Simple Approximate Solutions for Dynamic Response of Suspension System

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Abstract: - An approximate simplified analytic solution is proposed for the one DOF (degree of freedom) static and dynamic displacements alongside the stiffness (dynamic and static) and damping coefficients (minimum and maximum/critical values) of a parallel spring-damper suspension system connected to a solid mass-body gaining its energy by falling from height h. The analytic solution for the prescribed system is based on energy conservation equilibrium, considering the impact by a special G parameter. The formulation is based on the works performed by Timoshenko (1928), Mindlin (1945), and the U. S. army-engineering handbook (1975, 1982). A comparison between the prescribed studies formulations and current development has led to qualitative agreement. Moreover, quantitative agreement was found between the current prescribed suspension properties approximate value - results and the traditionally time dependent (transient, frequency) parameter properties. Also, coupling models that concerns the linkage between different work and energy terms, e.g., the damping energy, friction work, spring potential energy and gravitational energy model was performed. Moreover, approximate analytic solution was proposed for both cases (friction and coupling case), whereas the uncoupling and the coupling cases were found to agree qualitatively with the literature studies. Both coupling and uncoupling solutions were found to complete each other, explaining different literature attitudes and assumptions. In addition, some design points were clarified about the wire mounting isolators stiffness properties dependent on their physical behavior (compression, shear tension), based on Cavoflex catalog. Finally, the current study aims to continue and contribute the suspension, package cushioning and containers studies by using an initial simple pre – design analytic evaluation of falling mass- body (like cushion, containers, etc.).

Key-Words: - suspension, spring, damper, energy conservation, friction, displacement, stiffness, coupling.

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1 Introduction

Understanding the impact phenomenon of free fall body from height *h* has been studied by many researchers (all enlisted references). The problem has many applications in the packaging and automotive suspension industries.

The first to model vibrations by using advantage formulations of energy conservation was Timoshenko [1] in 1928 (pp. 74-76). He has suggested one DOF (degree of freedom) model (vibrating mass hanging on a spring and moves along the vertical direction only) of energy conservation between the kinetic energy ($mv^2/2$, m – mass weight, v - mass velocity) and the spring potential energy ($kx^2/2$ - $kx_{st}^2/2$, k-spring stiffness, x_{st} - static displacement) in order to evaluate the system frequency. About

two decades later in 1945, Mindling [2] has proposed his own one DOF model based on energy conservation for the free falling mass-body attached to a spring. During his derivation development, he considered an energetic balance between the impact force combined with the kinetic force and the gravitational potential energy. He has derived the expressions for the maximum impact force (P), maximum dynamic displacement (x_d) and the maximum acceleration factor (G).

Three decades later, during the years 1971-1975, the U.S. army, has published several studies [3] – [4] related to the cushioning and containers design, respectively; e.g. involved with analytic approximations for the acceleration, height and time during the shock damage. On the one hand, Ref. [3] deals with simple analytic approximation of impact acceleration dependent on time duration and height, Ref. [4] made a generalized model that predicts g-level response in terms of drop height, static stress, thickness of the cushion and temperature. Note that Ref. [5] is being an updated version of Ref. [3] published in 1982.

Based on the shock velocity and acceleration shapes (triangular saw, semi-sinusoidal acceleration shapes, velocity- time step and simple gravitational falling impact), both Cavoflex [6] and Endine [7] companies have developed energy model that links to the natural frequency to determine the wire rope isolators' suspension coefficients as dependent on the static and dynamic states. Note that Cavoflex company model has similar characteristics as current model; however, the stiffness is calculated by using the frequency parameter value.

Alternative methods to compute the shock dynamic response are mainly based on the full analytic and numerical solutions of the excitation equilibrium (motion) to determine the amplitudes and shock excitations that the suspension physical parameters are dependent on.

Many researchers, as will be mentioned here solved dynamic simple harmonic motion. Full analytic solution for the dynamic motion (mass, damper and spring) influenced by the sinusoidal excitations and gravitational effect in relative to the static state was given by Kaper [8]. Some examples dealing with falling objects have been investigated by French & Kirk (free falling mass-body on a located spring was measured experimentally: spring force and position vs. analytic method) [9], Wong (Board- falling by using one-dimensional spring model) [10], Nagurka & Huang [11] (falling bouncing ball modelled by mass-spring-damper transient model which includes contact time and impact force analyses).

Frequency-time dependent solution divided between static and dynamic states of different suspension products was performed and investigated by Tse *et al*. [12], Schwanen [13] in the context of wire spring modeling, Zhang [14] in the context of rubber isolators dynamic properties and Jazar [15] (different mechanical vibration elements).

Alternative solutions that concerns soft products (cushion and rubber vibration elements) modeling using stress-strain relations attached to energy or frequency expressions have been performed recently by Polukoshko *et al*. [16] and Ge [17], respectively. In addition, some methods based on lumped system or Lagrange energy solutions have been provided by Rajasekaran [18] and Bahreyni [19] in the context of earthquake and mass sensor micro-device modeling, respectively.

Enclosing the brief review, it should be mentioned that full frequent-time dependent solution of the well-known mass-spring-damper differential equation has been solved analytically and numerically by many researchers, for instance, Jacobsen and Ayre [20] (full analytic derivation for transient response to step and pulse reaction functions), Thomson [21] (continue the work of Mindlin [2] in the context of full frequent spring solution with damper), Constantinou *et al*. [22] (have developed relations for the suspensions physical properties in terms of energy expressions dependent on the frequency parameter in the context of earthquake absorbing response structures design), Ray *et al*. [23] (have developed approximate asymptotic expressions for the visco-elastic suspension system dependent on the energetic terms), Coleman [24] (analytic solution for step external input), Cruz-Duarte *et al*. [25] (have performed similar falling mass model as proposed in the current study although it was solved numerically). Moreover, analytic solution that assumes initial displacement appears in the opening of Escalante-Martínez *et al*. study [26] (p. 2) who also developed fractional developed differential equation of viscoelastic fluid in MR (magnetorheological) damper (solved numerically). Giraud [27] has developed advanced frequency lamped parameter – dependent solution for piezoelectric transducer. Of course, it should be mentioned there are many text books that deals with the dynamic vibrations solving the full motion frequency-dependent differential equation [28-33].

The current energy conservation approach is based upon the previous studies method [1]-[7]. Moreover, the current study considers the damping coefficient formulating an energy balance based on the force equilibrium states (static and dynamic) that accompanied with the G-acceleration dynamic factor to evaluate the suspension properties (displacements, spring stiffness, damping coefficient) during the mass-body free falling.

2 Analytic Problem Formulation

Suppose we have mass-body that is connected to a suspension system composed of spring with linear elasticity and a damper with neglected self-mass as shown in Fig. 1. Now, the system (mass, spring and damper) is rest falling from height h such as the energy conservation energy will be expressed as follows:

$$E_p = \underbrace{mg(h + x_d - x_{st})}_{\text{Gravitational energy}} = \underbrace{\frac{1}{2}k_d x_d^2 - \frac{1}{2}k_{st}x_{st}^2}_{\text{Spring potential energy}} = \underbrace{\frac{1}{2}mv^2}_{\text{Kinetic energy}} = E_k \quad (1)$$

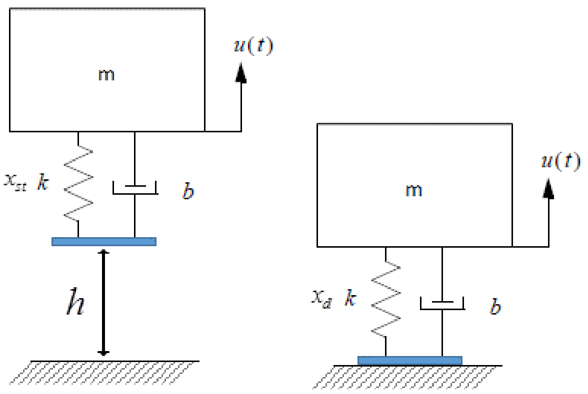


Fig. 1: Free falling of mass attached to a suspension system illustration for dynamic and static suspension states.

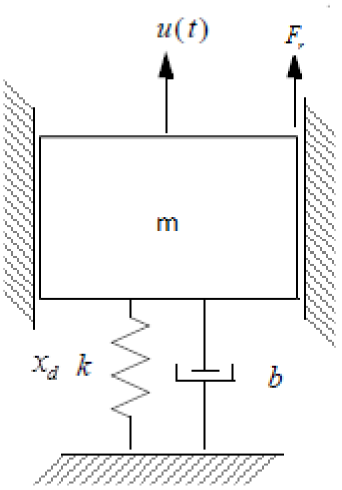


Fig. 2: Free falling of mass attached to a suspension system illustration with friction force influence (F_r).

The initial gravitational energy (E_p) term due to the falling mass body is mgh subtracted from the spring initial static displacement (x_{st}) and was added with the compressed dynamic displacement (x_d) to generate the expression $mg(h+x_d-x_{st})$. The potential energy equals to the kinetic energy (with velocity $v=\sqrt{2g\left(h-x_{st}\right)}\approx\sqrt{2gh}$ since $h>>x_{st}$) that in reversal/return equals to the potential energy difference of spring with linear elasticity ($E_{p-spring}=\frac{1}{2}k_d x_d^2-\frac{1}{2}k_{st}x_{st}^2$ alongside the dynamic k_d and static k_{st} stiffness coefficients, respectively).

Next step, we will define by using energy terms the following equality between the damping energy expression and the kinetic energy, by:

$$E_{p-dumper}=c_{\min}v\left(x_d-x_{st}\right)=\frac{1}{2}mv^2=E_k \quad (2)$$

where $c_{\min}$ is the minimum value of the damping coefficient, whereas the maximum or critical size will be defined by the relation $c_{\max}=Gg\frac{m}{v}\approx c_{critical}=2\sqrt{k_{st}m}$. The reason we have two kinds of distinct energy equalities for damper and spring (even though their mutual assembly configuration) is because each component act separately in relative to the kinetic and gravitational energies due their different functioning (spring is aided to absorb the gravitational energy while the damper should decelerate the kinetic energy) [5, p.5-8]. Although one should remember that in case of damping the dynamic oscillation, the frequency is dependent on the damping coefficient such as $\omega_d=\sqrt{k/m-\left(c/2m\right)^2}$, whereas the critical damping coefficient $c_{critical}=2\sqrt{km}$ and the natural frequency equals to $\omega_n=\sqrt{k/m}$ (see Yu and Wu [37]). The connection between the components is expressed by the distance difference $\Delta=x_d-x_{st}$. Moreover, the number of equations (5) should be compatible with the number of the unknowns' quantity ($k_d, k_{st}, x_{st}, x_d, c$).

Next step is to develop the shock (maximum dynamic force) and the static forces as follows:

$$\begin{aligned} F_d &= k_d x_d = Gmg \\ F_{st} &= k_{st} x_{st} = mg \end{aligned} \quad (3)$$

where G is the loading coefficient (ratio between dynamic and absorption weight) and the link between the spring dynamic and static stiffness coefficients will be assumed to behave linearly, by:

$$k_d=qk_{st}, \quad (4)$$

while the ratio q is constant. Substituting Eq. (3) and (4) into (1), yields the algebraic relations for the dynamic and static spring stiffness coefficients and displacements, respectively:

$$x_d = G\frac{2h}{G^2-2G+q}, \qquad x_{st} = q\frac{2h}{G^2-2G+q}$$
$$k_d = \left(\frac{G^2-2G+q}{2h}\right)mg, \quad k_{st} = \left(\frac{G^2-2G+q}{2h}\right)\frac{mg}{q}$$
$$c_{\min} = \frac{1}{2}\frac{mv}{x_d-x_{st}}, c_{\max} = Gg\frac{m}{v} \approx c_{critical} = 2\sqrt{km},$$
$$\zeta = c_{\min}/c_{critical} = \frac{v^2}{2Gg\left(x_d-x_{st}\right)} \approx \frac{1}{4}\frac{v}{x_d-x_{st}}\sqrt{\frac{m}{k_{st}}} \qquad (5)$$

In case we have a number of equal parallel springs (n) combination, relations (4) become:

$$x_d = G\frac{2h}{G^2-2G+q}, \qquad x_{st} = q\frac{2h}{G^2-2G+q}$$
$$k_d = \left(\frac{G^2-2G+q}{2h}\right)\frac{mg}{n}, \quad k_{st} = \left(\frac{G^2-2G+q}{2h}\right)\frac{mg}{qn}$$
$$c_{\min} = \frac{m}{2n}\frac{v}{x_d-x_{st}}, c_{\max} = \frac{m}{n}\frac{Gg}{v} \approx c_{critical} = 2\sqrt{k_{st}\frac{m}{n}},$$
$$\zeta = c_{\min}/c_{critical} = \frac{v^2}{2Gg\left(x_d-x_{st}\right)} \approx \frac{1}{4}\frac{v}{x_d-x_{st}}\sqrt{\frac{m}{nk_{st}}} \qquad (6)$$

where the dynamic and static forces fulfil $F_d = Gmg/n = F_{st}/G$. As one might observe the solution is not time dependent since we calculate the maximum value of the first energy step (peak) by the approximation method.

For the purpose of the current research extension, alternative solution that links between the energy absorption of both spring and damper might yield the following energy equation:

$$mg\left(h+x_d-x_{st}\right) = cv\left(x_d-x_{st}\right)+\frac{1}{2}k_d x_d^2-\frac{1}{2}k_{st}x_{st}^2 \qquad (7)$$

In turn, by assuming that $c = 2\zeta\sqrt{k_s m}$ alongside the previous assumptions and relations (3)-(4), becomes ($x_{st}, q \neq 0$):

$$\frac{v\zeta}{\sqrt{g}}\left(\frac{G}{q}-1\right)\sqrt{x_{st}}+\left(\frac{G^2-2G+q}{4q}\right)x_{st}-\frac{h}{2} = 0 \qquad (8)$$

where x_{st} should be solved numerically while all the other properties that are dependent on the static displacement should be derived using relations (3)-(4). In case $c = \frac{mg}{v}$ is substituted into Eq. (7) alongside relations (3)-(4), the following linear relations in the explicit form will be resulted:

$$x_{st} = \frac{2h}{\frac{G^2}{q}-1}, \quad k_{st} = \left(\frac{G^2}{q}-1\right)\frac{mg}{2h}$$
$$x_d = \frac{2hG}{G^2-q}, \quad k_d = \left(G^2-q\right)\frac{mg}{2h}$$
$$c_{\min} = \frac{m}{2}\frac{v}{x_d-x_{st}}, c_{\max} = \frac{mg}{v}G \approx c_{critical} = 2\sqrt{k_{st}m},$$
$$\zeta = c_{\min}/c_{critical} = \frac{v^2}{2Gg\left(x_d-x_{st}\right)} \approx \frac{1}{4}\frac{v}{x_d-x_{st}}\sqrt{\frac{m}{k_{st}}} \qquad (9)$$

The obtained coupled expressions (9) will be compared continually in the next section with literature reference calculations.

Another possible development for future study that will be mentioned here is the case of friction force participation (as illustrated in Fig. 2), the calculation should include the work of the friction force which is expressed by $F_r = \mu_d Q_n\left(x_d-x_s\right)$, where Q_n is the wall acting perpendicular normal force (gaining the maximum value of mg while being proportional to $mg\alpha,\ \alpha-\text{constant}$) and μ_d is the dynamic friction (whereas all friction parameters are assumed to be constants), respectively [34]. Adding the latter friction term into the dynamic equation (7) by replacing cv with $cv+\mu_d\alpha mg$ results with another non-linear algebraic equation for x_{st} that should be solved numerically in similar way to the previous equilibrium (8) (with the friction energy extension additive term $\frac{\mu_d\alpha}{2}\left(\frac{G}{q}-1\right)$):

$$\frac{v\zeta}{\sqrt{g}}\left(\frac{G}{q}-1\right)\sqrt{x_{st}}+\left(\frac{G^2-2G+1}{4q}\right)x_{st}+\frac{\mu_d\alpha}{2}\left(\frac{G}{q}-1\right)-\frac{h}{2} = 0 \qquad (10)$$

In case $c = \frac{mg}{v}$ is substituted into Eq. (9) alongside relations (3)-(4), the following linear relations in the explicit form are resulted:

$$x_{st} = \frac{2h}{\frac{G^2}{q} - 1 + \mu_d \alpha \left(\frac{G}{q} - 1 \right)}, \quad k_{st} = \left[\frac{G^2}{q} - 1 + \mu_d \alpha \left(\frac{G}{q} - 1 \right) \right] \frac{mg}{2h}$$
$$x_d = \frac{2hG}{G^2 - q + \mu_d \alpha (G - q)}, \quad k_d = \left[G^2 - q + \mu_d \alpha (G - q) \right] \frac{mg}{2h}$$
$$c_{\min} = \frac{m}{2} \frac{v}{x_d - x_{st}}, c_{\max} = \frac{mg}{v} G \approx c_{critical} = 2\sqrt{k_{st} m},$$
$$\zeta = c_{\min} / c_{critical} = \frac{v^2}{2Gg(x_d - x_{st})} \approx \frac{1}{4} \frac{v}{x_d - x_{st}} \sqrt{\frac{m}{k_{st}}} \tag{11}$$

Note that full accurate dynamic solution is presented by [34, 36 - 37].

Next step, a connection between the solved impact relations (5) and the following original well-known ODE (ordinary differential equation) dynamic equation accompanied with the boundary conditions will be presented:

$$m\ddot{x} + c\dot{x} + kx + mg = 0$$
$$\text{B.C.: } x(0) = 0 \text{ ; } \dot{x}(0) = -v_0 = -\sqrt{2gh} \tag{12}$$

where ω_n, ζ are the natural frequency and the damping ratio, respectively. x coordinate is measured in relative to the initial location of the mass suspension upper surface (cushion [4]) which considered to be positive upward [11].

Hence, the dynamic stiffness and the damping coefficient will be:

$$\omega_n = \sqrt{k/m} \rightarrow k_d = (2\pi f)^2 m/n;$$
$$c = 2\zeta\sqrt{km/n} \tag{13}$$

where f is the natural frequency [rounds/sec] and n represents the number of isolators, respectively. Note that in the case of perpendicular wall friction (Fig. 2 illustration), one should add the term $\mu\alpha mg$ to Eq. (12), whereas the friction coefficient is a function that might be varying in the range $\mu_s \le \mu \le \mu_d$ such as the accurate differential equation form as brought in [34, 36 - 37] is:

Dynamic motion: $m\ddot{x} + c\dot{x} + kx + mg + f_0 \operatorname{sgn}(\dot{x}) = 0;$

$f_0 = \alpha\mu_k mg$

Static state ($|F_r| < \alpha\mu_s mg$ and $|\dot{x}| < v_{slide}$):

$c\dot{x} + kx + mg - m\ddot{x} \le F_r;$ (14)

Note that a code in Python that might solve the problem is attached as a link in the Appendix section.

Although the full solution of the under damping system $(0 < \zeta < 1)$ (6) as a response to step function without friction is [11, 24, 26]:

$$x(t) = e^{-\omega_n \zeta t}\left[\frac{-v_0 + cg/2k}{\omega_d}\sin(\omega_d t) + x_0\cos(\omega_d t)\right] - x_0 \tag{15}$$

where v_0 is the initial velocity, $x_0 = \frac{mg}{k_s}$ is the initial state displacement, ω_n is the natural system resonance frequency, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the dynamic frequency, and $\zeta = \frac{c}{2\sqrt{km}}$ is the damping factor. Note that if $t \to \infty$, then the resulted static initial state is $x = -x_0$. Remark that a numerical code for solving Eq. (12) alongside main cases of dynamic behavior between two masses relationship are presented in the Appendix section. For the purpose of the forthcoming comparative discussion concerning the maximum acceleration value (to determine the G parameter), the second order analytic differentiation manipulation of Eq. (15) might lead to the following acceleration expression:

$$a = \frac{\partial^2 x}{\partial t^2} = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\omega_n \zeta t}\Big[(-2\zeta^3 \omega x_0 + 2\zeta^2 p + 2\zeta\omega x_0 - p)\sin(\omega_d t) + \sqrt{1-\zeta^2}(2\zeta^2\omega x_0 - 2\zeta p - \omega x_0)\cos(\omega_d t)\Big] = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\omega_n \zeta t}\left[\left(2\omega x_0 \zeta - p + \frac{\zeta^2 p}{1-\zeta^2}\right)(1-\zeta^2)\sin(\omega_d t) + \sqrt{1-\zeta^2}(2\zeta^2\omega x_0 - 2\zeta p - \omega x_0)\cos(\omega_d t)\right] = \frac{\omega_n p}{\sqrt{1-\zeta^2}} e^{-\omega_n \zeta t}[\cos(\omega_d t + \gamma)] \tag{16}$$

where $p = -v_0 + cg/2k$ and $\gamma = \frac{\left(\frac{2\omega x_0 \zeta}{p} + \frac{\zeta^2}{1-\zeta^2} - 1\right)\sqrt{1-\zeta^2}}{(2\zeta^2 - 1)\omega x_0/p - 2\zeta}$. Now, using [4] (trigonometric identity) and [11] ($x_0 \ll 1$) relations/assumptions on Eq. (9), the maximum value of the evaluated impact force and acceleration becomes [4]:

$$a_{max} = \left|\frac{\partial^2 x}{\partial t^2}\right| = \frac{\omega_n}{\sqrt{1-\zeta^2}} p,\ F = m a_{max},\ G = a_{max}/g \tag{17}$$

which is obtained for the critical natural frequency (ω_n) value, dependent on the stiffness to mass square root ratio ($\sqrt{k/m}$).

In case a friction term is involved, Eq. (15) becomes approximately:

$$x(t) = e^{-\omega_n \zeta t}\left[\frac{-v_0 + cg/2k}{\omega_d}\sin(\omega_d t) + x_0 \cos(\omega_d t)\right] - x_0 - \mu_d \alpha m g/k \tag{18}$$

The current literature cited equations that will be used to compare with the current formulations are:

$$\begin{aligned}
&[2,3,5]:\ x_d = \frac{2h}{G-2} \text{ (linear system) or } 0.09906\frac{h}{G} \text{ (non-linear system)};\\
&\quad x_{st} = \frac{W}{k_{st}} = \frac{W}{k_d}; k_d \square k_{st};\\
&\quad k_d = 2F_{impact}/x_d = 2ma_{impact}/x_d = 2hW/x_d^2 = W(G-2)^2/2h;\\
&\quad F_{impact} = ma_{impact}; a_{impact} = \frac{v^2}{2x_d};\\
&\quad f_n = \frac{1}{2\pi}\sqrt{g/x_s} = \frac{G}{2\pi}\sqrt{k_s/m} \approx \frac{2G}{\sqrt{h}}[Hz];\\
&\quad c_{critical} = 2\zeta k_s/\omega_n = 2\zeta\sqrt{k_s m}[N-\text{sec}/m];\\
&[4]:\ G = w_n\frac{p}{g}\left(1+\frac{1}{2}\zeta^2\right) \approx w_n\sqrt{2h/g}\left(1+\frac{1}{2}\zeta^2\right),\ p = \frac{cg}{2k} - v_0, \frac{cg}{2k} << 1; \zeta = \frac{c}{2\sqrt{km}}\\
&[6,38]:\ k_{st} = k_d/q;\ k_d = (2\pi f_n)^2 m/n; x_{st} = mg/k_{st};\\
&\quad x_d = mgG/k_d;\ q \text{ - dimensionless}\\
&[7]:\quad k_d = m(Gg/v)^2 \quad [N/m];\\
&\quad x_{st} = mg/k_{st} \quad [m];\\
&\quad x_d = v/\sqrt{k_d/m} \quad [m];\\
&\quad k_{st} = (2\pi f_n)^2 m \quad [N/m];\\
&\quad G \text{ - dimensionless.}
\end{aligned} \tag{19}$$

where $W = mg$.

3 Results and Discussion: Wire Mounting Damper Application

To exemplify the use of relations (5) and (9) versus relations (19) we will use the example appearing in the Cavoflex company catalogue [6] (p. 38). Suppose we have a container with sensitive equipment that should withstand falling from $h = 0.1[m] = 3.93701[inch]$ on a hard surface while $G = 8$ and the total mass equals to $m = 650[kg]$. The hit velocity is equal to $v = \sqrt{2gh} = 1.4[m/\text{sec}]$. In addition, the natural frequency is limited of not exceeding 12Hz.

Table 1. Suspension system parameters values comparison for example.

Ref.	x_d [mm]	x_{st} [mm]	k_{st} [MN/m]	k_d [MN/m]	$c_{critical}$ [kN-sec/m]	ω_n [rad/sec]	ζ *	G	m [kg]
[2,3,5]	33.33	5.56	1.148	1.148	54.6	84.306	0.44	8	650
[4]	--	7	0.91	--	48.64	51.1	0.44	8	650
[6] for q =1	13.8	1.72	3.7	3.7	98.08	75.4	1	8	650
[6] for q =2	13.8	3.45	1.85	3.7	69.35	75.4	1	8	650
[7]	18.5	3.13	2.04	3.7	72.83	75.4	0.81	8	650
Current uncoupling explicit solution Eq. (5) for q=1	32.65	4.1	1.56	1.56	63.69	48.92	0.44	8	650
Current uncoupling explicit solution Eq. (5) for q=2	32	8	0.79	1.59	45.52	34.80	0.44	8	650
Current coupling explicit solution Eq. (9) for q=1	25.4	3.175	2	2	72.11	55.4	0.56	8	650
Current coupling explicit solution Eq. (9) for q=2	25.81	6.45	0.98	1.96	50.48	38.83	0.645	8	650

*The calculation was performed using relation $\zeta = c_{\min} / c_{critical} = \dfrac{v^2}{2Gg\left(x_d - x_{st}\right)}$

Table 1 results alongside relations (5) and (9) might lead to the following comprehensions:

- Numerically, the obtained results are in the same order as the literature results. However, current calculation values of Eqs. (5) are quantitatively agree with Refs. [2, 3, 5] and partially with static calculation [4] (data is not sufficient available on the dynamic stiffness coefficient calculation) and [7]. The reason for numerical differences are derived due to the direct dependency on the natural frequency oscillation that increases the dynamic stiffness and reduces the dynamic distance [6, 7] (reduced reliance only on basic energy conservation, but considering full frequency-time energetic behaviour) while the current calculation concentrating on basic energy conservation. Accordingly, the isolator stiffness dynamic and static properties values (expressions (5) and (9)) have different design requirements (displacements are smaller for larger stiffness values and reversal) than current uncoupling results based on Refs. [6, 7] (although the numeric values are close quantitatively to the coupling solution (9)). Also, an acceptable agreement was found between Ref. [4] and Refs. [2, 3, 5].

- Moreover, the coupling approximate solution (Eq. (9)) values are quantitatively close to the non-coupling solution for the parameter values $q=1$ and $q=2$, respectively. The distinction between solutions (5) and (9) is derived due to the participation of the damping energy potential term in the energy balance equation, which in turn causes to the static and dynamic displacement reduction in relative to the uncoupling case (5).

- One might observe in relations (5) that the height of the falling (h) has a great influence alongside other coefficients (q and G impact intensity), in determining the static and dynamic displacement (x_{st}, x_d), damping (c) and stiffness (k_{st}, k_d). The falling height can be expressed in the automotive industry applications (falling into a pit or bump obstacle) [40 - 41] or in aspects of storing a package falling from the height of a heavy truck or a level [2, 5, 10, 40 - 41].

- The prominent advantageous of expression solutions (5), (6), (9) ,(11), (16), and (18) for different type of suspension state is that it enables to make initial simple pre - design evaluation of falling mass- body (like cushion, containers, etc.) and to compare it with advanced experimental and numerical - simulation tools.

- Note that G value should be a summation of two (displacement) states: dynamic and static, such as the total G value is equal to the dynamic state value added to the static state; e.g. $G = G_d + 1$.

- In cases where the loading magnitude reach to its critical value (over 1 ton), then the dynamic stiffness value has different value than the static one ($k_d \neq k_{st}$, maximum compression ratio: $k_d = 2k_{st}$) [6, 39]. Accordingly, the compression stiffness always greater than the dynamic stiffness ($k_{st,compression} > k_{d,compression}$) due its mechanical mechanism enables the spring shrinkage/compression, resulting the spring stiffness diminishing until extreme state of suspension buckling is likely to occur (maximum coil loading durability).
- In the case of tension loading, the opposite case of compression will be fulfilled ($k_{d,tension} > k_{st,tension}$, but mostly with smaller stiffness ratio $k_{d,tension}/k_{st,tension} \approx 1.5$) [39].
- In case of shear configuration, the suspension stiffness behaves as the tension case ($k_{d,shear} \approx 1.8k_{st,shear}$) until it reaches to its critical loading. Afterwards, the state flips to the compressive stiffness case, so it becomes ($k_{st,shear} > k_{d,shear}$); for the maximum loading configuration the stiffness values are close such as $k_{st,shear} \approx k_{d,shear}$.

Finally, examining the Cavoflex catalog tables [6], it can be observed that most of the differences between the tables are expressed in the dynamic and static displacements and spring stiffness coefficients:

A. On the one hand, the static displacement numerical value is substantially different (ratio of 2 times). On the other hand, the static displacement is greater than dynamic state in the case of $q = 2$.
B. In the context of dynamic and static spring stiffness values, dependent on the loading parameter (q) parameter. The stiffness coefficients in the case $q = 1$ were found to be significantly higher than the case $q = 2$.

4 Conclusion

In this study we present a general approximate framework for analytically calculating the suspension impact parameters (stiffness, displacement and damping coefficients - (minimum and maximum/critical values)) in the dynamic and static states, based on simple energy conservation and force equalities. Next, comparison with literature has revealed that the obtained results are in the same order. Moreover, current calculation values were found to agree numerically with some of the references, which have common assumptions or conditions.

In addition, some design points were clarified about the wire mounting isolators stiffness properties dependent on their physical behaviour (compression, shear tension), based on Cavoflex company catalogue. We also mentioned briefly the wall friction phenomena, analysing it both numerically and analytically. Moreover, a coupling energy equation was derived dependent on the damping energy, friction work (with and without cases), spring potential energy and gravitational energy and was found to agree qualitatively with the uncoupled case and some literature studies. Both solutions, coupling and uncoupling, have been revealed to complete each other and might explain different literature attitudes and assumptions.

Finally, appendix section concerning the numerical modelling of simple suspension systems variations of two mass bodies is brought for the reader use alongside free code.

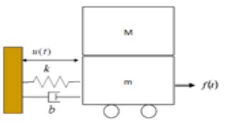
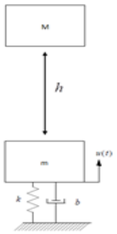
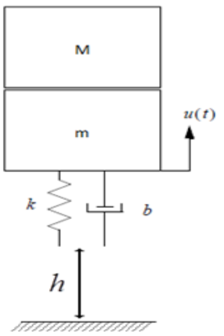
I strongly believe that the prescribed approximations could be beneficial to understand the impact phenomena of cushion, containers and suspension systems design to achieve simple and immediate results on the complete (overall) suspension system properties for the static and dynamic states (displacements, stiffness, damping and frequency parameters).

Appendix: A Simple suspension systems variations of two mass bodies – mass loading vs. Carrier

- Simple suspension systems variations of two mass bodies; mass loading vs. Carrier models are presented and demonstrated using Matlab/Octave code.

- Table of possible cases variations to evaluate dynamic response of suspension systems vibrations of two mass bodies: Mass loading vs. Carrier is presented.
- The target of this document is to evaluate in each case of masses relation the corresponding values of k (stiffness, spring coefficient) and b (damping, damper coefficient) by using dynamic numerical analysis.
- Applications example: Carrier vs. its cargo in different situations (loading, unloading, travelling, etc.)
- A computer program code in Python language that includes the friction influence could be found in the following link: stackoverflow.com/questions/56754421/solving-a-system-of-mass-spring-damper-and-coulomb-friction

Table 2. Dynamic case for two masses relation and their mathematical representation

Case Description	Schematic description	Dynamic Equation	Boundary conditions
Side horizontally impact force $f(t)$ acting on carrier m loaded with mass M		$(m+M)\ddot{x}+kx+b\dot{x}=f(t)$ $\omega_n=\sqrt{k/(M+m)}$ $b=2\zeta\sqrt{k(M+m)}$	$x(0)=0$ *or* $x(0)=0.02$ (some chosen value) $\dot{x}(0)=v_0=0$ $f(t)$ impact force should be a constant value or step value that should be daecaying over time
The loading M falls from rest and height h on the carrier m		$(m+M)x+kx+bx+(M+m)g=0$ $\omega_n=\sqrt{k/(M+m)}$ $b=2\zeta\sqrt{k(M+m)}$	$x(0)=0$ $\rightarrow$ (Energy equation + Elastic Momentum equation) $\dot{x}(0)=v_0=\frac{M}{m+M}\sqrt{2gh}$
The loaded mass m with loading mass M falls from rest and height h		$(m+M)x+kx+bx+(M+m)g=0$ $\omega_n=\sqrt{k/(M+m)}$ $b=2\zeta\sqrt{k(M+m)}$	$x(0)=0$ $\rightarrow$ (Energy equation) $\dot{x}(0)=v_0=\sqrt{2gh}$

Matlab/Octave code

```
clc
clear all
close all
flag=1;
zeta=??;
g=-9.81;              %[m/s^2]
k=??;                 %[N/m]
M=??;                 %[kg]


m=??;                 %[kg]
c=2*zeta*sqrt(k*(m+M));  %[Ns/m]
w_n=sqrt(k/(m+M))        %[rad/sec]
t=0:??:??;          %[sec]  t = [time t_0: increment : t_final]   %starting time for impact detection
%%%%
function F=F_sin(A,w,D,t)
F=A*sin(w*t)+D;
endfunction
%%%

if flag == 1     %Unloading case - both masses are moving together horizontally - under horizontal impact force
x0=??;               %[m]
v0=0;                %[m/sec]
```

```
delta=??;                    %[N]
D=delta;                     %[N]
A=0;                         %[N]
elseif flag == 2  %Loading case - both masses are moving together after free fall of mass M on the carrier m
x0=0;                        %[m]
v0=(M./(M+m))*sqrt(-2*g*h0);%[m/sec]
D=(m+M)*g;                 %[N]
A=0;                       %[N]
elseif flag == 3 %Both masses are free falling together on the suspension system
x0=??;                     %[m]
v0=sqrt(-2*g*h0);            %[m/sec]
D=(m+M)*g;                  %[N]
A=0;                       %[N]
elseif flag == 4  %Carrier m loaded with mass M is vibrated along a given way under sinusoidal force
c=2*zeta*sqrt(k*(m+M));  %[Ns/m]
w_n=sqrt(k/(m+M))         %[rad/sec]
v0=0;                      %[m/sec]
h0=??;                     %[m]
A=-??*(m+M)*g;              %[N]
w=2*pi*??;                 %[rad/s]
D=(m+M)*g;                  %[N]
endif
x0=[h0 v0];                        %x0 = [Initial Position [m], Initial Velocity[m/sec]]
sistema=@(t,x)[x(2);(-c*x(2)-k*x(1)+F_sin(A,w,D,t))/(m+M)];
[t,x] = ode45(sistema,t,x0);
accel=(-c*x(:,2)-k*x(:,1)+F_sin(A,w,D,t))/m;
%%%Plot%%%
figure(1)
plot(t,x(:,1),'b');xlabel('time[sec]');ylabel('Position [m]');grid on;
figure(2)
plot(t,3.6*x(:,2),'r');xlabel('time[sec])');ylabel('Velocity [Km/sec]');grid on;
figure(3)
plot(t,-accel./g,'g');xlabel('time[sec]');ylabel('G - units');grid on;
```

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Data Availability

All data, models, and code generated or used during the study appear in the submitted article.

Conflict of Interest

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