Archimedean Copulas and Several Methods on Constructing Generators

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Abstract: The main aim of this paper is to propose new methods in constructing generators for Archimedean copulas (AC). After reviewing some construction methods of AC generators, three general methods are proposed to construct new generators. These new methods are based on any convex and decreasing functions on [0; 1] and for these forms several examples are provided.

KeyWords: Archimedean copulas, Dependence, Generator, Convex function

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1. Introduction

For the first time Sklar [19] used the word copula, as a function which allows us to combine univariate distributions to obtain a joint distribution. Namely, copula is a function $C : [0,1]^2 \to [0,1]$ which satisfies the following conditions:

(a) For every u, v in [0, 1], $C(u,0) = 0 = C(0,v)$, and $C(u,1) = u$, $C(1,v) = v$;

(b) For every u_1, u_2, v_1, v_2 in [0, 1] such that $u_1 \leq u_2$, and, $v_1 \leq v_2$, $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$.

One of the most important classes of copulas is known as Archimedean copulas(AC). AC are very easy to construct. Many parametric families belong to this class and they have a great variety of different dependence structures. In addition, the Archimedean representation allows us to reduce the study of a multivariate copula to a single univariate function. AC originally appeared in the study of probabilistic metric spaces, where they were studied as part of the development of a probabilistic version of the triangle inequality [17]. Basic properties of AC are presented below. More information could be found in [14].

Let φ be a convex continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1) = 0$. The pseudo-inverse of φ is the function $\varphi^{[-1]}$ given by

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{(-1)}(t), & 0 \leq t \leq \varphi(0) \\ 0, & \varphi(0) \leq t \leq \infty \end{cases} \quad (1)$$

Copulas of the form $C_\varphi(u,v) = \varphi^{[-1]}(\varphi(u)+\varphi(v))$, for every u, v in $[0,1]$, are called AC and the function φ is called a generator (additive generator) of the copula. If $\varphi[0] = \infty$, we say that φ is a strict generator. In this case, $\varphi^{[-1]} = \varphi^{(-1)}$ and $C_\varphi(u,v) = \varphi^{(-1)}(\varphi(u) + \varphi(v))$ is said to be a strict Archimedean copula.

In this study, we review some constructions of AC generators in the literature, then we propose three general methods to construct generators of AC. We provide a wide range of examples for these forms. Moreover, we express several useful Lemma to extend AC.

The rest of this paper is constructed as follows. Section 2, reviews several construction methods of AC. In section 3, we propose three general forms to construct generators in AC, then we provide several examples.

2. Several Construction Methods of AC

AC find a wide range of applications, mainly because of: (1) The ease with which they can be constructed, (2) The great variety of families of copulas which belong to this class, and (3) The many nice properties possessed by the members of this class. Hence there are more efforts for the construction of this class of copulas in the literature. In this section we review some of them. Let Φ be the set of all additive generators of binary AC.

Klement et al. [8] have mentioned a construction method for AC as follows:
Let $f: [0,1] \to [0,1]$ be a concave increasing bijection. Then, for any additive generator $\varphi \in \Phi$, $(\varphi \circ f)(x) = \varphi(f(x))$ is an additive generator from Φ. Generalization of this construction (f need not be a bijection) can be found in [4], [5]. Pekárová [15] proposed new generator for the case, $f: [0,1] \to [0,1]$ is an absolutely monotone bijection (i.e., all derivatives of f on (0,1) exist and they are non-negative). Then for any $\varphi \in \Phi$ also

$$f \circ \varphi \in \Phi. \tag{2}$$

Bacigál et al. [1], proposed three construction methods for AC as follows:
1. Let $f: [0,\infty] \to [0,\infty]$ be a concave increasing bijection. Then for any $\varphi \in \Phi$ we have,

$$f \circ \varphi \in \Phi. \tag{3}$$

2. Let $\varphi_1, ..., \varphi_k \in \Phi$ be additive generators, and $c_1, ..., c_k \in [0,1]$, such that $\sum_{i=1}^{k} c_i = 1$. Then

$$\varphi = \sum_{i=1}^{k} c_i \varphi_i \in \Phi. \tag{4}$$

We note that (4) also holds for any $\varphi_i \in \Phi$ and $c_i \in (0,\infty)$, $i = 1, ..., k$.
3. Let $\varphi_1, ..., \varphi_k \in \Phi$ be additive generators, and $c_1, ..., c_k \in [0,1]$, such that $\sum_{i=1}^{k} c_i = 1$. Then $\varphi \in \Phi$ where $\varphi^{(-1)} = \sum_{i=1}^{k} c_i \varphi_i^{(-1)}$ (i.e. pseudo-inverse of φ is a convex combination of pseudo-inverse of $\varphi_1, ..., \varphi_k$.

In [10], the review of the existing transforms and several properties of AC generators are given as follows,
I. (left composition (LC)) If $\varphi \in \Phi$ and $f: [0,+\infty) \to [0,+\infty)$ is an increasing concave function with $f(0) = 0$, the function defined by

$$\varphi^T(t) = (f \circ \varphi)(t) \tag{5}$$

is a well defined Archimedean generator function. Furthermore, φ^T is strict, iff, φ is strict. We note that here f needs to be an increasing concave bijection (then it is same as the result (3) in [1]). Also we mentioned right composition in [8].
II. If $\theta \in (0,1)$ and $\varphi \in \Phi$, the function defined by

$$\varphi^T(t) = \varphi(\theta t) - \varphi(\theta) \tag{6}$$

is a well defined Archimedean copulas generator function. Furthermore, φ^T is strict iff, φ is strict. Under the same supposing, Mesiar et al. [9] show that, conditional copula $C_{(\theta)}$ ($= C_{[\theta]}$) of the Archimedean copula C with generator φ, is again Archimedean copula, and also it has a generator of the form (6), see also a related study in [7].
III. If $\varphi_1,\ \varphi_2$ are two generators with $(\varphi_2')^2 \leq \varphi_2''$, then function defined by $\varphi^T(t) = \varphi_1(e^{-\varphi_2(\theta)})$, is a well defined Archimedean generator function. Furthermore, φ^T is strict iff, φ_1 and φ_2 are strict.
IV. Let α and β be two positive constants and $\varphi_1,\ \varphi_2$ be two generators, then function defined by

$$\varphi^T(t) = \alpha\varphi_1(t) + \beta\varphi_2(t) \tag{7}$$

is a well defined Archimedean generator function. Furthermore, φ^T is strict iff, φ_1 and φ_2 are strict. For a thorough discussion and analysis of the tail dependence behavior and limiting behavior of the four types of generator transforms, as well as various illustrations, we refer to Michiels and De Schepper [11]. Michiels et al. [12] proposed a new and advantageous method for constructing bivariate Archimedean copula families, based on the function $\lambda(t) = \frac{\varphi(t)}{\varphi'(t)}$, $t \in [0,1]$ where φ is a generator for the Archimedean copulas.

Junker and May [6] have a study related with AC generators and they provided several examples of transformations some of which had previously appeared in the literature. As the results which we have discussed above, totaly conclude Junker and May's results, hence we just express two numbers of their examples: Let $\varphi \in \Phi$, then for any $a \in (1,\infty)$

$$\varphi^T(t) = a^{\varphi(t)} - 1 \tag{8}$$

and for any $a \in (0,1)$

$$\varphi^T(t) = a^{-\varphi(t)} - 1 \tag{9}$$

are AC generators. It has been shown that, if $\varphi \in \Phi$, then for any $\lambda \in [1,\infty)$, $\varphi^\lambda \in \Phi$. For details see [18].

Recently there are new comments on generators of copulas and aggregation functions and we recommend interested readers to [13] and also [2], [3]. In this paper we point on these models in the next sections.

3. New Family of Archimedean Copulas

In this section, by considering the results of previous sections, we propose three general form for generators of AC and then we show that these forms also include several well known families of AC.

Let $h: [a,b] \to]-\infty,\infty]$ be an arbitrary convex continuous, strictly decreasing function. Since $[0,1] \subseteq [a,b]$, we get $g = h|_{[0,1]}$. So $g: [0,1] \to]-\infty,\infty]$ is an convex continuous, strictly decreasing function.

Let $f: Rang(g) = [g(1), g(0)] \to [0,\infty]$, we investigate the cases which $\varphi = f \circ g$ is an additive generator. If f is convex, continuous strictly increasing function, then for any $x, y \in [0,1]$ and any $\lambda \in [0,1]$,

$$\begin{aligned}
\varphi(\lambda x + (1-\lambda)y) &= f(g(\lambda x + (1-\lambda)y)) \quad f\ is\ increasing \\
&\leq f(\lambda g(x) + (1-\lambda)g(y)) \quad f\ is\ convex \\
&\leq \lambda f(g(x)) + (1-\lambda)f(g(y)) \\
&= \lambda\varphi(x) + (1-\lambda)\varphi(y).
\end{aligned} \tag{10}$$

Namely, φ is convex, continuous strictly decreasing function on $[0,1]$, and so $\varphi \in \Phi$.

Let $f: [0,1] \to [0,1]$ be a concave, continuous strictly increasing function. Then similar to (10) we can see that $\varphi = g \circ f$ is convex, continuous strictly decreasing function on $[0,1]$. We recall that $f: [0,1] \to [0,1]$, $f(x) = x^\theta$, for any $\theta \in$

$[1,\infty)$, is convex, continuous strictly increasing function and for any $\theta \in (0,1)$ is concave, continuous strictly increasing function, and so $\varphi \in \Phi$.

By relaying on the above discussion, we propose three general form for AC generators in the following Lemma:

Lemma 3.1: Let $h : [a,b] \to]-\infty,\infty]$ be an arbitrary convex continuous, strictly decreasing function on D where $[0,1] \subseteq D = [a,b]$, we get $g = h|_{[0,1]}$, and then
1) $\varphi(t) = g(\theta t) - g(\theta)$, for any $\theta \in D - \{0\}$, is a generator in AC.
2) $\varphi(t) = g(t^\theta) - g(1)$ is a generator in AC, for any $\theta \in (0,1)$.
3) For the defined $g(t)$, if for all $t \in [0,1]$, $g(t) \geq 0$, then $\varphi(t) = g^\theta(t) - g^\theta(1)$ is a generator in AC, for any $\theta \in [1,\infty)$.
Evidently this Lemma gives only a sufficient condition for θ. Now we are interested in the cases where g is differentiable. We have summarized our result in the following remark.

Remark 3.2: Let $g(t)$ for any $t \in [0,1]$ be an arbitrary convex continuous, strictly decreasing function and also, differentiable for any $t \in [0,1]$ exclusion on n points $x_i \in [0,1]$, $i = 1,...,n$, where $0 = x_0 < ... < x_n < x_{n+1} = 1$. Then
1) $\varphi(t) = g(\theta t) - g(\theta)$, for any $\theta \in D - \{0\}$, is a generator in AC.
2) $\varphi(t) = g(t^\theta) - g(1)$ is a generator in AC, for any θ in the following set:

$$sup\{\theta \in (0,\infty)|\ \theta t g''(t) + (\theta - 1)g'(t) > 0\} \tag{11}$$

for all $t \in [0,1]$, where g' and g'' exist.
3) For the defined $g(t)$, if for all $t \in [0,1]$, $g(t) \geq 0$, then $\varphi(t) = g^\theta(t) - g^\theta(1)$ is a generator in AC, for any θ in the following set

$$sup\{\theta \in (0,\infty)|\ g''(t) + (\theta - 1)g'^2(t)/g(t) > 0\}. \tag{12}$$

for all $t \in [0,1]$, where g' and g'' exist.
It is clear that in all three cases, φ is strict iff, $g(0) = \infty$ and if $g(t)$ for any $t \in [0,1]$ is differentiable, then calculations will be more simple. Let's investigate several examples:

Example 3.3: Let $g(t) = 1/atan(t)$, as this function is convex and decreasing on $[0,\infty)$, so with regard to Remark 3.2, $\varphi(t) = 1/atan(\theta t) - 1/atan(\theta)$, $\varphi(t) = 1/atan(t^\theta) - 1/atan(1)$ and $\varphi(t) = 1/atan^\theta(t) - 1/atan^\theta(1)$ are generators in AC for any $\theta \in (0,\infty)$.

Example 3.4: Let $g(t) = csc(t)$. This function is convex and decreasing on $[0,\pi/2]$, then Remark 3.2 guarantees that, $\varphi(t) = csc(\theta t) - csc(\theta)$, is a generator in AC for any $\theta \in (0,\pi/2]$. Also $\varphi(t) = csc(t^\theta) - csc(1)$ and $\varphi(t) = csc^\theta(t) - csc^\theta(1)$ are generators in AC for any $\theta \in (0,\infty)$.

Example 3.5: Let $g(t) = csch(t)$. This function is convex and decreasing on $[0,\infty)$, therefore $\varphi(t) = csch(\theta t) - csch(\theta)$, $\varphi(t) = csch(t^\theta) - csch(1)$ and $\varphi(t) = csch^\theta(t) - csch^\theta(1)$ are generators in AC for any $\theta \in (0,\infty)$.

Example 3.6: Let $g(t) = cot(t)$. This function is convex and decreasing on $[0,\pi/2]$. Hence, $\varphi(t) = cot(\theta t) - cot(\theta)$, is a generator in AC for any $\theta \in (0,\pi/2]$, also $\varphi(t) = cot(t^\theta) - cot(1)$ and $\varphi(t) = cot^\theta(t) - cot^\theta(1)$ are generators in AC for any $\theta \in (0,\infty)$.

Example 3.7: Let $g(t) = coth(t)$. This function is convex and decreasing on $[0,\infty)$. Hence $\varphi(t) = coth(\theta t) - coth(\theta)$, $\varphi(t) = coth(t^\theta) - coth(1)$ and $\varphi(t) = coth^\theta(t) - coth^\theta(1)$ are generators in AC for any $\theta \in (0,\infty)$.

Example 3.8: Let $g(t) = \frac{1}{t}$, this function is convex and decreasing on $[0,\infty)$, hence for the first form we get $\varphi(t) = \frac{1}{\theta}(\frac{1}{t} - 1)$, and for other two forms we get $\varphi(t) = \frac{1}{t^\theta} - 1$, for any $\theta \in (0,\infty)$. Clearly these forms, introduce only Clayton family generators.

Example 3.9: Let $g(t) = 1 - t$, this function is convex and decreasing on $[0,\infty)$, so for the first form we get generators of the form $\varphi(t) = \theta(1 - t)$, where $\theta \in (0,\infty)$. For the second form, we have $\varphi(t) = 1 - t^\theta$, where $\theta \in (0,1]$. Clearly these two forms are Clayton family generators. For the third form, we get $\varphi(t) = (1 - t)^\theta$, where it is a generator for any $\theta \in [1,\infty)$. This generator had been cited in [14] as generator of the 4.2.2 family.

Example 3.10: Let $g(t) = cot(\pi t/2)$, this function is convex and decreasing on $[0,1]$, so via the first form we get $\varphi(t) = cot(\theta\pi t/2) - cot(\theta\pi/2)$, for any $\theta \in (0,1]$. Via the second form we get $\varphi(t) = cot(\pi t^\theta/2)$ where $\theta \in (0,\infty)$. Also the third form presents $\varphi(t) = cot^\theta(\pi t/2)$, where $\theta \in [1,\infty)$. This family had been cited as Cot-copula in [16].

Example 3.11: Let $g(t) = (t-1)^2 - 1$, this function is convex and decreasing for any $[0,1]$, therefore for the first form, we get generators of the form, $\varphi(t) = (\theta t - 1)^2 - (\theta - 1)^2$, for any $\theta \in (0,1]$. For the second form, we have $\varphi(t) = (t^\theta - 1)^2$, where $\theta \in (0,1]$, which is Clayton family generator. The third form, as $g(1) = -1 < 0$, is not applicable.

Example 3.12: Let $g(t) = e^{-t}$, this function is convex and decreasing on $[0,\infty)$, hence for the first and third form, we get generators of the form, $\varphi(t) = e^{-\theta t} - e^{-\theta}$, for any $\theta \in (0,\infty)$. For the second form, we have $\varphi(t) = e^{-t^\theta} - e^{-1}$, where $\theta \in (0,1]$.

Example 3.13: Let $g(t) = e^{1/t}$, this function is convex and decreasing on $[0,\infty)$, hence for the first form, we get generators of the form, $\varphi(t) = e^{1/(\theta t)} - e^{1/\theta}$, for any $\theta \in (0,\infty)$. For the second form, we have $\varphi(t) = e^{t^{-\theta}} - e$, where $\theta \in (0,\infty)$. This is generator of the family that had been cited in [14] as 4.2.20 family. For the third form, we get $\varphi(t) = e^{\theta/t} - e^\theta$, for any $\theta \in (0,\infty)$. This is also generator of the family that had been cited in [14] as 4.2.19 family.

Example 3.14: Let $g(t) = -log(t)$, this function is convex and decreasing on $[0,\infty)$, therefore for the first form, we get generators of the form, $\varphi(t) = -log(t)$. For the second form, we have $\varphi(t) = -\theta log(t)$, where $\theta \in (0,\infty)$. Clearly, these two generators, generate only product copula. For the third form, we get $\varphi(t) = (-log(t))^\theta$, for any $\theta \in [1,\infty)$. This is generator of the Gumbel family.

Example 3.15: Let $g(t) = 1 - log(t)$, this function is convex and decreasing on $[0,\infty)$, therefore for the first form, we get generators of the form, $\varphi(t) = -log(t)$. For the second form, we have $\varphi(t) = -\theta log(t)$, where $\theta \in (0,\infty)$. Clearly, these generators, generate only product copula. For the third form, we get $\varphi(t) = (1 - log(t))^\theta - 1$, for any $\theta \in (0,\infty)$. This is generator of the family that had been cited in [14] as 4.2.13 family.

Now we want to discuss about structure of the proposed new forms of generators by Lemma 3.1 in the Archimedean conditional copulas. We recall that, Mesiar et al. in [9] show that, conditional copula $C_{(\theta)}$ $(= C_{[\theta]})$ of the Archimedean copula C with generator φ, is again Archimedean copula, and also it has a generator of the form

$$\varphi^T(t) = \varphi(\theta t) - \varphi(\theta) \quad (13)$$

where $\theta \in (0,1)$. Structure of the generators in the Archimedean conditional copulas, for the proposed new forms of generators by Lemma 3.1 are summarized in the following Lemma whose proof is a matter of calculation only and therefore omitted.

Lemma 3.16: Let $g(t)$ is an arbitrary convex continuous, strictly decreasing function on $D = [a, b]$, where, $[0, 1] \subseteq D$. Then
1) $\varphi(t) = g(\theta t) - g(\theta)$, for any $\theta \in D - \{0\}$, generators an Archimedean conditional copula.
2) $\varphi(t) = g(at^\theta) - g(a)$ is a generator in Archimedean conditional copulas, for any $0 < a < 1$ and $\theta \in (0, 1)$.
3) For the defined $g(t)$, if for all $t \in [0, 1]$, $g(t) \geq 0$, then $\varphi(t) = g^\theta(at) - g^\theta(a)$ is a generator in Archimedean conditional copulas, for any $0 < a < 1$ and $\theta \in [1, \infty)$. It is clear that in all three cases, φ is strict iff, $g(0) = \infty$.
In the cases which g is differentiable, we summarized our result in the following remark.

Remark 3.17: Let $g(t)$ for any $t \in [0, 1]$ is arbitrary convex continuous, strictly decreasing function and also, it is differentiable for any $t \in [0, 1]$ exclusion on n points $x_i \in [0, 1]$, $i = 1, ..., n$, where $0 = x_0 < ... < x_n < x_{n+1} = 1$. Then
1) $\varphi(t) = g(\theta t) - g(\theta)$, for any $\theta \in D - \{0\}$, generators an Archimedean conditional copula.
2) $\varphi(t) = g(at^\theta) - g(a)$ is a generator in Archimedean conditional copulas, for any $0 < a < 1$ and θ in the following set

$$sup\{\theta \in (0, \infty)|\ \theta t g^{''}(t) + (\theta - 1)g'(t) > 0\} \quad (14)$$

for all $t \in [0, 1]$, where g' and g'' exist.
3) For the defined $g(t)$, if for all $t \in [0, 1]$, $g(t) \geq 0$, then $\varphi(t) = g^\theta(at) - g^\theta(a)$ is a generator in Archimedean conditional copulas, for any $0 < a < 1$ and θ in the following set

$$sup\{\theta \in (0, \infty)|\ g^{''}(t) + (\theta - 1)g'^2(t)/g(t) > 0\} \quad (15)$$

for all $t \in [0, 1]$, where g' and g'' exist. It is clear that in all three cases, φ is strict iff, $g(0) = \infty$.

Example 3.18: Let $g(t) = \frac{1}{t}$, this function is convex and decreasing on $[0, \infty)$, so in the first form, for any $\theta \in (0, \infty)$ we get $\varphi(t) = \frac{1}{\theta}(\frac{1}{t} - 1)$, with other two forms, for any $a \in (0, 1)$ and $\theta \in (0, \infty)$, we get $\varphi(t) = \frac{1}{a}(\frac{1}{t^\theta} - 1)$. Clearly all these three forms, introduce only Clayton family generators. We recall that only Clayton family are stable under univariate conditioning (for more details, see related study by Jágr et al. [7]).

Example 3.19: Let $g(t) = 1 - t$, this function is convex and decreasing on $[0, \infty)$. For the first form we get generators for Archimedean conditional copulas in the form $\varphi(t) = \theta(1 - t)$, where $\theta \in (0, \infty)$. For the second form, we have $\varphi(t) = a(1 - t^\theta)$, where $0 < a < 1$ and $0 < \theta \leq 1$. Clearly these two forms are generators of the Clayton family which is stable under univariate conditioning. For the third form, we get $\varphi(t) = (1 - at)^\theta - (1 - a)^\theta$, where generates Archimedean conditional copulas, for any $0 < a < 1$ and $\theta \in [1, \infty)$.

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