 41 *The biφ- fractional Riemann-Liouville derivative of order α is denoted and given by*

$$[D^{\mathrm{bi}\varphi(\alpha)}f](x)$$
$$= \frac{d^{\mathrm{bi}\varphi(n)}}{d\varphi^{-1}(x)^n}\varphi(\frac{1}{\Gamma(n-\alpha)}\int_a^x (\varphi^{-1}(x) - \varphi^{-1}(t))^{n-\alpha-1}\varphi^{-1}(f(\varphi^{-1}(t)))d\varphi^{-1}(t)).$$
$$= \varphi([D^{(\alpha)}_{\varphi^{-1}(t)}f](x)), \Re(n-\alpha) > 0.$$

As examples,

$$[D^{\pi(\alpha)}f](x) = \frac{d^{\pi(n)}}{d\ln x^n}\int_a^x (f(\ln t)^{\frac{(\ln\frac{x}{t})^{n-\alpha-1}}{\Gamma(n-\alpha)}})^{d\ln t},$$

$\Re(n-\alpha) > 0$ (36)

$[D^{\mathrm{bi}\star(\alpha)}f](x) =$
$\frac{d^{\mathrm{bi}\star(n)}}{d\tanh^{-1}x^n}\int_a^x f(\tanh^{-1}t)d^{\star(\alpha)}\tanh^{-1}t\,.$ (37)

(32)

Definition 42 *The biφ- fractional Caputo derivative of order α is denoted and given by*

$$[{}^{C}D^{\mathrm{bi}\varphi(\alpha)}f](x) = \varphi(\tfrac{1}{\Gamma(n-\alpha)}\int_a^x (\varphi^{-1}(x) - \varphi^{-1}(t))^{\alpha-1}\varphi^{-1}(f^{\mathrm{bi}\varphi(n)}(\varphi^{-1}(t)))d\varphi^{-1}(t))$$
$$= \varphi([{}^{C}D^{(\alpha)}_{\varphi^{-1}(t)}f](x)), \Re(n-\alpha) > 0$$

Remark 43 *One can see that the Hadamard fractional derivative is the logarithm of the* *bigeometric RL derivative of e^f. That is, it is a RL derivative on the manifold $\mathbb{I}$ under the diffeomorphism φ. This hints that many of the fractional derivatives that are defined may indeed be a RL derivative on a given manifold, from a differential geometric point of view.*

$[\,^{C}D^{\pi(\alpha)}f](x) =$

$$\int_a^x (f^{\pi(n)}(\ln t)^{\frac{(\ln\frac{x}{t})^{n-\alpha-1}}{\Gamma(n-\alpha)}})^{d\ln t}, \qquad (40)$$

$[\,^{C}D^{\mathrm{bi}\star(\alpha)}f](x) =$

$$\int_a^x f^{\star(n)}(\tanh^{-1}t)d^{\star(\alpha)}\tanh^{-1}t \qquad (41)$$

These definitions also allows one to calculate various φ- and biφ-fractional derivatives, and integrals. Hence, all the results that holds true for Caputo and Riemann-Liouville differintegrals are also true under the influence of the homeomorphism φ. For a more general case, consider a function $f: \mathbb{I}_1 \to \mathbb{I}_2$, where $\mathbb{I}_1, \mathbb{I}_2 \subseteq \mathbb{R}$ are ordered fields equipped with the usual metric and their field structure are based on the algebraic operations similar to those in the beginning of the second section under the influence of the homeomorphisms $\varphi: \mathbb{R} \to \mathbb{I}_1$, $\eta: \mathbb{R} \to \mathbb{I}_2$. Then, we can define the bi(φ,η)-derivative in a similar fashion, where

$$f^{\mathrm{bi}(\varphi,\eta)}(x) = \lim_{y\to x}\eta\left(\frac{\eta^{-1}\circ f(x) - \eta^{-1}\circ f(y)}{\varphi^{-1}(x) - \varphi^{-1}(y)}\right)$$

$$= \eta\left(\frac{(\eta^{-1}\circ f)'(x)}{\varphi^{-1}(x)'}\right) \qquad (42)$$

And the bi(φ,η)-integral,

$$\int_a^b f(x)d^{\mathrm{bi}(\varphi,\eta)}x = \eta\left(\int_a^b \varphi^{-1}(x)'\eta^{-1}\circ f(x)dx\right) \qquad (43)$$

Remark 44 *The domains of the homeomorphisms may be a subset of the real numbers.*

Example 45 *Consider $\varphi: \mathbb{R} \to \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $\eta: \mathbb{R} \to (0,\infty)$ defined by $\varphi(x) = arctanx$, $\eta(x) = e^x$. Then we have,*

$$f^{\mathrm{bi}(\varphi,\eta)}(x) = \exp\left(\frac{f('x)}{f(x)sec^2(x)}\right)$$

and,

$$\int_a^b f(x)d^{\mathrm{bi}(\varphi,\eta)}x = \exp\left(\int_a^b sec^2(x)\ln f(x)dx\right)$$

Where the tangent function is on domain $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Remark 46 *Many other fractional calculi may be defined under the influence of a diffeomorphism defined as a composition of finitely many diffeomorphisms.*

5 Conclusion

In this paper, the very basic definitions of fractional calculus are established in the relatively new φ-calculi and biφ-calculi, which are promising to be of great use. Indeed, they give an interpretation of the so-called ψ −fractional calculus under the scope of the discussed subject, as Remark 43 mentions. This paper also reveals a new form of the discussed calculi as seen in the first section which is useful in proofs. We have also arrived at an important link which in future papers will make establish relations between φ-calculi and biφ-calculi in a smooth and practical way as well as a relation to the Newtonian versions, where various analogs such as the φ −gamma function.

References:

[1] Kochubei A, Luchko Y, Handbook of Fractional Calculus with Applications, de Gruyter, Berlin, 2019.

[2] Agamirza Bashirov, Emine Mısırlı, Y¨ucel Tando˘gdu, and Ali ¨Ozyapıcı. On modeling with multiplicative differential equations. Applied Mathematics-A Journal of Chinese Universities, 26(4):425–438, 2011. Michael Grossman and Robert Katz. Non-Newtonian Calculus. Non-Newtonian Calculus, 1972.

[3] Dick Stanley. A multiplicative calculus. Problems, Resources, and Issues in Mathematics Undergraduate Studies, 9(4):310–326, 1999.

[4] Agamirza Bashirov, Emine Kurpınar, and Ali ¨Ozyapıcı. Multiplicative calculus and its applications. Journal of mathematical analysis and applications, 337(1):36–48, 2008.

[5] Agamirza E Bashirov and Sajedeh Norozpour. On an alternative view to complex calculus. Mathematical Methods in the Applied Sciences, 41(17):7313–7324, 2018.

[6] Svetlin G Georgiev. Focus on Calculus. Nova Science, Hauppauge, NY, May 2020.

[7] Raymond Mortini. The f`aa di bruno formula revisited. Elemente der Mathematik, 68(1):33–38, 2013. 15

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Both authors contributed equally.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding is granted.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.