Stability Property of the Boundary Equilibria of an Ecological Model of Mutualism Between Two Species with a Mortal Predator

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Abstract: - This article investigates the stability property of the boundary equilibria of an ecological model of mutualism between two species with a mortal predator. The model was proposed by Srinivasarao Tote (On an ecological model of mutualism between two species with a mortal predator, Applications and Applied Mathematics: An International Journal, 15(2)(2020): 1309-1322). We first give two numeric examples to show that the main results of Tote may not be correct. Then, by applying the standard comparison theorem, we obtain a set of sufficient conditions which ensure the global attractivity of the predator-washed state. We also demonstrate that the second mutual species washed state is unstable. Our results complement and supplement the main results of Srinivasarao Tote.

Key-Words: Predator; Prey; Equilibrium; Mutualism; Stability

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1 Introduction

Srinivasarao Tote [1] proposed the following ecological model of mutualism between two species with a mortal predator:

$$\begin{aligned} \frac{dx}{dt} &= x(1-x+\alpha_{12}y), \\ \frac{dy}{dt} &= ry(1-y+\alpha_{21}x)-\frac{(1-p)yz}{v_1+(1-p)y}, \\ \frac{dz}{dt} &= z\Big(-v_2+\frac{v_3(1-p)y}{v_1+(1-p)y}\Big), \end{aligned} \tag{1}$$

where $x(t)$, $y(t)$, and $z(t)$ denote the population density of the two mutualism species and a predator species at time t, respectively. The predator feeds on the second mutualism species y according to the Holling II functional response. The system admits six equilibria: $E_1(0,0,0)$, $E_2(0,1,0)$, $E_3(1,0,0)$, $E_4(\hat{x},\hat{y},0)$, $E_5(0,\bar{x},\bar{y})$, and $E_6(x^*,y^*,z^*)$. The author showed that E_1, E_2, and E_3 are all unstable. Concerned with the existence, locally asymptotical stability, and global stability of E_4 and E_5, the author obtained the following results (see Theorem 3.2-3.7 in [1] for more detailed expression).

Theorem A. *Predator washed state E_4 exists only when $\alpha_{12}\alpha_{21}<1$, where*

$$\hat{x}=\frac{1+\alpha_{12}}{1-\alpha_{12}\alpha_{21}},\quad \hat{y}=\frac{1+\alpha_{21}}{1-\alpha_{12}\alpha_{21}}. \tag{2}$$

Theorem B. *Second mutual species washed state E_5 exists only when $v_3>v_2$, $(v_3-v_2)>v_1v_2$, where*

$$\begin{aligned} \overline{y} &= \frac{v_1v_2}{(1-p)(v_3-v_2)}, \\ \overline{z} &= \frac{rv_3v_1[(1-p)(v_3-v_2)-v_1v_2]}{(1-p)^2(v_3-v_2)^2}. \end{aligned} \tag{3}$$

Theorem C. *The boundary steady state E_4 is always stable in xy-direction.*

Theorem D. *The equilibrium point E_4 is globally stable in the interior R_+^2 of the xy-plane.*

Theorem E. *The equilibrium point E_4 is globally asymptotically stable in the interior R_+^3.*

Theorem F. *If $v_1v_3+v_1v_2>(1-p)(v_3-v_2)$, the boundary steady state $E_5(0,\bar{x},\bar{y})$ is stable in yz-plane.*

Theorem G. *Along the conditions stated in Theorem 3.5, the equilibrium point E_5 is globally asymptotically stable in the interior R_+^2 of the yz-plane.*

Theorem H. *Along the conditions stated in Theorem 3.5, the equilibrium point E_5 is globally asymptotically stable in the interior of R_+^3.*

Now, from Theorem A, C, D, and E, one could easily see that if those Theorems hold, it will follow that E_4 is globally stable if the inequality $\alpha_{12}\alpha_{21}<1$ holds. Such a result is too good to be true. Indeed, let us consider the following example.

Example 1.1. Consider the following system

$$\begin{aligned} \frac{dx}{dt} &= x(1-x+0.5y), \\ \frac{dy}{dt} &= ry(1-y+0.5x) \\ &\quad -\frac{(1-0.25)yz}{0.5+(1-0.25)y}, \\ \frac{dz}{dt} &= z\Big(-0.25+\frac{0.5(1-0.25)y}{0.5+(1-0.25)y}\Big). \end{aligned} \tag{4}$$

Here, corresponding to system (1), we choose $\alpha_{12} = \alpha_{21} = 0.5, p = 0.25, v_1 = 0.5, v_2 = 0.25, v_3 = 0.5$. By simple computation, we have $\alpha_{12}\alpha_{21} = \frac{1}{4} < 1$, That is, condition in Theorem A, C, D, and E is satisfied. It follows from Theorem A, C, D, and E, that the predator washed state $E_4(\frac{4}{3}, \frac{4}{3}, 0)$ is globally asymptotically stable in the interior R^3_+. Numeric simulations (Figures 1-3) show that in this case, $E_4(\frac{4}{3}, \frac{4}{3}, 0)$ is not globally asymptotically stable, since $z(t)$ is not approach to 0 as $t \to +\infty$.

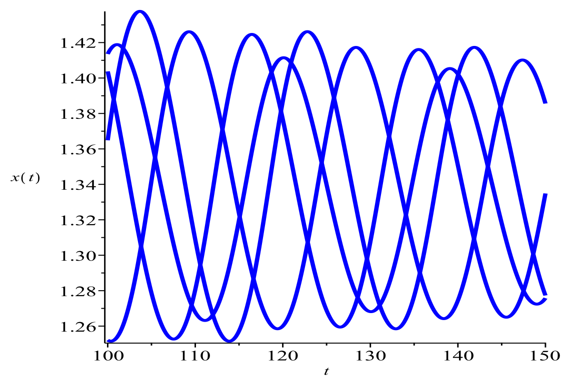

Figure 1: Dynamic behaviors of the first component x in system (4) with the initial condition $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$, and $(2, 2, 2)$, respectively.

On the other hand, from Theorems B, F, G, and H, one could easily see that if those Theorems hold, it will follow that E_5 is globally stable if the inequalities $v_3 > v_2$, $(1-p)(v_3-v_2) > v_1v_2$, and $v_1v_3 + v_1v_2 > (1-p)(v_3-v_2)$ hold. However, we say that this is impossible. Indeed, let us consider the following example.

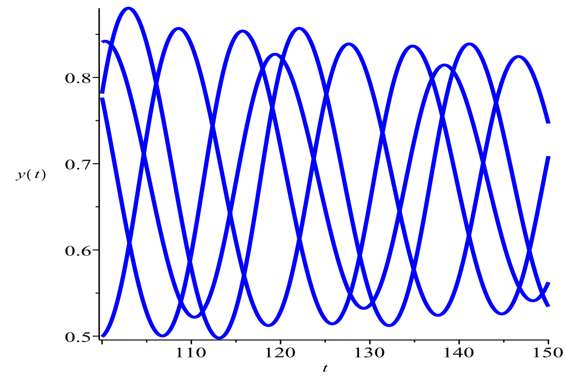

Figure 2: Dynamic behaviors of the second component x in system (4) with the initial condition $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$, and $(2, 2, 2)$, respectively.

Example 1.2. Consider the following system

$$\begin{aligned} \frac{dx}{dt} &= x(1-x+0.5y), \\ \frac{dy}{dt} &= ry(1-y+0.5x) \\ &\quad -\frac{(1-0.5)yz}{0.25+(1-0.5)y}, \\ \frac{dz}{dt} &= z\Big(-0.25+\frac{0.5(1-0.25)y}{v_1+(1-0.5)y}\Big). \end{aligned} \tag{5}$$

Here, corresponding to system (1), we choose $\alpha_{12} = \alpha_{21} = 0.5, p = 0.5, v_1 = 0.25, v_2 = 0.25, v_3 = 0.5$. By simple computation, we have

$$v_3 = 0.5 > 0.25 = v_2,$$
$$(1-p)(v_3-v_2) = \tfrac{1}{8} > \tfrac{1}{16} = v_1v_2,$$
$$v_1v_3 + v_1v_2 = \tfrac{3}{16} > \tfrac{1}{8} = (1-p)(v_3-v_2).$$

That is, all the conditions in Theorems B, F, G, and H are satisfied. It follows from Theorem Theorems B, F, G, and H that the second mutual species washed state $E_5(0, \bar{x}, \bar{y})$ is globally asymptotically stable in the interior R^3_+. Numeric simulations (Figures 4-6) show that in this case, $E_5(0, \bar{x}, \bar{y})$ is not globally asymptotically stable, since $x(t)$ is approach to 1 as $t \to +\infty$.

Above two examples show that at least Theorem E and H in [1] may not be correct. Hence, we should revisit the stability property of the equilibrium E_4 and E_5. This paper aims to put forward some studies on this direction. Indeed, we will prove the following results.

Theorem 1.1 *Assume that*

$$\alpha_{12}\alpha_{21} < 1 \tag{6}$$

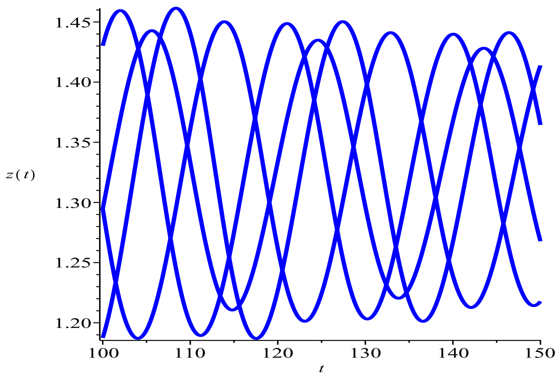

Figure 3: Dynamic behaviors of the third component x in system (4) with the initial condition $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$, and $(2, 2, 2)$, respectively.

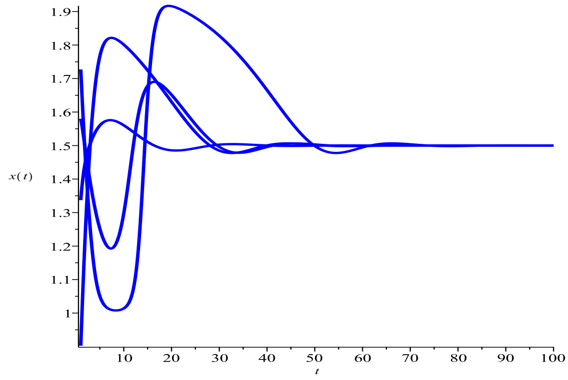

Figure 4: Dynamic behaviors of the first component x in system (5) with the initial condition $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$, and $(2, 2, 2)$, respectively.

and

$$-v_2 + \frac{v_3(1-p)\hat{y}}{v_1 + (1-p)\hat{y}} < 0 \tag{7}$$

are satisfied, then the predator washed state E_4 is globally attractive in the interior R^3_+.

Remark 1.1. Theorem 1.1 shows that to ensure the globally attractivity of E_4, additional condition (7) is needed.

Theorem 1.2 *The second mutual species washed state $E_5(0, \bar{x}, \bar{y})$ is unstable.*

Remark 1.2. Theorem 1.2 shows that Theorem B and H in [1] is incorrect.

The rest of the paper is organized as follows. We will prove Theorem 1.1 in the next section. We end this work with a brief discussion. For more works on the predator-prey system or mutualism model, one could refer to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] and the references cited therein.

2 Proof of the Main Results

Lemma 2.1[11] *If $a > 0, b > 0$ and $\dot{x} \geq b - ax$, when $t \geq 0$ and $x(0) > 0$, we have*

$$\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}.$$

If $a > 0, b > 0$ and $\dot{x} \leq b - ax$, when $t \geq 0$ and $x(0) > 0$, we have

$$\limsup_{t\to+\infty} x(t) \leq \frac{b}{a}.$$

Now let us consider the system

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x + \alpha_{12} y), \\ \frac{dy}{dt} &= ry(1 - (1+\delta)y + \alpha_{21} x), \end{aligned} \tag{8}$$

where $\delta \geq 0$ is nonnegative constant.

Lemma 2.2 *Assume that $\alpha_{12}\alpha_{21} < 1$, then system (8) admits a unique positive equilibrium $A(\hat{x}_\delta, \hat{y}_\delta)$, where*

$$\hat{x}_\delta = \frac{\alpha_{12} + 1 + \delta}{1 + \delta - \alpha_{12}\alpha_{21}},$$
$$\hat{y}_\delta = \frac{\alpha_{21} + 1}{1 + \delta - \alpha_{12}\alpha_{21}},$$

which is globally asymptotically stable.

Proof. Let us consider the following Lyapunov function

$$\begin{aligned} V(x, y) &= \left(x - \hat{x} - \hat{x}_\delta \ln \tfrac{x}{\hat{x}_\delta}\right) \\ &\quad + \frac{\alpha_{12}}{r\alpha_{21}}\left(y - \hat{y}_\delta - \hat{y}_\delta \ln \frac{y}{\hat{y}_\delta}\right), \end{aligned}$$

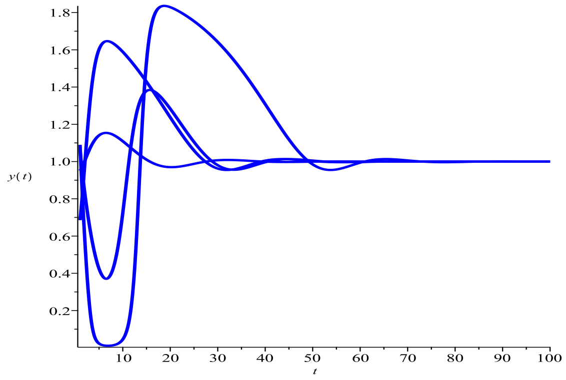

Figure 5: Dynamic behaviors of the second component y in system (5) with the initial condition $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$, and $(2, 2, 2)$, respectively.

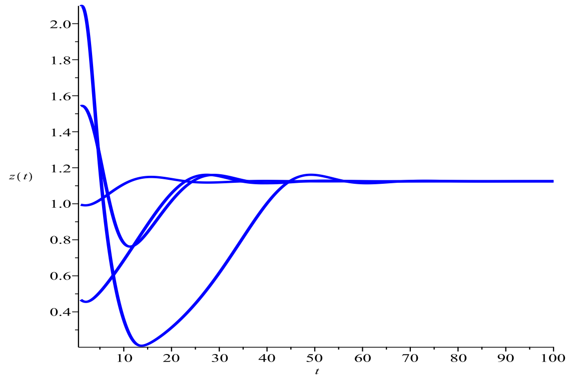

Figure 6: Dynamic behaviors of the third component z in system (5) with the initial condition $(x(0), y(0), z(0)) = (0.5, 0.5, 0.5)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$, and $(2, 2, 2)$, respectively.

Calculating the derivative of V along the solution of the system (8), we have

$$\begin{aligned}
&\frac{dV}{dt}\\
=\;& \Big(1-\frac{\hat{x}_\delta}{x}\Big)\Big(1-x+\alpha_{12}y\Big)x\\
&+\frac{\alpha_{12}}{r\alpha_{21}}\Big(1-\frac{\hat{y}_\delta}{y}\Big)\Big(1-(1+\delta)y+\alpha_{21}x\Big)y\\
=\;& \Big(x-\hat{x}_\delta\Big)\Big(-(x-\hat{x}_\delta)+\alpha_{12}(y-\hat{y}_\delta)\Big)\\
&+\frac{\alpha_{12}r}{r\alpha_{21}}\Big(y-\hat{y}_\delta\Big)\Big(-(1+\delta)(y-\hat{y}_\delta)\\
&+\alpha_{21}(x-\hat{x}_\delta)\Big)y\\
=\;& -(x-\hat{x}_\delta)^2\\
&+2\alpha_{12}(x-\hat{x}_\delta)(y-\hat{y}_\delta)\\
&-\tfrac{\alpha_{12}}{\alpha_{21}}(1+\delta)(y-\hat{y}_\delta)^2\\
=\;& -(x-\hat{x}_\delta, y-\hat{y}_\delta)A\begin{pmatrix} x-\hat{x}_\delta\\ y-\hat{y}_\delta\end{pmatrix},
\end{aligned}$$

where

$$A=\begin{pmatrix} 1 & \alpha_{12}\\ \alpha_{12} & \frac{\alpha_{12}}{\alpha_{21}}(1+\delta)\end{pmatrix}.$$

According to the hypothesis $\alpha_{12}\alpha_{21} < 1$, $\frac{dV}{dt} < 0$ strictly for all $x, y > 0$ except for the positive equilibrium $(\hat{x}_\delta, \hat{y}_\delta)$, where $\frac{dV}{dt} = 0$. Thus, $V(x, y)$ satisfies Lyapunov's asymptotic stability theorem, and the positive equilibrium $(\hat{x}, \hat{y})$ of the system (8) is globally asymptotically stable. The proof of Lemma 2.2 is now complete.

Remark 2.1. For the case $\delta = 0$, Srinivasarao Tote [1] also attempted to demonstrate this conclusion, for more information, see the proof of Theorem 3.4 in [1], however his argument is flawed because he choose $l_1 = -\dfrac{r\alpha_{21}}{\alpha_{12}}$, which prevents V from being positive definite.

Proof of Theorem 1.1. For $\varepsilon > 0$ enough small, from (7), the continuity of the function $\frac{ax}{b+cx}$, and the sign-preserving properties of the continuous function, the following inequality holds:

$$-v_2 + \frac{v_3(1-p)(\hat{y}+\varepsilon)}{v_1+(1-p)(\hat{y}+\varepsilon)} < 0. \tag{9}$$

From the first and second equations of the system (1) and the positivity of the solution of the system (1), we have

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x+\alpha_{12}y),\\
\frac{dy}{dt} &\le ry(1-y+\alpha_{21}x).
\end{aligned} \tag{10}$$

Now let us consider the system

$$\begin{aligned}
\frac{dv_1}{dt} &= v_1(1-v_1+\alpha_{12}v_2),\\
\frac{dv_2}{dt} &= rv_2(1-v_2+\alpha_{21}v_1).
\end{aligned} \tag{11}$$

It follows from (6) and Lemma 2.2 that the positive equilibrium $A(\hat{x}, \hat{y})$ of the system (11) is globally asymptotically stable, where

$$\hat{x} = \frac{1+\alpha_{12}}{1-\alpha_{12}\alpha_{21}}, \quad \hat{y} = \frac{1+\alpha_{21}}{1-\alpha_{12}\alpha_{21}}. \tag{12}$$

That is, for any positive solution $(v_1(t), v_2(t))$ of the system (11), one has

$$\lim_{t\to+\infty} v_1(t) = \hat{x}, \quad \lim_{t\to+\infty} v_2(t) = \hat{y}.$$

Let $(x(t), y(t), z(t))$ be any positive solution of system (1) with positive initial condition (x_0, y_0, z_0), and let $(v_1(t), v_2(t))$ be the positive solution of system (11) with the initial condition $(v_1(0), v_2(0)) = (x_0, y_0)$, then it follows from (10), (11) and the differential inequality theory that

$$x(t) \le v_1(t), \;\; y(t) \le v_2(t) \quad \text{for all } \; t \ge 0,$$

and therefore that

$$\limsup_{t\to+\infty} x(t) \le \hat{x}, \;\; \limsup_{t\to+\infty} y(t) \le \hat{y}. \tag{13}$$

Thus, for aforementioned $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$y(t) < \hat{y} + \varepsilon \quad \text{for all} \quad t \ge T_1. \tag{14}$$

From the third equation of system (1), for $t > T_1$, we have

$$\begin{aligned} \frac{dz}{dt} &= z\Big(-v_2 + \frac{v_3(1-p)y}{v_1 + (1-p)y}\Big) \\ &\le z\Big(-v_2 + \frac{v_3(1-p)(\hat{y}+\varepsilon)}{v_1 + (1-p)(\hat{y}+\varepsilon)}\Big). \end{aligned} \tag{15}$$

It then follows from (9) that

$$z(t) \le z(T)\exp\Big\{B(t-T)\Big\} \to 0 \quad \text{as} \quad t \to +\infty. \tag{16}$$

where

$$B = -v_2 + \frac{v_3(1-p)(\hat{y}+\varepsilon)}{v_1 + (1-p)(\hat{y}+\varepsilon)}.$$

Hence, for aforementioned $\varepsilon > 0$, there exists a $T_2 > T_1$, such that

$$z(t) < \frac{v_1}{1-p}\varepsilon \quad \text{as} \quad t \ge T_2. \tag{17}$$

From (17) and the first and second equation of system (1), for $t > T_2$, we have

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x + \alpha_{12}y), \\ \frac{dy}{dt} &= ry(1 - y + \alpha_{21}x) - \frac{(1-p)yz}{v_1 + (1-p)y} \\ &\ge ry(1 - y + \alpha_{21}x) - \frac{(1-p)yz}{v_1} \\ &\ge ry(1 - y + \alpha_{21}x) - \frac{(1-p)y\frac{v_1}{1-p}\varepsilon}{v_1} \\ &= ry(1 - (1+\varepsilon)y + \alpha_{21}x). \end{aligned} \tag{18}$$

Now let us consider the system

$$\begin{aligned} \frac{dw_1}{dt} &= w_1(1 - w_1 + \alpha_{12}w_2), \\ \frac{dw_2}{dt} &= rw_2(1 - (1+\varepsilon)w_2 + \alpha_{21}w_1). \end{aligned} \tag{19}$$

It follows from (6) and Lemma 2.2 that the positive equilibrium $A(\hat{x}_\varepsilon, \hat{y}_\varepsilon)$ of system (19) is globally asymptotically stable, where

$$\begin{aligned} \hat{x}_\varepsilon &= \frac{\alpha_{12} + 1 + \varepsilon}{1 + \varepsilon - \alpha_{12}\alpha_{21}}, \\ \hat{y}_\varepsilon &= \frac{\alpha_{21} + 1}{1 + \varepsilon - \alpha_{12}\alpha_{21}}, \end{aligned} \tag{20}$$

That is, for any positive solution $(w_1(t), w_2(t))$ of the system (19), one has

$$\lim_{t\to+\infty} w_1(t) = \hat{x}_\varepsilon, \;\; \lim_{t\to+\infty} w_2(t) = \hat{y}_\varepsilon.$$

Let $(x(t), y(t), z(t))$ be any positive solution of system (1) with positive initial condition $(x_{T_2}, y_{T_2}, z_{T_2})$, and let $(w_1(t), w_2(t))$ be the positive solution of system (19) with the initial condition $(w_1(T_2), w_2(T_2)) = (x_{T_2}, y_{T_2})$, then it follows from (18), (19) and the differential inequality theory that

$$x(t) \ge w_1(t), \;\; y(t) \ge w_2(t) \quad \text{for all } \; t \ge T_2,$$

and therefore that

$$\liminf_{t\to+\infty} x(t) \ge \hat{x}_\varepsilon, \;\; \liminf_{t\to+\infty} y(t) \ge \hat{y}_\varepsilon. \tag{21}$$

(13) combines with (21) leads to

$$\begin{aligned} \hat{x}_\varepsilon \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le \hat{x}, \\ \hat{y}_\varepsilon \le \liminf_{t\to+\infty} y(t) \le \limsup_{t\to+\infty} y(t) \le \hat{y}. \end{aligned} \tag{22}$$

Noting that ε is any enough small positive constant, letting $\varepsilon \to 0$ in the (22) leads to

$$\lim_{t\to+\infty} x(t) = \hat{x}, \;\; \lim_{t\to+\infty} y(t) = \hat{y}. \tag{23}$$

So, (16) together with (22) shows that the predator washed state E_4 is globally attractive in the interior R^3_+.

This completes the proof of Theorem 2.1.

Proof of Theorem 1.2. It is enough to show that $x(t)$ is impossible approach to 0 as $t \to +\infty$. Indeed, noting that the first equation in the system (1) is independent of the predator species, from the positivity of the solution and the first equation of the system (1), we have

$$\frac{dx}{dt} = x(1 - x + \alpha_{12}y) \ge x(1 - x). \tag{24}$$

Applying Lemma 2.1 to this inequality leads to

$$\liminf_{t\to+\infty} x(t) \geq 1.$$

Therefore, there exists a $T > 0$ such that

$$x(t) > \frac{1}{2} \quad \text{as} \quad t \geq T.$$

This means that $x(t)$ is impossible approach to 0 as $t \to +\infty$.
The proof of Theorem 1.2 is now complete.

3 Numeric Simulations

Now let us consider the following example.

Example 3.1. Consider the following system

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x + 0.5y), \\ \frac{dy}{dt} &= ry(1 - y + 0.5x) \\ &\quad -\frac{(1-0.25)yz}{0.5 + (1-0.25)y}, \\ \frac{dz}{dt} &= z\Big(-1 + \frac{0.5(1-0.25)y}{0.5+(1-0.25)y}\Big). \end{aligned} \tag{25}$$

Here, corresponding to system (1), we choose $\alpha_{12} = \alpha_{21} = 0.5, p = 0.25, v_1 = 0.5, v_3 = 0.5, v_2 = 1$. By simple computation, from $\alpha_{12}\alpha_{21} = \frac{1}{4} < 1$, we have $\hat{x} = 2, \hat{y} = 2$, thus,

$$\begin{aligned} & -v_2 + \frac{v_3(1-p)\hat{y}}{v_1 + (1-p)\hat{y}} \\ = & -1 + \frac{0.5 \times 0.75 \times 2}{0.5 + (1-0.25) \times 2} \\ = & \ 0.625 < 0. \end{aligned} \tag{26}$$

That is, condition (6) in Theorem 1.1 is satisfied. It follows from Theorem 1.1 that the predator washed state $E_4(\frac{4}{3}, \frac{4}{3}, 0)$ is globally attractive in the interior R^3_+. Numeric simulations (Figures 7-9) show that in this case, $E_4(2, 2, 0)$ is globally attractive.

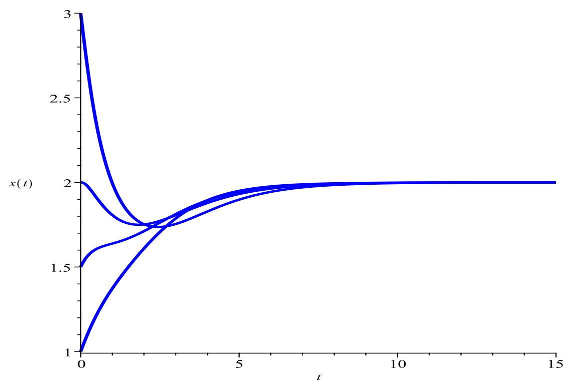

Figure 7: Dynamic behaviors of the first component x in system (25) with the initial condition $(x(0), y(0), z(0)) = (3, 3, 3)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$, and $(2, 2, 2)$, respectively.

4 Discussion

Recently, Srinivasarao Tote [1] proposed an ecological model of mutualism between two species with a mortal predator. The authors investigated the local and global stability of the equilibria. However, it may be because the system is three-dimensional, and the Lyapunov function constructed by the author is only related to two variables, which leads to some very absurd results. In this paper, with the help of the comparison principle of the differential equation, we gave sufficient conditions to ensure the global attractivity of E_4, and proved that E_5 is impossible to be stable. A more interesting fact: from the numerical simulation of Example 1.1, the system (1.1) may have a periodic solution, that is, the system may exhibit Hopf bifurcation in some situation, but we cannot strictly prove this conjecture at present. This requires further work in the future.

It must be noted that for the cooperative system or commensalism system, so far, as far as the author knows, no limit cycle has been found (see [13]-[22] and the references cited therein). After adding the predator population to the system, the system can have a periodic solution as was shown in Example 1.1, which is obviously a fascinating phenomenon. It seems very necessary to consider the multi-species ecological model including both predator and mutualism relationship.

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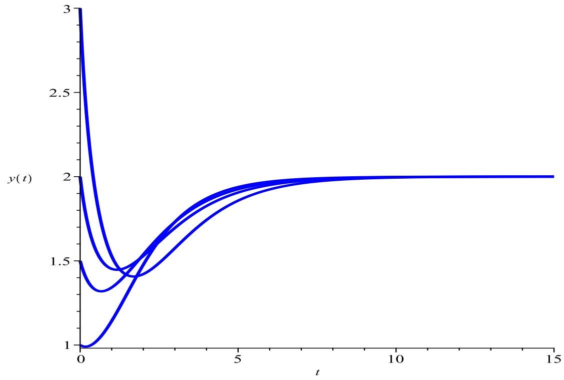

Figure 8: Dynamic behaviors of the second component y in system (25) with the initial condition $(x(0), y(0), z(0)) = (3, 3, 3)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$, and $(2, 2, 2)$, respectively.

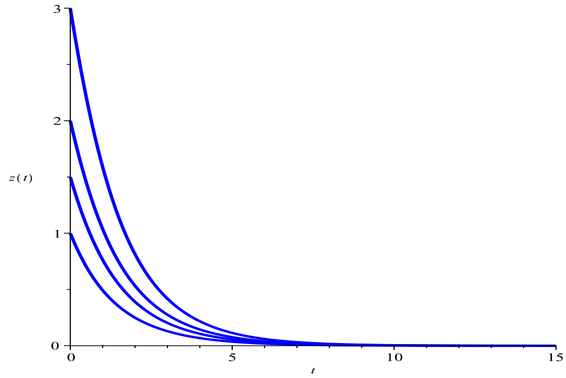

Figure 9: Dynamic behaviors of the third component z in system (25) with the initial condition $(x(0), y(0), z(0)) = (3, 3, 3)$, $(1, 1, 1)$, $(1.5, 1.5, 1.5)$, and $(2, 2, 2)$, respectively.

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Xiaoran Li carried out the computation and wrote the draft.
Qin Yue carried out the simulation.
Fengde Chen was responsible for the proposing of the problem.

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