Generalized 'Useful' Converse Jensen's Inequality with Data Illustration

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Abstract: - In the present communication, we give the converse of generalized 'useful' Jensen inequality and show that some recently reported inequalities are simple consequences of those results that have been established for a long time. We also include a new improvement of the proposed inequality of Jensen as well as changes to some associated outcomes, where generalized 'useful' converse of the Inequality of Jensen is presented and implementations related to it are given in the theory of information. Finally, it is shown with the help of numerical data that inequalities hold well both for convex and concave functions.

Key-words: - Probability distribution; Jensen's inequality; utility distribution; 'useful' Information measure; Strongly convex function; 'useful' converse Jessen's inequality.

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1 Introduction

Let $\triangle_k^+ = \{C = (c_1, c_2, \dots \dots c_k);\ c_j \geq 0, \sum_{j=1}^{k} c_j = 1\}$, be a set of all possible discrete Chance distributions of a random variable Y having utility distribution $U = \{(u_1, u_2, \dots \dots u_k);\ u_j > 0\ \forall j\}$ attached to each $C \in \Delta_k^+$ such that $u_j > 0$ is the utility of an event having the chance of occurrence $c_j > 0$.

Let $U = (u_1, u_2, \dots \dots u_k)$ be the set of positive real numbers, where u_j is the utility or importance of the outcome y_j. In general, the utility is independent of the likelihood of encoding the source symbol y_j, that is c_j.
The source of information is thus given by

$$\begin{bmatrix} y_1, y_2 \dots \dots \dots y_k \\ c_1, c_2 \dots \dots \dots c_k \\ u_1, u_2 \dots \dots \dots u_k \end{bmatrix} \tag{1}$$

Where $u_j > 0, 0 < c_j \leq 1, \sum_{j=1}^{k} c_j = 1,$ is called a utility information scheme. Corresponding to the scheme (1) Belis and Guiasu [1] gave the following measure of information:

$$H(C;U) = -\sum_{j=1}^{k} u_j c_j \log\ c_j \tag{2}$$

The above measure (2) is called 'useful information' and it reduces to Shannon's information [14] when utilities are ignored, as seen following:

$$H(C) = -\sum_{j=1}^{k} c_j \log\ c_j \tag{3}$$

By using various postulates, many authors have defined the entropy of Shannon. Using essential assumptions that were deduced by Fadeev [7], Chandy and Mcliod [2], Kendall [9], Khinchin [10] made Shannon's argument more accurate. Tverberg [16], etc., was further defined by the entropy of Shannon by considering various sets of postulates. Simic [15] depicts a continuous series of real numbers adhering to a defined finite interval with a defined upper global bound, as well as demonstrating with examples how this technique can be used to establish the converse of several key inequalities. For strongly convex and strongly mid-convex functions, the counterparts of the converse Jensen inequality were presented by Klaricic & Nikodem [11]. Approximation theory, mathematical economics, and optimization theory all benefit from strong convex functions. Many of their qualities have been documented in the Literature for isotonic linear functionals Dragomir [6] has given a reverse of Jensen's inequality.

In several fields of mathematics, convex functions play an essential role to Rashid et al. [13] and Ge-Jile et al. [8]. They are particularly useful in the study of optimization issues, where they have a variety of useful features. The convex function is an open set, for example, it contains just one minimum. Convex functions continue to meet similar characteristics in infinite-dimensional spaces given acceptable additional assumptions, they are the most well-known basic aspect in the calculus of variations as a result. The convex enables control of the measured data of a random variable that is always bounded above by the convex function's expected value in probability theory. Jensen's inequality, as well as Holder's inequality and Arithmetic–Geometric mean inequality, may be derived from this conclusion.

Convexity is something we encounter all the time and in a variety of ways. The most typical scenario is our standing stance, which is safe as long as our center of gravity's vertical projection is contained inside of the convex envelope of our feet! Convexity also has a significant influence on our daily lives due to its diverse uses in industry, business, health, art, and other fields. Problems with optimal resource allocation and non-cooperative game equilibria are also present. Because a convex function has a convex set as its basis, the theory of convex functions falls within the umbrella of convexity. Nonetheless, it is a significant theory in and of itself, as it affects practically all fields of mathematics.

The graphical analysis is most often the initial issue that necessitates the acquaintance with this theory. This is an opportunity to learn about the second derivative proof of concavity, which is a useful tool for detecting convexity. The difficulty of identifying the extremal values of functions with many variables, as well as the application of Hessian as a higher dimensional generalization of the second derivative, follows. The next step is to go on to optimization issues in infinite-dimensional spaces, however full of technological complexity required to solve such issues, the fundamental concepts are quite comparable to those behind only one variable example.

We would like to highlight the introduction and study of strongly convex functions, which play a crucial contribution to information theory and related fields. Many authors, for instance, strongly convex functions were used to explaining the one-of-a-kind presence of a possible answer to nonlinear supplementary problems. In the study of iterative approaches, the convergence towards tackling variational inequalities and equilibrium difficulties, strongly convex functions were also critical. Using strongly convex functions, Nikodem and Pales [12] explore the crucial explanation of inner product spaces, which is an innovative and unique application.

For convex functions, we obtained the following converse of generalized 'useful' Inequality of Jensen's that reduce the inequality given by S. S. Dragomir and N. M. Ionescu in [4]:

$$0 \le \frac{\sum_{j=1}^{k} u_j c_j g(y_j)}{\sum_{j=1}^{k} u_j c_j} - g\left(\frac{\sum_{j=1}^{k} u_j c_j y_j}{\sum_{j=1}^{k} u_j c_j}\right) \le \frac{\sum_{j=1}^{k} u_j c_j y_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j} - \frac{\sum_{j=1}^{k} u_j c_j y_j}{\sum_{j=1}^{k} u_j c_j} \frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j} \quad (4)$$

Suppose that E° is the interior of the interval E, and $g: E \subseteq \boldsymbol{R} \to \boldsymbol{R}$ is differentiable convex on E°, $y_j \in E^{\circ}$, and $\sum_{j=1}^{k} c_j = 1, c_j > 0$ $(j = 1,\dots,k)$. If g on E is strictly convex, then iff $y_1 = \cdots = y_k$ the equality case holds in (4). The above measure reduces to Dragomir [5], when 'utilities are ignored. Several applications of this can be found in Dragomir and Goh [3]. The key contribution of this research is to highlight refining of the converse of generalized 'useful' inequality of Jensen's defined in (4).

2 New Improvements

In this section, we have given some lemma and their proofs where utilities are attached to probabilities of a differentially convex function and differentially strictly convex function and basic results that will be needed in this correspondence.

Lemma 2.1 Suppose a differentiable convex function on E°, defined as $g: E \subseteq \boldsymbol{R} \to \boldsymbol{R}$, and $u_j > 0$ are the utilities attached to probabilities and $y_j \in E^{\circ}, c_j > 0$ $(j = 1,\dots,k)$ with $\sum_{j=1}^{k} c_j = 1$, then we have the inequality

$$\frac{\sum_{j=1}^{k} u_j c_j g(y_j)}{\sum_{j=1}^{k} u_j c_j} \le \frac{\sum_{j=1}^{k} u_j c_j y_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j} + \inf_{y \in E^{\circ}} \left(g(y) - y \frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right) \quad (5)$$

Proof. The following inequality hold for all $y, z \in E^{\circ}$, if g is differentiable convex on E°:

$$g(y) - g(z) \geq g'(z)(y - z) \qquad (6)$$

Again, we get the next inequality if we multiply with $c_j > 0$ and choose in (6), $z = y_j (j = 1,\dots,k)$, and sum over j from 1 up to k.

$$g(y) - \frac{\sum_{j=1}^{k} u_j c_j g(y_j)}{\sum_{j=1}^{k} u_j c_j} \geq y \frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j} - \frac{\sum_{j=1}^{k} u_j c_j y_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}$$

it is equal to the following inequality

$$g(y) - y \frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j} + \frac{\sum_{j=1}^{k} u_j c_j y_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j} \geq \frac{\sum_{j=1}^{k} u_j c_j g(y_j)}{\sum_{j=1}^{k} u_j c_j} \qquad (7)$$

For each $y \in E^{\circ}$.
We deduce (5), taking the infimum over $y \in E^{\circ}$.
The following outcome relates to the refining of Dragomir-Ionescu (4). It can be noted that (5) reduces the inequality given by Dragomir [5] when utilities are ignored.

Theorem 2.1 Suppose a differentiable convex function on E°, defined as $g: E \subseteq \boldsymbol{R} \to \boldsymbol{R}$, and $u_j > 0$ are the utilities attached to probabilities and $y_j \in E^{\circ}, c_j > 0 \ (j = 1,\dots,k)$ with $\sum_{j=1}^{k} c_j = 1$, then

$$0 \leq \frac{\sum_{j=1}^{k} u_j c_j g(y_j)}{\sum_{j=1}^{k} u_j c_j} - g\left(\frac{\sum_{j=1}^{k} u_j c_j y_j}{\sum_{j=1}^{k} u_j c_j}\right)$$
$$\leq \inf_{y \in E^{\circ}} \left(g(y) - y \frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right) + \frac{\sum_{j=1}^{k} u_j c_j y_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j} - g\left(\frac{\sum_{j=1}^{k} u_j c_j y_j}{\sum_{j=1}^{k} u_j c_j}\right)$$
$$\leq \frac{\sum_{j=1}^{k} u_j c_j y_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}. - \frac{\sum_{j=1}^{k} u_j c_j y_j}{\sum_{j=1}^{k} u_j c_j} . \frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j} \qquad (8)$$

Proof. In the above inequality (8) the second inequality is followed by the first inequality in (5). It's indeed obvious that

$$\inf_{y \in E^{\circ}} \left(g(y) - y \frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right) \leq g(\bar{y}) - \bar{y} \frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j},$$

where $\bar{y} = \frac{\sum_{j=1}^{k} u_j c_j y_j}{\sum_{j=1}^{k} u_j c_j} \in E^{\circ}$ and therefore the last portion of (8) is then proven.
We may utilize the following result for applications.
Lemma 2.2 Suppose a differentiable strictly convex function on E°, defined as $g: E \subseteq \boldsymbol{R} \to \boldsymbol{R}$, and $u_j > 0$ are the utilities attached to probabilities and $y_j \in E^{\circ}, c_j > 0 \ (j = 1,\dots,k)$ with $\sum_{j=1}^{k} c_j = 1$, then

$$\frac{\sum_{j=1}^{k} u_j c_j g(y_j)}{\sum_{j=1}^{k} u_j c_j} \leq \frac{\sum_{j=1}^{k} u_j c_j y_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j} + g\left((g')^{-1} - \left(\frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right)\right) - (g')^{-1}\left(\frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right). \frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j} \qquad (9)$$

where $(g')^{-1}$ represents the opposite function of the derivative g' defined on $g'(E^{\circ})$. If $y_1 = \cdots = y_k$, then equality case holds in (9).

Proof. Define the function $f: E^{\circ} \to \boldsymbol{R}, f(y) = g(y) - y \frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}$. Consequently, f is differentiable on E° and then

$$f'(y) = g'(y) - \frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j} \qquad (10)$$

If $f'(y) = 0$, and $y \in E^{\circ}$ then the above equation is equivalent to as follows

$$g'(y) = \frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j} \qquad (11)$$

and since $\frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j} \in g'(E^{\circ}), g'$ is one-to-one, existence strictly increasing on E°, The equation (11) therefore has a unique solution $y_0 \in E^{\circ}$ alternatively to E° given by

$$y_0 = (g')^{-1}\left(\frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right) \in E^{\circ}, \qquad (12)$$

The derivative of g' defined on $g'(E^{\circ})$, where $(g')^{-1}$ is the inverse function of the derivative.

Let, $f'(y) > 0$ if $y > y_0, y \in E^\circ$ and $f'(y) < 0$ if $y < y_0, y \in E^\circ$, then it follows that

$$\inf_{y\in E^\circ} f(y) = f(y_0)$$
$$= g\left((g')^{-1}\left(\frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right)\right)$$
$$-(g')^{-1}\left(\frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right).\left(\frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right).$$

By using (4), we deduce (9). It is clear that (9) reduces the inequality given by Dragomir [5] when utilities are ignored. The equality case assumes the strict convexity of E° and specifics are omitted. It is now feasible to disclose the next step in the Dragomir-Ionescu process (4).

Theorem 2.2 Suppose a differentiable strictly convex function on E°, defined as $g: E \subseteq \boldsymbol{R} \to \boldsymbol{R}$, and $u_j > 0$ are the utilities attached to probabilities and $y_j \in E^\circ, c_j > 0$ $(j = 1, \ldots, k)$ with $\sum_{j=1}^{k} c_j = 1$, then

$$0 \le \frac{\sum_{j=1}^{k} u_j c_j g(y_j)}{\sum_{j=1}^{k} u_j c_j} - g\left(\frac{\sum_{j=1}^{k} u_j c_j y_j}{\sum_{j=1}^{k} u_j c_j}\right) \quad (13)$$
$$\le \frac{\sum_{j=1}^{k} u_j c_j y_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}$$
$$+g\left((g')^{-1} - \left(\frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right)\right)$$
$$-(g')^{-1}\left(\frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right).\left(\frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right)$$
$$-g\left(\frac{\sum_{j=1}^{k} u_j c_j y_j}{\sum_{j=1}^{k} u_j c_j}\right)$$
$$\le \frac{\sum_{j=1}^{k} u_j c_j y_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j} - \frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\frac{\sum_{j=1}^{k} u_j c_j y_j}{\sum_{j=1}^{k} u_j c_j}$$

Iff $y_1 = \cdots = y_k$, the equality holds in (13).

Proof. By lemma 2.1 and theorem 2.1, the proof is obvious.

Remark 2.1 In Lemma 2.2, we note that there is double inequality with the assumptions

$$g\left(\frac{\sum_{j=1}^{k} u_j c_j y_j}{\sum_{j=1}^{k} u_j c_j}\right) \le \frac{\sum_{j=1}^{k} u_j c_j g(y_j)}{\sum_{j=1}^{k} u_j c_j}$$
$$\le \frac{\sum_{j=1}^{k} u_j c_j y_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}$$
$$+g\left(-(g')^{-1} - \left(\frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right)\right)$$
$$-\left(\frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right).(g')^{-1}\left(\frac{\sum_{j=1}^{k} u_j c_j g'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right) \quad (14)$$

with equality iff $y_1 = \cdots = y_k$.
In case, if f is a strictly concave and differentiable function, then

$$f\left(\frac{\sum_{j=1}^{k} u_j c_j y_j}{\sum_{j=1}^{k} u_j c_j}\right) \ge \frac{\sum_{j=1}^{k} u_j c_j f(y_j)}{\sum_{j=1}^{k} u_j c_j}$$
$$\ge \frac{\sum_{j=1}^{k} u_j c_j y_j f'(y_j)}{\sum_{j=1}^{k} u_j c_j} +$$
$$f\left(-(f')^{-1}\left(-\frac{\sum_{j=1}^{k} u_j c_j f'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right)\right)$$
$$+\left(\frac{\sum_{j=1}^{k} u_j c_j f'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right).(f')^{-1}\left(-\frac{\sum_{j=1}^{k} u_j c_j f'(y_j)}{\sum_{j=1}^{k} u_j c_j}\right) \quad (15)$$

The proof of (15) follows by (14) choosing $g = -f$, with equality iff $y_1 = \cdots = y_k$.

3 Numerical and Graphical Illustration

In this section, we give a numerical result that will further strengthen our results (14) and (15).
Suppose a differentiable strictly convex function on $E^\circ = g(y) = y^2$, defined as $g: E \subseteq \boldsymbol{R} \to \boldsymbol{R}$, and let $u_1 = 2, u_2 = 3, u_3 = 4, u_4 = 5, u_5 = 6, u_6 = 7$ are the utilities that are attached to probabilities $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = \frac{1}{6}$ with $\sum_{j=1}^{6} c_j = 1$. Now we take the convex function $g(y) = y^2$ is an example of one. The 'useful' Jensen's inequality asserts that we must discover the value of the 'useful Jensen inequality for convex function $g\left(\frac{\sum_{j=1}^{k} u_j c_j y_j}{\sum_{j=1}^{k} u_j c_j}\right) \le \frac{\sum_{j=1}^{k} u_j c_j g(y_j)}{\sum_{j=1}^{k} u_j c_j}$ in Table 1, we measure the numerical value for taking examples of the convex function, the value of $g\left(\frac{\sum_{j=1}^{k} u_j c_j y_j}{\sum_{j=1}^{k} u_j c_j}\right) = 0.34$ and the value of $\frac{\sum_{j=1}^{k} u_j c_j g(y_j)}{\sum_{j=1}^{k} u_j c_j} = 2.53$ which holds the inequality (14) and (15). A bar graph for the convex function

is shown in Figure 1. If we choose any two locations on the graph of a function and draw a line segment between them, the entire segment is above the graph, then the function is convex. The function, on the other hand, is considered to be concave if the line segment always sits below the graph. To put it another way, $g(y)$ is convex only, if $-g(y)$ is concave.

Table 1. The function evaluated at the expectation

y	y^2	c	u	$c*y$	$c*y^2$	$g\left(\frac{\sum_{j=1}^{k} u_j c_j y_j}{\sum_{j=1}^{k} u_j c_j}\right)$	$\left(\frac{\sum_{j=1}^{k} u_j c_j\, g(y_j)}{\sum_{j=1}^{k} u_j c_j}\right)$
1	1	1/6	2	0.17	0.17	0.03	0.03
2	4	1/6	3	0.33	0.67	0.06	0.11
3	9	1/6	4	0.50	1.50	0.08	0.25
4	16	1/6	5	0.67	2.67	0.11	0.44
5	25	1/6	6	0.83	4.17	0.14	0.69
6	36	1/6	7	1.00	6.00	0.17	1.00
						=0.34	=2.53

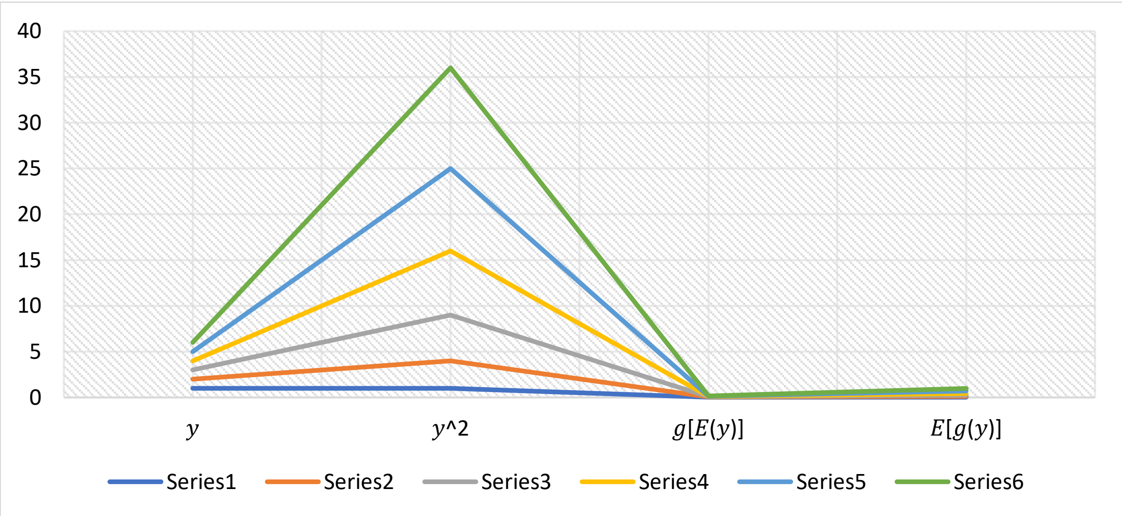

Figure 1. Bar graph representation of inequality for the convex function.

4 Discussion

In this paper, we have mainly worked on a differentiable convex function in which a utility test is done as well as we have displayed the result in terms of a differentiable function and its related result also, we have shown the result in terms of Jensen's inequalities. Belis and Guiasu gave a measure of information for the discrete probability distribution of random variables that are required for information theory.

5 Conclusion

This paper introduced the converse of generalized 'useful' Jensen inequality. Our work reinforces Dragomir's fundamental result through a stronger and more generalized 'useful' inequality of Jensen, which can be used further in information theory. With the help of numerical data, it is shown that inequality holds for both convex and concave functions.

References:

[1] Belis, M. and S. Guiasu. (1968). A quantitative-qualitative measure of information in Cybernetics System, *IEEE Trans. Inform. Theory IT* 14: 593–594.

[2] Chandy, T.W. And J.B. Mcliod (1960). On a functional equation, *Proc. Edinburgh Math's* 43: 7–8.

[3] Dragomir, S. S., C. J. Goh (1996). A counterpart of Jensen's discrete inequality for differentiable convex mappings and applications in information theory, *Math. Comput. Modelling* 24(2): 1–11.

[4] Dragomir, S. S., N. M. Ionescu (1994). Some converse of Jensen's inequality and applications, *Anal. Num. Theor. Approx* 23: 71–78.

[5] Dragomir, S.S. (2001, a). On a converse of Jensen's Inequality, *Univ. Beograd. Publ. Elektrotehn. Fak.Ser. Mat.* 12: 48–51.

[6] Dragomir, S.S. (2001, b), on a reverse of Jensen's inequality for isotonic linear functionals, *journal of Inequalities in Pure and Applied Mathematics*, volume 2, issue 3, article 36.

[7] Fadeev, D.K. (1956). On the concept of entropies of finite probabilistic scheme, (Russian) *Uspchi Math. Nauk* 11: 227–231.

[8] JiLe G., H. Rashid., Farooq F., and Sultana, S. (2021). Some Inequalities for a New Class of Convex Functions with Applications via Local Fractional Integral, *Journal of Function Spaces*, vol. 2021, https://doi.org/10.1155/2021/6663971.

[9] Kendall, D.G. (1964). Functional equations in information theory, *Z.Wahrs Verw.Geb* 2: 225–229.

[10] Khinchin, A.I. (1957). Mathematical Foundations of Information Theory, *Dover Publications* New York.

[11] Klaricic Bakula, M., & Nikodem, K. (2016). On the converse Jensen inequality for strongly convex functions. *Journal of Mathematical Analysis and Applications*, 434(1):516–522. https://doi: 10.1016/j.jmaa.2015.09.032.

[12] Nikodem, K.; Pales, Z.S. (2011). Characterizations of inner product spaces by strongly convex functions. *Banach J. Math. Anal.*, vol. 1: 83–87.

[13] Rashid S., T. Abdeljawad, F. Jarad, and M. A. Noor (2019). Some estimates for generalized Riemann-Liouville fractional integrals of exponentially convex functions and their applications, *Mathematics*, vol.7(9): 807.

[14] Shannon, C.E. (1948). A mathematical theory of communication, *Bell System Technical Journal* 27: 379–423(Part I): 623–656(Part II).

[15] Simic, S. (2009). On a Converse of Jensen's Discrete Inequality. *Journal of Inequalities and Applications*, (1), 153080. doi:10.1155/2009/153080.

[16] Tverberg, H. (1958). A new derivation of the information function, *Math. Scand* 6: 297–298.