Nonexistence Results of Global Solutions for Fractional Order Integral Equations on the Heisenberg Group

ABD ELHAKIM LAMAIRIA
Department of Mathematics and Informatics
University of Tebessa
ALGERIA

Abstract: We consider the fractional order integral equation with a time nonlocal nonlinearity $^{c}\mathbf{D}^{\beta}_{0|t}(u)+(-\Delta_{\mathbb{H}})^{m}(u)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\omega)^{\alpha-1}|u(\omega)|^p d\omega$, posed in $(.,t)\in\mathbb{H}\times(0,\infty)$, supplemented with an initial data $u(.,0)=u_0(.)$, where $m>1$, $p>1$, $0<\beta<1$, $0<\alpha<1$, and $^{c}\mathbf{D}^{\beta}_{0|t}$ denotes the caputo fractional derivative of order β, and $\Delta_{\mathbb{H}}$ is the Laplacian operator on the $(2N+1)$-dimensional. Heisenberg group $\mathbb{H}$. Then, we prove a blow up result for its solutions.

Keywords: Riemann-Liouville, Heisenberg group, Laplace operator, Hilbert, space

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1. Introduction

In this paper, we investigate the higher-order semilinear parabolic equation with nonlocal in time nonlinearity of the following form:

$$\begin{cases} ^{c}\mathbf{D}^{\beta}_{0|t}(u)+(-\Delta_{\mathbb{H}})^{m}(u)=\mathbf{I}^{\alpha}_{0|t}|u(t)|^p, \\ \eta=(x,y,\tau)\in\mathbb{H},\ t>0 \end{cases} \tag{1.1}$$

subject to the initial data

$$u(\eta,0)=u_0(\eta),$$

Where $\mathbf{I}^{\alpha}_{0|t}\psi$ is the Riemann–Liouville fractional integral of order $(0<\alpha<1)$ defined for a continuous function $\psi(t), t>0$,

$$(\mathbf{I}^{\alpha}_{0|t}\psi)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\omega)^{\alpha-1}\psi(\omega)d\omega,$$

Here, $\Gamma(.)$ stands for the gamma function.
First, for the sake of the reader, we give some known facts about the Heisenberg group $\mathbb{H}$ and the operator $\Delta_{\mathbb{H}}$. For their proof and more information, we refer for example to $[1,4,5,11,19]$. The Heisenberg group $\mathbb{H}$, whose elements are $\eta=(x,y,\tau)$ is the Lie group $(\mathbb{R}^{2N+1},\circ)$ with the group operation " $\circ$ " defined by

$$\eta\circ\tilde{\eta}=(x+\tilde{x},y+\tilde{y},\tau+\tilde{\tau}+2(<x,\tilde{y}>-<\tilde{x},y>)),$$

where $<.,.>$ is the usual inner product in $\mathbb{R}^N$, The laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ is obtained from the vector fields $X_i=\partial_{x_i}+2y_i\partial_\tau$ and $Y_i=\partial_{y_i}+2x_i\partial_\tau$, by

$$\Delta_{\mathbb{H}}=\sum_{i=1}^N\left(X_i^2+Y_i^2\right),$$

explicitly, we have

$$\Delta_{\mathbb{H}}=\sum_{i=1}^N\left(\frac{\partial^2}{\partial x_i^2}+\frac{\partial^2}{\partial y_i^2}+4y_i\frac{\partial^2}{\partial x_i\partial\tau}-4x_i\frac{\partial^2}{\partial y_i\partial\tau}+4(x_i^2+y_i^2)\frac{\partial^2}{\partial\tau^2}\right),$$

A natural group of dilitations on $\mathbb{H}$ is given by

$$\delta_\gamma(\eta)=\left(\gamma x,\gamma y,\gamma^2\tau\right),\ \gamma>0,$$

whose Jacobian determinant is γ^Q where

$$Q=2N+2$$

is the homogeneous dimension of $\mathbb{H}$.
The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of $\mathbb{H}$ and homogeneous with respect to the dilatations δ_γ. More precisely, we have

$$\Delta_{\mathbb{H}}(u(\eta\circ\tilde{\eta}))=(\Delta_{\mathbb{H}}u)(\eta\circ\tilde{\eta}),\ \Delta_{\mathbb{H}}(u\circ\delta_\gamma)=\gamma^2(\Delta_{\mathbb{H}}u)\circ\delta_\gamma\ \ \eta,\tilde{\eta}\in\mathbb{H}.$$

The natural distance from η to the origin is

$$|\eta|_{\mathbb{H}}=\left(\tau^2+\left(\sum_{i=1}^N\left(x_i^2+y_i^2\right)\right)^2\right)^{\frac{1}{4}}.$$

Now,we call sub-elliptic gradient

$$\nabla_{\mathbb{H}}=(X,Y)=(X_1,...,X_N,Y_1,...,Y_N),$$

A remarkable property of the Kohn Laplacian is that a fundamental solution of $-\Delta_{\mathbb{H}}$ with pole at zero is given by

$$\Gamma(\eta)=\frac{C_\Lambda}{|\eta|_{\mathbb{H}}^{\Lambda-2}},$$

where C_Λ is a suitable positive constant.
A basic role in the functional analysis on the Heisenberg group is played by the following Sobolev-type inequality

$$\|v\|^2_{\Lambda^*}=c\|\nabla_{\mathbb{H}}v\|_2^2,\forall v\in C_0^\infty(\mathbb{H}^N),$$

where $\Lambda^* = \frac{2\Lambda}{\Lambda-2}$ and c is a positive constant.
This inequality ensures in particular that for every domain Ω the function

$$\|v\| \leq \|\nabla_{\mathbb{H}} v\|_2,$$

is a norm on $C_0^\infty(\Omega)$. We denote by $S_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to this norm; $S_0^1(\Omega)$ becomes a Hilbert space with the inner product

$$< u, v >_{S_0^1} = \int_\Omega < \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} v >,$$

Fractional powers of sub-elliptic Laplacians. Here, we recall a result on fractional powers of sub-Laplacian in the Heisenberg group. Let $N(t,x)$ be the fundamental solution of $\Delta_{\mathbb{H}} + \frac{\partial}{\partial t}$. For all $0 < \beta < 4$, the integral

$$R_\beta(x) = \frac{1}{\Gamma\left(\frac{\beta}{2}\right)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} N(t,x)dt,$$

converges absolutely for $x \neq 0$. If $\beta < 0, \beta \neq 0, -2, -4, \ldots$,then

$$\tilde{R}_\beta(x) = \frac{\frac{\beta}{2}}{\Gamma\left(\frac{\beta}{2}\right)} \int_0^{+\infty} t^{\frac{\beta}{2}-1} N(t,x)dt,$$

defines a smooth function in $\mathbb{H} - \{0\}$, since $t \mapsto N(t,x)$, vanishes of infinite order as $t \to 0$ if $x \neq 0$. In addition, $\tilde{R}_\beta$ is positive and $\mathbb{H}$-homogeneous of degree $\beta - 4$.

Theorem:
For every $v \in S(\mathbb{H})$ (Schwartz's class), we have $(-\Delta_{\mathbb{H}})^s \in L^2(\mathbb{H})$ and

$$(-\Delta_{\mathbb{H}})^s = \int_{\mathbb{H}} (v(x \circ y) - v(x) - \chi(y) < \nabla_{\mathbb{H}} v(x), y >) \tilde{R}_{-2s}(y)dy,$$

where χ is the characteristic function of the unit ball $B_\rho(0,1)$, ($\rho(x) = R_{2-\alpha}^{\frac{-1}{2-\alpha}}(x)$, $0 < \alpha < 2$, ρ is an $\mathbb{H}$-homogeneous norm in $\mathbb{H}$ smooth outside the origin).

2. Preliminaries

2.1 Definition

(Riemann-Liouville fractional derivatives)

Let $f \in AC[a,b], -\infty < a < b < +\infty$,[1] The Riemann-Liouville left- and right-sided fractional derivatives of order $\alpha \in (0,1)$ are, respectively, defined by

$$\mathbf{D}_{a|t}^\alpha f(t) = \frac{d}{dt} \mathbf{I}_{a|t}^{1-\alpha} f(t)$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} f(\tau)d\tau, \quad t > a \tag{2.1}$$

[1] let $AC[a,b]$ be the space of functions f which are absolutely continuous on $[a,b]$.
$AC^n[a,b] = \left\{ f : [a,b] \to \mathbb{C} \text{ and } \left(D^{n-1}f\right)(x) \in AC[a,b] \quad \left(D = \frac{d}{dx}\right)\right\}$.
In particular, $AC^1[a,b] = AC[a,b]$,

and

$$\mathbf{D}_{t|b}^\alpha f(t) = -\frac{d}{dt} \mathbf{I}_{t|b}^{1-\alpha} f(t)$$

$$= -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (\tau - t)^{-\alpha} f(\tau)d\tau, \quad t < b \tag{2.2}$$

2.2 Definition

(Riemann-Liouville fractional integrals)

Let $f \in L^1(a,b), -\infty < a < b < +\infty$, The Riemann-Liouville left- and right-sided fractional integrals of order $\alpha \in (0,1)$ are, respectively, defined by

$$\mathbf{I}_{a|t}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{-(1-\alpha)} f(\tau)d\tau, \quad t > a \tag{2.3}$$

and

$$\mathbf{I}_{t|b}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{-(1-\alpha)} f(\tau)d\tau, \quad t < b \tag{2.4}$$

2.3 Definition

For $0 < \alpha < 1$, the Caputo derivative of order α for a differentiable function $f : [0, \infty) \to \mathbb{R}$ can been written as

$${}^c\mathbf{D}_{a|t}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} f'(\tau)d\tau, \ t > a \tag{2.5}$$

It is clear that

$${}^c\mathbf{D}_{a|t}^\alpha f(t) = \mathbf{D}_{a|t}^\alpha \left[f(t) - f(0)\right],$$

Finally, taking into account the following integration by parts formula:

$$\int_a^b f(t)\, \mathbf{D}_{a|t}^\alpha g(t)\, dt = \int_a^b \mathbf{D}_{t|b}^\alpha f(t)\, g(t)\, dt.$$

2.4 Proposition

For $0 < \alpha < 1, -\infty < a < b < +\infty$, we have the following identities

$$\mathbf{D}_{a|t}^\alpha \mathbf{I}_{a|t}^\alpha f(t) = f(t), \ t \in (a,b)$$

for all $f \in L^r(a,b), 1 \leq r \leq \infty$
and

$$-\mathbf{D}\mathbf{D}_{t|b}^\alpha f = \mathbf{D}_{t|b}^{1+\alpha} f,$$

for all $f \in AC^2[a,b]$, where $\mathbf{D} = \frac{d}{dt}$.

For $\rho \gg 1$ and $0 < \alpha < 1$. Let

$$f(t) = \begin{cases} \left(1 - \frac{t}{T}\right)^\rho, & 0 < t \leq T, \\ 0, & t \geq T, \end{cases} \tag{2.6}$$

$$\mathbf{D}_{t|T}^\alpha f(t) = \frac{(1-\alpha+\rho)\Gamma(\rho-1)}{\Gamma(2-\alpha-\rho)} T^{-\alpha} \left(1 - \frac{t}{T}\right)^{\rho-\alpha},$$

and

$$\int_0^T f(t)^{\frac{-p'}{p}} |\mathbf{D}_{t|T}^\alpha f(t)|^{p'} = CT^{1-p'\alpha},$$

3. Nonexistence Results

3.1 Definition

(Weak solution).Let $T > 0$, a locally integrable function $u \in C\left([0,T], L^1_{loc}(Q_T) \cap L^p_{loc}(Q_T)\right)$ is called a local weak solution of (1.1) in Q_T ($Q_T = \mathbb{H} \times [0,T]$) subject to the initial data $u_0 \in L^1_{loc}(\mathbb{H})$ if the equality

$$\int_{Q_T} u_0 \mathbf{D}^{\beta}_{t|T}\varphi d\omega + \int_{Q_T} \varphi \mathbf{I}^{\alpha}_{0|t}|u|^p d\omega = \int_{Q_T} u\mathbf{D}^{\beta}_{t|T}\varphi d\omega + \int_{Q_T} u(-\Delta_{\mathbb{H}})^m \varphi d\omega$$

, is satisfied for any φ be a smooth test function $\varphi \in C_0^\infty(Q_T))$ witn

$$\varphi(.,T) = 0, \quad \varphi \geq 0, \quad d\omega = d\eta dt$$

and the solution is called global if $T = +\infty$.

3.2 Theorem

Let $p > 1$, and

$$p < p_c = \frac{(2N+2)\beta + 2m}{(2N+2)\beta + 2m(1-\alpha)},$$

(c for critical)

Then, (1.1) does not have a nontrivial global weak solution.

3.3 Proposition

Consider a convex function $F \in C^2(\mathbb{R})$. Assume that $\varphi \in C_0^\infty(\mathbb{R}^{2N+1})$, then

$$F'(\varphi)(-\Delta_{\mathbb{H}})^m\varphi \geq (-\Delta_{\mathbb{H}})^m F(\varphi),$$

In particular, if $F(0) = 0$ and $\varphi \in C_0^\infty(\mathbb{R}^{2N+1})$, then

$$\int_{\mathbb{R}^{2N+1}} F'(\varphi)(-\Delta_{\mathbb{H}})^m\varphi d\eta \geq 0.$$

Let us mention that hereafter we will use inequality (2.1) for $F(\varphi) = \varphi^l, \quad l \gg 1, \quad \varphi \geq 0$, in this case it reads

$$l\varphi^{l-1}(-\Delta_{\mathbb{H}})^m\varphi \geq (-\Delta_{\mathbb{H}})^m\varphi^l, \tag{1}$$

We need the following Lemma taken from [32].

3.4 Lemma

Let $f \in L^1(\mathbb{R}^{2N+1})$ and $\int_{\mathbb{R}^{2N+1}} f d\eta \geq 0$. Then there exists a test function $0 \leq \varphi \leq 1$, such that

$$\int_{\mathbb{R}^{2N+1}} f\varphi d\eta \geq 0. \tag{2}$$

Proof of theorem :.

The proof is done by contradiction. Suppose that u is a global bounded weak solution. First we Choose the test function. For this aim, we shall use a non-negative smooth function ϕ which was constructed in [20].

$$\phi(\xi) = \begin{cases} 1 & if \quad 0 \leq \xi \leq 1. \\ \searrow & if \quad 1 \leq \xi \leq 2, \\ 0 & if \quad \xi \geq 2, \end{cases} \tag{3}$$

$$\varphi_1(\eta) = \phi\left(\frac{\tau^2 + |x|^4 + |y|^4}{R^4}\right), \quad \eta = (x,y,\tau) \in \mathbb{H}$$

$$\varphi_2(t) = \begin{cases} \left(1 - \dfrac{t}{T}\right)^\rho, & 0 < t \leq T, \\ 0, & t \geq T, \end{cases} \quad \rho \gg 1$$

$$\varphi(\eta,t) = \mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\tilde{\varphi}(\eta,t) = \varphi_1^l(\eta)\, \mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\varphi_2\left(\frac{t}{R^{\frac{2m}{\beta}}}\right), \quad R > 0$$

Let, $Q = \mathbb{H} \times \left[0, TR^{\frac{2m}{\beta}}\right]$,

Using the Definition 3.1, we obtain

$$\int_Q u_0 \mathbf{D}^{\beta}_{t|TR^{\frac{2m}{\beta}}} \mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\tilde{\varphi}(\eta,t)d\eta dt + \int_Q \mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\tilde{\varphi}(\eta,t)\mathbf{I}^{\alpha}_{0|t}|u|^p d\eta dt$$
$$= \int_Q u\mathbf{D}^{\beta}_{t|TR^{\frac{2m}{\beta}}} \mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\tilde{\varphi}(\eta,t)d\eta dt + \int_Q u(-\Delta_{\mathbb{H}})^m \mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\tilde{\varphi}(\eta,t)d\eta dt,$$

A simple computation yields

$\mathbf{D}^{\beta}_{t|TR^{\frac{2m}{\beta}}}\left(\mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\tilde{\varphi}\right) = \mathbf{D}^{\alpha+\beta}_{t|TR^{\frac{2m}{\beta}}}\tilde{\varphi}$, we obtain

$$c(TR^{\frac{2m}{\beta}})^{1-(\alpha+\beta)}\int_{\mathbb{H}} u_0\varphi_1^l(\eta)d\eta + \int_Q \tilde{\varphi}|u|^p d\eta dt$$
$$= \int_Q u\varphi_1^l(\eta)\mathbf{D}^{\alpha+\beta}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})d\eta dt + \int_Q u(-\Delta_{\mathbb{H}})^m\varphi_1^l(\eta)\mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})d\eta dt,$$

The application of inequality (3.1)

$$l\varphi_1^{l-1}(-\Delta_{\mathbb{H}})^m\varphi_1 \geq (-\Delta_{\mathbb{H}})^m\varphi_1^l,$$

implies that

$$\int_Q |u|^p\tilde{\varphi}d\eta dt$$
$$\leq l\int_Q u\varphi_1^{l-1}(\eta)(-\Delta_{\mathbb{H}})^m\varphi_1(\eta)\mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})d\eta dt + \int_Q u\varphi_1^l(\eta)\mathbf{D}^{\alpha+\beta}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})d\eta dt,$$

For estimating the second member of the above inequality, we write

$$\int_Q u\varphi_1^{l-1}(\eta)(-\Delta_{\mathbb{H}})^m\varphi_1(\eta)\mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})d\eta dt$$

$$= \int_Q u\tilde{\varphi}^{\frac{1}{p}}\varphi_1^{l-1}(\eta)(-\Delta_{\mathbb{H}})^m\varphi_1(\eta)\mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})\tilde{\varphi}^{\frac{-1}{p}}d\eta dt.$$

According to ϵ-Young inequality

$$XY \le \epsilon X^p + C(\epsilon)Y^{p'}, \quad p+p' = pp',$$

we have

$$\int_Q u\varphi_1^{l-1}(\eta)(-\Delta_{\mathbb{H}})^m\varphi_1(\eta)\mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})d\eta dt \le$$

$$\epsilon\int_Q |u|^p\tilde{\varphi}d\eta dt$$

$$+C_1(\epsilon)\int_Q \varphi_1^{(l-1)p'}(\eta)|(-\Delta_{\mathbb{H}})^m\varphi_1(\eta)\mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})|^{p'}\tilde{\varphi}^{\frac{-p'}{p}}d\eta dt,$$

In the same way, we get

$$\int_Q u\varphi_1^{l}(\eta)\mathbf{D}^{\alpha+\beta}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})d\eta dt \le$$

$$\epsilon\int_Q |u|^p\tilde{\varphi}d\eta dt$$

$$+C_2(\epsilon)\int_Q |\varphi_1^{l}(\eta)\mathbf{D}^{\alpha+\beta}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})|^{p'}\tilde{\varphi}^{\frac{-p'}{p}}d\eta dt,$$

Now, when ϵ is small, and $C = \max\{C_1(\epsilon), C_2(\epsilon)\}$ we obtain

$$\int_Q |u|^p\tilde{\varphi}d\eta dt \le$$

$$C\left\{\int_Q \varphi_1^{(l-1)p'}(\eta)|(-\Delta_{\mathbb{H}})^m\varphi_1(\eta)\mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})|^{p'}\tilde{\varphi}^{\frac{-p'}{p}}d\eta dt\right.$$

$$\left.+\int_Q |\varphi_1^{l}(\eta)\mathbf{D}^{\alpha+\beta}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})|^{p'}\tilde{\varphi}^{\frac{-p'}{p}}d\eta dt\right\},$$

as

$$\tilde{\varphi}^{\frac{-p'}{p}}(\eta,t) = \varphi_1^{\frac{-p'}{p}l}(\eta)\varphi_2^{\frac{-p'}{p}}(\frac{t}{R^{\frac{2m}{\beta}}}), \quad p' = \frac{p}{p-1}$$

we have

$$\int_Q |u|^p\tilde{\varphi}d\eta dt \le$$

$$C\left\{\int_Q \varphi_1^{(l-p')}(\eta)\varphi_2^{\frac{-1}{p-1}}(\frac{t}{R^{\frac{2m}{\beta}}})|(-\Delta_{\mathbb{H}})^m\varphi_1(\eta)\mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})|^{p'}d\eta dt\right.$$

$$\left.+\int_Q \varphi_1^{l}(\eta)\varphi_2^{\frac{-1}{p-1}}(\frac{t}{R^{\frac{2m}{\beta}}})|\mathbf{D}^{\alpha+\beta}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})|^{p'}d\eta dt\right\},$$

We apply the change of next variables $\tilde{\tau} = \frac{\tau}{R^2}$, $\tilde{x} = \frac{x}{R}$, $\tilde{y} = \frac{y}{R}$, $\tilde{t} = \frac{t}{R^{\frac{2m}{\beta}}}$, then we put

$$\Omega = \left\{\tilde{\eta} = (\tilde{x},\tilde{y},\tilde{\tau}) \in \mathbb{H};\ 0 \le \tilde{\tau}^2 + |\tilde{x}|^4 + |\tilde{y}|^4 \le 2\right\}$$

.

if we put

$$\mathcal{A} = \int_Q \varphi_1^{(l-p')}(\eta)\varphi_2^{\frac{-1}{p-1}}(\frac{t}{R^{\frac{2m}{\beta}}})$$

$$\times|(-\Delta_{\mathbb{H}})^m\varphi_1(\eta)\mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})|^{p'}d\eta dt,$$

$$\mathcal{B} = \int_Q \varphi_1^{l}(\eta)\varphi_2^{\frac{-1}{p-1}}(\frac{t}{R^{\frac{2m}{\beta}}})|\mathbf{D}^{\alpha+\beta}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})|^{p'}d\eta dt,$$

we get

$$\mathcal{A} = \int_0^{TR^{\frac{2m}{\beta}}} \varphi_2^{\frac{-1}{p-1}}(\frac{t}{R^{\frac{2m}{\beta}}})|\mathbf{D}^{\alpha}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})|^{p'}dt$$

$$\times\int_{\mathbb{H}} \varphi_1^{(l-p')}(\eta)|(-\Delta_{\mathbb{H}})^m\varphi_1(\eta)|^{p'}d\eta$$

$$= R^{\frac{2m}{\beta}-\frac{2mp\alpha}{\beta(p-1)}}$$

$$\times\int_0^T \varphi_2^{\frac{-p'}{p}}\left(\tilde{t}\right)|\mathbf{D}^{\alpha}_{\tilde{t}|T}\varphi_2\left(\tilde{t}\right)|^{p'}d\tilde{t}$$

$$\times R^{2N+2}$$

$$\times\int_\Omega \phi^{(l-p')}\left(\tilde{\tau}^2+|\tilde{x}|^4+|\tilde{y}|^4\right)|(-\Delta_{\mathbb{H}})^m\phi\left(\tilde{\tau}^2+|\tilde{x}|^4+|\tilde{y}|^4\right)|^{p'}d\tilde{x}d\tilde{y}d\tilde{\tau}$$

$$= CT^{1-\frac{p\alpha}{p-1}}R^{2N+2+\frac{2m}{\beta}-\frac{2mp\alpha}{\beta(p-1)}}$$

$$\times\int_\Omega \phi^{(l-p')}\left(\tilde{\tau}^2+|\tilde{x}|^4+|\tilde{y}|^4\right)|(-\Delta_{\mathbb{H}})^m\phi\left(\tilde{\tau}^2+|\tilde{x}|^4+|\tilde{y}|^4\right)|^{p'}d\tilde{x}d\tilde{y}d\tilde{\tau},$$

and

$$\mathcal{B} = \int_0^{TR^{\frac{2m}{\beta}}} \varphi_2^{\frac{-1}{p-1}}(\frac{t}{R^{\frac{2m}{\beta}}})|\mathbf{D}^{\alpha+\beta}_{t|TR^{\frac{2m}{\beta}}}\varphi_2(\frac{t}{R^{\frac{2m}{\beta}}})|^{p'}dt\int_{\mathbb{H}}\varphi_1^{l}(\eta)d\eta$$

$$= R^{\frac{2m}{\beta}-\frac{2mp(\alpha+\beta)}{\beta(p-1)}}\int_0^T \varphi_2^{\frac{-p'}{p}}\left(\tilde{t}\right)|\mathbf{D}^{\alpha+\beta}_{\tilde{t}|T}\varphi_2\left(\tilde{t}\right)|^{p'}d\tilde{t}$$

$$\times R^{2N+2}\int_\Omega \phi^{l}\left(\tilde{\tau}^2+|\tilde{x}|^4+|\tilde{y}|^4\right)d\tilde{x}d\tilde{y}d\tilde{\tau}$$

$$= CT^{1-\frac{p(\alpha+\beta)}{p-1}}R^{2N+2+\frac{2m}{\beta}-\frac{2mp(\alpha+\beta)}{\beta(p-1)}}$$

$$\times\int_\Omega \phi^{l}\left(\tilde{\tau}^2+|\tilde{x}|^4+|\tilde{y}|^4\right)d\tilde{x}d\tilde{y}d\tilde{\tau},$$

in the last

$$\int_Q |u|^p\tilde{\varphi}d\eta dt \le C\left\{\mathcal{A}+\mathcal{B}\right\} \le \mathcal{C}R^{2N+2+\frac{2m}{\beta}-\frac{2mp\alpha}{\beta(p-1)}} \quad (4)$$

Now, if $2N + 2 + \frac{2m}{\beta} - \frac{2mp\alpha}{\beta(p-1)} < 0 \Leftrightarrow p < p_c$ by letting $R \to +\infty$ in (3.4), we obtain

$$\int_Q |u|^p d\eta dt = 0 \Rightarrow u \equiv 0,$$

this is a contradiction.

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