Exact solution of the optimal control problem of coordinating a supplier-manufacturer supply chain in advanced geometric concepts

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Abstract: Supply chain coordination deals with collaborative efforts of supply chain parties and making globally-optimal decisions that can improve overall performance and efficiency of the entire supply chain. In many situations, the problem of supply chain coordination requires formulation of a continuous time optimal control model, in which optimal solution is identified approximately through numerical estimation. Therefore, in this paper, a novel approach was presented for optimal control problem by developing a new formulation based on advanced ingredients of differential and Poisson geometry. Thus, the exact optimal solution of control problem can be obtained using an analytical methodology that converts the Hamilton-Jacobi-Bellman partial differential equation (PDE) into a reduced Hamiltonian system. The proposed approach was applied to the problem of coordinating supplier development programs in a two-echelon supply chain comprising of a single supplier and a manufacturing firm. For further illustrating applicability and efficiency of the proposed methodology, a numerical example was also provided. The proposed approach offers unique advantages and can be applied to find the exact solution of optimal control models in various optimization problems.

Key-Words: Optimal Control Problem, Hamiltonian System, First Integral, Supply Chain Coordination, Supplier Development

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1 Introduction

In recent years, supply chain coordination has become a major issue in supply chain management and received much attention from both supply chain researchers and practitioners [1].

Supply chain coordination implies collaborative efforts of supply chain members working together towards mutually-defined goals and activities, including supplier development, coordination with suppliers and customers, etc. [2].

It is concerned with making globally-optimal supply chain decisions that can benefit all supply chain members, instead of individual decisions [3].

In the recent years, supply chain coordination has become a major issue in supply chain management and has received a great deal of attention both from researchers and practitioners in the field of supply chain [4].

Supply chain coordination implies collaborative efforts of supply chain members working together to achieve the mutually-defined goals and activities including supplier development, coordination with suppliers and customers, etc. [5]. It is concerned with making globally-optimal supply chain decisions that can be useful for all supply chain members, instead of individual decisions [6].

Supply chain coordination plays a critical role in improving the overall performance of supply chain and the lack of coordination among supply chain partners may reduce efficiency and result in undesirable consequences in supply chain operations. Therefore, the centralized decision-making and various mechanisms are used by supply chain partners including revenue sharing, risk sharing, synchronized operation, etc., to achieve coordination purposes [7].

In many industries, manufacturing firms develop strategic, long-term relationships with their suppliers by implementing and supporting supplier development programs [8].

The goal is improving performance and capabilities of the suppliers to meet short- and long-term supply needs of manufacturing firm, which in turn results in improving operational performance in terms of cost, quality, delivery, etc. [9].

Such a strong relationship between manufacturers and suppliers enhances the overall efficiency and profitability of both parties and helps to create sustainable competitive advantage [10].

Despite potential benefits of supplier development programs, they might be unattractive for suppliers, because suppliers might be reluctant to modify their internal processes and instead pursue their own objectives [11]. Since, success of supplier development program depends on mutual recognition and aligned objectives, coordination between supplier and manufacturer is required [8], [12]. Thus, optimal decision on supplier development is characterized by a solution for the problem of supply chain coordination.

Many problems of supply chain coordination which were mentioned above, involve formulating and solving a continuous time optimal control model with an equation of incomplete Hamiltonian system, in which the exact optimal solution cannot be obtained, and instead it should be approximately estimated by numerical analysis (e.g., [8], [13]).

Therefore, in this paper, a novel analytical solution approach is presented based on differential and Poisson geometry by reformulating and converting the original problem to a reduced Hamiltonian system ([14-19]. The proposed approach is applied to obtain optimal solution to the problem of coordinating supplier development in a two-echelon supply chain comprising of a single supplier and single manufacturer.

2 Problem of coordinating supplier development

2.1 Problem description and formulation

We consider the problem of coordinating supplier development in a two-echelon supply chain as presented in the study by Proch et al., [8]. Supply chain comprises of a single supplier and a single manufacturing firm where manufacturer assembles components from supplier and sells the final products to the market. The goal is identifying the optimal decision of supplier development investment.

A centralized decision-making process is assumed and supply chain is considered as an integrated system ,in which all parameters including the optimal amount of effort invested in supplier development are simultaneously chosen. This decision-making process ensures efficiency of the entire system and opts for the optimum level of supplier development, i.e., maximizes the total profit of the supply chain. Variables and parameters for this model are summarized in Table 1.

Table 1. Parameters and Decision Variables ([8])

Parameters/ Variables	Description
a	Prohibitive price (e.g. maximum willingness to pay)
b	Price elasticity of the commodity
c_M	Manufacturer's unit production cost
c_{SD}	Supply cost per unit charged by the supplier
c_0	Supplier's unit production cost at the begininig of the contract period
$x(t)$	The measurement of the efforts invested in the supplier development
m	The supplier learning rate
$c_S(x)$ $c_S(x) = c_0 x^m$ $m = \dfrac{\ln(\theta)}{\ln \chi};$ $\theta \in [0,1], \chi > 1$	Supplier production cost
r	The supplier fixed profit margin
$u(t)$	The effort at time t
$\omega(t)$	Capicity limit of (resource availability in terms of time, man power or budget)

The profit function $J^{SC} : L^1([0,T), \mathbf{R}) \to \mathbf{R}$ of the set of measurable functions and the model of efforts invested in supplier development are defined by the following problem:

$$\begin{aligned} &J^{SC} \\ &:= \int_0^T \frac{\left(a - c_M - c_0 x^m(t)\right)^2 - r^2}{4b} - c_{SD} u(t)\,dt, \\ &\textit{subject to } \dot{x} = u; \\ &u : [0,T) \to [0,\omega), \\ &x(0) = x_0 = 1. \end{aligned} \tag{1}$$

The centralized collaboration strategy should be determined such that, the accumulated profit function (1) is maximized. Using the maximum principle applied to the optimal control problem (1) with the Hamiltonian function of

$$H(t,x,u,\lambda) = \frac{\left(a - c_M - c_0 x^m(t)\right)^2 - r^2}{4b} - c_{SD}u(t) + \lambda(t)u(t), \quad (2)$$

switching time t^* can be obtained by the solution to

$$\frac{\partial H}{\partial u}\left(x^*, u^*(t), \lambda(t)\right) = -c_{SD} + \lambda(t) = 0 .$$

Then, as investigated in a previous study [8], t^* is obtained by numerical analysis from the following equation:

$$\frac{mc_0\left(1+\omega t^*\right)^{m+1}\left(a - c_M - c_0\left(1+\omega t^*\right)^m\right)\left(t^* - T\right)}{2b} = c_{SD} \quad (3)$$

More details on the above fromulation have been given in the previous study [8].

2.2. Conversion of the model based on the proposed methodology

The optimization problem given in Equation (1) is a common form in many problems of supply chain coordination. It results in an equation with different parameters for switching time and the optimal control function, which can be only evaluated by numerical estimation. In fact, there exists only one equation with different parameters (Equation (3)).

The case where the Hamiltonian H **Σφάλμα! Δεν έχει οριστεί σελιδοδείκτης.** is linear in control u is of special interest. Especially, it is a simple situation to handle when H **Σφάλμα! Δεν έχει οριστεί σελιδοδείκτης.** is plotted against u **Σφάλμα! Δεν έχει οριστεί σελιδοδείκτης.** either as a positively-or negatively-sloped straight line, since the optimal control is always to be found at a boundary of u. Thus, the only task is determining this boundary. Moreover, this case serves to highlight how a complex situation in the calculus of variations has now become easily manageable in optimal control theory.

This simple approach apparently results in elimination of some equations of the Hamiltonian system in the mentioned coordination optimization problem. For example, because accurate determination of the capacity limit $\omega = \omega(t)$ of $u(t)$ in the problem is not critical to our discussion, it is exogenously assessed to be feasible to the problem.

However, given the proposed approach, we consider all the functions and parameters in the system along with their actual effect. Thus, it will be possible to incorporate more variables in the coordination optimization model. This can be implemented by considering some variables as multiple functions and then, the Hamiltonian function as a function of these variables and their derivatives.

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2.3. Solution method

According to Equation (4), we can rewrite the corresponding Hamiltonian function (2) as

$$H = H(\lambda, x, u, d) = d\left(p(d(t)) - c_M - c_{SC}\right) - c_{SD}u(t) + \lambda(t)u(t), \quad (4)$$

with production quantity of $d(t) = \frac{a - c_M - c_{SC}}{2b}$, and price distribution of

$$p(d) = p(d(t)) = a - bd = a + c_M + c_{SC}$$

$$c_{SC} = r + c_0 x^m$$

Step 1 *(Hamiltonian System):* We have the Hamiltonian system

$$\begin{cases} \frac{\partial H}{\partial x} = \frac{-2mc_0 x^{m-1}(t)\left(a - c_M - c_0 x^m(t)\right)}{4b} = -\frac{d\lambda}{dt}, \\ \frac{\partial H}{\partial d} = \dot{d}\left(a - bd - c_M - c_{SC}\right) - bd = -\frac{du}{dt}, \\ \frac{\partial H}{\partial \lambda} = u = \frac{dx}{dt}, \quad \frac{\partial H}{\partial u} = -c_{SD} + \lambda = \dot{d}, \end{cases}$$

which can be written as follows

$$\lambda(t)=\frac{mc_0x^{m-1}(t)\left(a-c_M-c_0x^m(t)\right)}{2b}, \tag{5}$$

$$\dot{d}(t)=c_{SD}+\lambda(t), \tag{6}$$

$$\dot{u}(t)=\dot{d}\left(-a+bd+c_M+c_{SC}\right)+bd\cdot \tag{7}$$

Step 2 *(First Integrals)*: According to Equation (5), we have

$$\begin{aligned}&\lambda(t)\\&=\lambda\left(t^*\right)\\&-\int_t^{t^*}\frac{mc_0x^{m-1}(s)\left(a-c_M-c_0x^m(s)\right)}{2b}ds\\&=\lambda\left(t^*\right)\\&-\frac{mc_0}{2b}\left(a-c_M\right)\left(I_{m-1}\left(t^*\right)-I_{m-1}(t)\right)\\&+\frac{mc_0^2}{2b}\left(I_{2m-1}\left(t^*\right)-I_{2m-1}(t)\right)\end{aligned} \tag{8}$$

where, $I_m(s)=\int_0^s x^m(k)dk$. Also, according to Equation (6), we have

$$\begin{aligned}d(t)&=d\left(t^*\right)-\int_t^{t^*}\left(\lambda(s)-c_{SD}\right)ds\\&=d\left(t^*\right)-\left(t-t^*\right)c_{SD}-\int_t^{t^*}\lambda(s)ds\cdot\end{aligned} \tag{9}$$

Then, substituting this into Equation (7) results in

$$\begin{aligned}&u(t)=u\left(t^*\right)\\&+\frac{mc_0}{4b}\left(-a+c_M+r\right)\left(I_{m-1}\left(t^*\right)-I_{m-1}(t)\right)\\&+\frac{mc_0^2}{4b}\left(I_{2m-1}\left(t^*\right)-I_{2m-1}(t)\right)\\&-\frac{1}{2}\left(a-c_M-r\right)\left(t^*-t\right)\\&+\frac{c_0}{2}\left(I_m\left(t^*\right)-I_m(t)\right)\end{aligned} \tag{10}$$

Finally, for $x(t)=1+\omega t$, $t\in[0,t^*]$, as expressed in Equation (8), we conclude that

$$\begin{aligned}&\lambda(t)\\&=\lambda\left(t^*\right)\\&-\frac{c_0\left(a-c_M\right)}{2b\omega}\left(\left(1+\omega t^*\right)^m-\left(1+\omega t\right)^m\right)\\&+\frac{c_0^{\,2}}{4b\omega}\left(\left(1+\omega t^*\right)^{2m}-\left(1+\omega t\right)^{2m}\right)\end{aligned} \tag{11}$$

In addition, based on Equation (9), we have:

$$\begin{aligned}&d(t)=d\left(t^*\right)\\&+\frac{c_0\left(a-c_M\right)}{2b\omega}\left(1+\omega t^*\right)^m\left(t^*-t\right)\\&-\frac{c_0\left(a-c_M\right)}{2b(m+1)\omega^2}\left(\left(1+\omega t^*\right)^{m+1}-\left(1+\omega t\right)^{m+1}\right)\\&-\frac{c_0^{\,2}}{4b\omega}\left(\left(1+\omega t^*\right)^{2m}\left(t^*-t\right)\right)\\&+\frac{c_0^{\,2}}{4b(2m+1)\omega^2}\left(\left(1+\omega t^*\right)^{2m+1}-\left(1+\omega t\right)^{2m+1}\right)\end{aligned} \tag{12}$$

Finally, Equation (10) results in

$$\begin{aligned}&u(t)\\&=u\left(t^*\right)\\&+\frac{c_0}{4b\omega}\left(-a+c_M+r\right)\left(\left(1+\omega t^*\right)^m-\left(1+\omega t\right)^m\right)\\&+\frac{mc_0^{\,2}}{8(m+1)\omega b}\left(\left(1+\omega t^*\right)^{2m+2}-\left(1+\omega t\right)^{2m+2}\right)\\&-\frac{1}{2}\left(a-c_M-r\right)\left(t^*-t\right)\\&+\frac{c_0}{2\omega(m+1)}\left(\left(1+\omega t^*\right)^{m+1}-\left(1+\omega t\right)^{m+1}\right)\end{aligned} \tag{13}$$

Step 3 *(Reduction)*: Following Step 2, we have

$$\left(-c_{SD}+p\right)\frac{\partial H}{\partial p}=u\frac{\partial H}{\partial u}$$

Then, its first integral is

$$u=\pm\sqrt{2p^2-2c_{SD}}$$

and Equation (4) is reduced to

$$\begin{aligned}&H(p,x,d)\\&=d\left(p\left(d(t)\right)-c_M-c_{SC}\right)\pm\sqrt{2p^2-2c_{SD}}\left(-c_{SD}+p\right)\end{aligned}$$

2.4. Numerical example and discussion

For further illustrating applicability and superiority of the proposed methodology, a numerical example is presented. Data for the example are adopted from the study by Proch et al., [10]. We apply the proposed approach and the exact solution algorithm presented in the current research to obtain the results and compare them with those obtained from numerical estimation. It helps to evaluate performance and efficiency of the proposed algorithm and analyze quality of the obtained solution against a reference solution. Characteristics of the numerical example are given in Table 2.

Table 2. Parameter Values for Numerical Analysis (Adopted from Proch et al. [8])

T	a	b
60	200	0.01
r	c_{SD}	ω
15	100000	1
c_M	c_0	m
70	100	-0.1

For numerical analysis of the problem using the given parameter values, from Equation (11), we obtain

$$\lambda(t^*) = \lambda(T) - \frac{c_0(a - c_M)}{2b\omega}\left((1+\omega T)^m - (1+\omega t^*)^m\right)$$

$$+\frac{c_0}{4b\omega}\left((1+\omega T)^{2m} - (1+\omega t^*)^{2m}\right)$$

Since $\lambda(t^*) = c_{SD}$, then we have

$$100000 = 0 - \frac{100(200-70)}{0.02}\left((1+60)^{-0.1}\left(1+t^*\right)^{-0.1}\right)$$

$$+\frac{100}{0.04}\left((1+60)^{-0.2} - \left(1+t^*\right)^{-0.2}\right)$$

resulting in $t^* = 9.844$.

Substituting the identified value in Equation (11), we have

$$\lambda(t) = -25655 - 0650000(1+t)^{-0.1} - 250000(1+t)^{-0.2}$$

Since

$$d(t^*) = \frac{\left(a - c_M - \left(rc_0 x^{*m}\right)\right)}{2b} = -19348.65$$

then based on Equation (12), we conclude that

$$d(t) = -19348.65 + 512146.73(9.844 - t)$$

$$-722222.22\{8.54 - (1+t)^{0.9} - 155203.71(9.844 - t)$$

$$+312500\left(6.73 - (1+t)^{0.8}\right)\}$$

Also, according to Equation (13), we obtain

$$u(t) = u(t^*) - 287500\left(0.78 - (1+t)^{-0.1}\right)$$

$$+13888.88\left(73 - (1+t)^{1.8}\right)$$

$$-57.5(9.88 - t) + 55.55\left(8.54 - (1+t)^{0.9}\right)$$

The approximate value of t^* is equal to 9.212, as obtained numerically in the study by Proch et al., [8]. However, the analytical solution algorithm developed herein provides a better answer as it yields a bigger objective value. The difference between the results is due to elimination of some equations of the Hamiltonian system, which is also a prevalent practice to find the answer to the optimal control model in coordination optimization problems.

Using our proposed methodology, the value of switching t^* was obtained as 9.844, which is clearly better than the result obtained in the study by Proch et al., [8] for the presented maximization control problem. In the previous works (e.g., [8], [12-13]), the optimal solution has been identified by eliminating some critical equations. Thus, some important characteristics of the problem should be overlooked. In fact, an accurate determination of $\lambda(t)$, $d(t)$ and $u(t)$ variables has been exogenously assessed to be feasible or they should be approximately identified. But in the proposed method, in actual inspection, we consider the variables as multiple functions and then, the Hamiltonian function as a function of these variables and their derivatives.

3. Conclusions

In this paper, a novel methodology was presented to find the exact optimal solution of the general continuous time optimal control problem by developing a novel reformulation drawing upon differential and Poisson geometry. For this purpose, we applied geometric notions about symmetric groups and first integrals to reduce the order of the Hamiltonian system. The proposed approach and solution method was applied to supply chain coordination problem in a two-echelon supply chain with the objective of finding the optimal decision of supplier development investment. We obtained the exact optimal solution and the optimum switching time for corresponding coordination problem with a single supplier and single manufacturing firm.

The main advantage of the proposed methodology is that it outperforms the numerical estimation approach which is prevalent in solving the optimal control models in coordination optimization problems. The proposed methodology converts the original problem to the system of fully Hamiltonian equations with equations as equal as variables. It provides the analytic optimal solution and, thus, it yields better results than those obtained through numerical estimation. The proposed approach can be also successfully applied to solve optimal control models in other optimization problems.

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