Approximate Controllability

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Abstract: The objective of this work is to study the approximate controllability with constraints on the control of the heat equation. To demonstrate the approximate controllability, it suffices to use Hahn Banach theorem and notice that the orthogonal of the set of reachable states at the moment t=0 is reduced to zero. Finally we adapted the method of duality (primal-dual) of Fenchel-Rockafellar to characterize the optimal control.

Key–Words: Heat equation, Approximate controllability, Optimal control, Penalty method.

Received: September 18, 2019. Revised: February 1, 2020. Accepted: February 27, 2020. Published: March 3, 2020.

1 Introduction

The notion of controllability is of great importance in mathematical control theory. It is closely related to the theory of minimal realization and optimal control. Many fundamental problems of control theory such as pole-assignment, stabilisability and optimal control may be solved under assumption that system is controllable. Roughly speaking, controllability means, that it is possible to steer a dynamic system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. To be brief, if a dynamic system is controllable, all modes of the system can be excited from the input. Controllability of a system permits the choice of state feedback resulting in desirable properties of closed-loop system. There are many different definitions and criteria of controllability that depend both on state equation and constraints on the control. Most of the criteria, which can be met in literature, are formulated for finite dimensional systems. It should be pointed out that many unsolved problems still exit as far as controllability of infinite dimensional systems is concerned, in order to fill this gap. In the case of infinite dimensional systems two basic concepts of controllability can distinguished. There are exact and approximate controllability. This is strongly related to the fact that in infinite dimensional spaces there exist linear subspaces, which are not closed. Exact controllability enables to steer the system to arbitrary final state while approximate controllability means that system can be steered to arbitrary small neighborhood of final state. In other words approximate controllability gives the possibility of steering the system to the states which form the dense subspace in the state space. Taking this into account it is obvious that exact controllability is essentially stronger notion than approximate controllability. In other words, exact controllability always implies approximate controllability. The converse statement is generally false. However, in the case of infinite dimensional systems exact controllability appears rather exceptionally. On the other hand, it should be stressed that in the case of finite dimensional systems notions of exact and approximate controllability coincide. The main purpose of this work is to formulate and prove the approximate controllability of the parabolic system "equation of heat" and characterize the optimal control. Using the semi-group theory and in general the properties of functional analysis and operators, we are going to prove that the system is controllable in an approximate way if and only if the set of all the reachable states is dense in the space of states. Finally, we are going to present a simple example illustrating the general theory. In the example, the computable conditions necessary and sufficient for the approximate controllability of the dynamic system with linear distributed parameters described by a partial differential equation of the heat type will be presented.

2 Preliminary

2.1 Optimal control

We consider the system (5) and we give a cost function

$J(u, q)$ (1)

Definition 1 *The optimal control of the system (5)-(1) consists in finding a couple-state* $(\tilde{u}, \tilde{q}) \in L^2(\mathcal{O} \times (0,T)) \times L^2(\mathcal{Q})$ *solution of the problem*

$$\begin{cases} \min J(u,q), \\ (u,q)\, verifying(5). \end{cases}$$

2.2 Penalty method

Penalty is a simple concept to transform an optimization problem with constraints to a problem or a series of optimization problems without constraints.

The different techniques of Penalty are often based on the following principal.

We replace the problem

$$(\mathcal{P}) : \begin{cases} \inf\limits_{x \in \Lambda} f(x), \end{cases}$$

where Λ is a part of a vector space E and $f : E \longrightarrow R$ is a function, by one or more problem(s) of the type

$$(\mathcal{P}_\varepsilon) : \begin{cases} \inf\limits_{x \in E} J_\varepsilon(x), \end{cases}$$

where J_ε is a function defined by

$$J_r(x) = f(x) + \varepsilon p(x).$$

With $\varepsilon > 0$ and $p : E \longrightarrow R$ is a function chosen according to constraints.

2.3 Duality of Fenchel-Rockafellar

In this part, we adapt the **HUM** method to our control problem. For all penalty parameter $\varepsilon > 0$, we compute the control that minimizes the penalized HUM functional F_ε given by $F_\varepsilon = \frac{1}{2}\|u\|^2_{L^2(\mathcal{O}\times(0,T))} + \frac{1}{2}\|y(T; y_0, u)\|^2_{L^2(\Omega)}$,

where y is the solution to (4) . We can find the argument relating the null/approximate controllability and this kind of functional. Using the Fenchel-Rockafellar theory, we know that the minimum of F_ε is equal to the opposite of the minimum of J_ε, the so-called dual functional, relative to the first one we will call the primal problem, defined for all $\rho^0 \in L^2(\Omega)$ by $J_\varepsilon = \frac{1}{2}\|\rho\|^2_{L(\mathcal{Q})} + \frac{\varepsilon}{2}\|\rho^0\|^2_{L(\Omega)} + \langle y(T; y_0, 0), \rho^0 \rangle_{L^2(\Omega)}$,

where ρ is the solution to the adjoint system (16). moreover the minimizers u_ε and ρ^0_ε of the functional F_ε and J_ε respectively, are related through the equality $u_\varepsilon = 1_{\mathcal{O}}\rho_\varepsilon$,where ρ_ε is the solution to the backward System (16) with the initial data $\rho(0) = \rho^0_\varepsilon$. A simple computation leads to $\nabla J_\varepsilon(\rho^0) = \Lambda\rho^0 + \varepsilon\rho^0 + y(T; y_0, 0)$,

with the Gramian operator Λ defined as follows

$$\begin{array}{rccc} \Lambda : & L^2(\Omega) & \longrightarrow & L^2(\Omega) \\ & \rho^0 & \longrightarrow & z(T) \end{array},$$

where z is the solution of the following backward and forward systems

$$\begin{cases} \rho' - \Delta\rho &=& 0 & in\mathcal{Q}, \\ \rho(0) &=& \rho^0 & in\Omega, \\ \rho &=& 0 & on\Sigma, \end{cases} \tag{2}$$

and

$$\begin{cases} -z' - \Delta z &=& \rho\chi_{\mathcal{O}} & in\mathcal{Q}, \\ z(T) &=& 0 & in\Omega, \\ z &=& 0 & on\Sigma. \end{cases} \tag{3}$$

Then the minimizer u_ε of F_ε will be computed with the help of the minimizer ρ^0_ε of J_ε which is the solution of the linear problem

$$(\Lambda + \varepsilon)_{\rho^0_\varepsilon} = -y(T; y_0, 0).$$

Remark 2 *The theory of duality ensures that if* J *is convex and coercive then both primal and dual problems admit the same optimal value.*

3 Approximate Controllability

This section studies the equations of heat through two broad domains (control theory and optimization). From a given controllability problem we demonstrate the existence and uniqueness of an optimal solution (in a sense to be specified).

What does it mean to control a system of equations? In general, to control a system of equations, we start from a system admitting a unique solution on which we have the choice of one of parameters that we will call control.

The system thus becomes over determined and we will try, as much as possible, to find the control that will allow, for example, to reach a target or to optimize a data.

The controllability that interests us is the study of controlled systems, i.e. dynamic systems (depending on time noted t)on which we act using a command (or control).

The objective is then to bring the system from an initial state (at the initial instant $t = t_0$) close to a final state (at a moment $t = T$), which is called approximate controllability.

We study the approximate controllability of the following problem

$$\begin{cases} y' - \Delta y &=& 0 & in & \mathcal{Q} = \Omega \times]0,T[, \\ y(0) &=& y_0 & in & \Omega, \\ y &=& 0 & on & \Sigma = \Gamma \times]0,T[. \end{cases}$$

3.1 Problem presentation

Let Ω an open of R^n representing the geometric domain of the system$(n = 1, 2, 3 for applications)$ and let $T > 0$, we assume that the border is smooth.

We consider a distributed parameter system described by the system state equation

$$\left\{ \begin{array}{rcll} y' - \Delta y & = & 0 & in \quad \mathcal{Q} = \Omega \times]0, T[\,, \\ y(0) & = & y_0 & in \quad \Omega, \\ y & = & 0 & on \quad \Sigma = \Gamma \times]0, T[\,. \end{array} \right. \tag{4}$$

Let $\mathcal{O} \subset \Omega$, the open $\mathcal{O}$ is the set of observations .

We must consider that $\mathcal{O}$ is "small" . And we put $U = L^2(\mathcal{O} \times [0, T])$.

3.1.1 The adjoint state

We define $q = q(x, t)$ solution of

$$\left\{ \begin{array}{rcll} -q' - \Delta q & = & (h + u)\,\chi_{\mathcal{O}} & in \quad \mathcal{Q}, \\ q(T) & = & 0 & in \quad \Omega, \\ q & = & 0 & in \quad \Sigma. \end{array} \right. \tag{5}$$

Where $(.)'$ is the partial derivative w.r.t time. $h \in U$ and $\chi_{\mathcal{O}}$ denotes the characteristic function of $\mathcal{O}$. It is well known that parabolic problem (5) admits a unique solution, this function depends on u which is to be determined. (See for example [8]).

We will use the notation

$$q = q(x, t; u)\,. \tag{6}$$

To say that the solution q of (5) depends on the command u which plays a particular role.

More specifically, let $\varepsilon > 0$, we would like to choose u in order to reach the following objective.

Let h be a given function in U, we look for a variable control $u \in U$ such that

$$\|u\|_U = minimum. \tag{7}$$

And such that if $q = q(x, t; u)$ is the unique solution of (5), then

$$\|q(., t; u)\|_{L^2(\Omega)} \leq \varepsilon in \Omega. \tag{8}$$

Remark 3 *Condition (7) expresses that we "move away as little as possible" (in the sense of L^2) from h.*

The role of u is to guarantee the approximate controllability property (8).

Problem (5) and (8) is a problem of approximate controllability.

3.2 Approximate controllability and density theorem

We are therefore looking for u such that if $q = q(x, t; u)$ is the solution of (5) we have

$$\left\{ \begin{array}{l} \|q(., t; u)\|_{L^2(\Omega)} \leq \varepsilon, \\ \|u\|_U = min. \end{array} \right.$$

For conditions (8) and (7) to be satisfied, it is sufficient that for every $\varepsilon > 0$, there is a function $u \in U$ such that

$$\|q(., 0; u)\|_{L^2(\Omega)} \leq \varepsilon.$$

For this purpose, system (5) is divided into two systems

$$\left\{ \begin{array}{rcll} -q_1' - \Delta q_1 & = & h\chi_{\mathcal{O}} & in \quad \mathcal{Q}, \\ q_1(T) & = & 0 & in \quad \Omega, \\ q_1 & = & 0 & on \quad \Sigma. \end{array} \right. \tag{9}$$

and

$$\left\{ \begin{array}{rcll} -z' - \Delta z & = & u\chi_{\mathcal{O}} & in \quad \mathcal{Q}, \\ z(T) & = & 0 & in \quad \Omega, \\ z & = & 0 & on \quad \Sigma. \end{array} \right. \tag{10}$$

So $z = q - q_1$.

Function q_1 is therefore given, we are looking for u so that $z = z(u)$ can check

$$\|z(0, u) + q_1(0)\|_{L^2(\Omega)} \leq \varepsilon. \tag{11}$$

If we consider here that
u =control function,
$z = z(u)$ =state of a (new) system.

Then (11) and (7) is a controllability problem. For all $\varepsilon > 0$ we are looking for a control u of the control space U allowing to approach to within ε , in a finite time, the state of 0 (at the "initial" moment $t = T$) until the state $-q_1(0)$ (at the "final" moment $t = 0$), with minimum expense for u, in the sense of$\|u\|_{L^2} = \min$ (see [5]).

The main result is as follows

Theorem 4 *For $\varepsilon > 0, h \in U$, there is a control u and a state q such that (5) and (8) are verified. In addition to this, there is a unique couple $(\tilde{u}, \tilde{q})$ with u of minimum norm in U, i.e. such that (5),(4) and (7) are verified.*

Before proving this theorem it should be noted that

1. The demonstration of this theorem based on the following result.

   There is $u \in U$ such that if z a solution of (10), we have $F(0) = \{z(x,0); u \in U\}$.

   To demonstrate density, it is sufficient to use Hahn Banach's theorem and notice that $F^{\perp}$ is reduced to zero $\left(F^{\perp} = \{0\}\right)$.

   Let $\rho^0 \in F^{\perp}$ and ρ solutions of $\rho' - \Delta\rho = 0, \rho(0) = \rho^0, \rho = 0$ on Σ.

   Remark 5 *Since $u = -h$ gives $q \equiv 0$, so (10), the previous problem always admits one solution and only one so that the real problem is to calculate the optimal u.*

2. In brief, the problem of the existence of a control is the same as solving the following optimization problem

$$(\mathcal{P}) : \left\{ \min_{\Lambda} \|u\|_{L^2} \right\}, \tag{12}$$

   where

$$\Lambda = \left\{ \begin{array}{l} us.t \left\{ \begin{array}{rcll} -q' - \Delta q &=& (h+u)\chi_{\mathcal{O}} & in\mathcal{Q}, \\ q(T) &=& 0 & in\Omega, \\ q &=& 0 & on\Sigma, \end{array} \right. \\ \quad with \|q(.,0;u)\|_{L^2(\Omega)} \leq \varepsilon. \end{array} \right\}$$

   The constraints domain of the problem $(\mathcal{P})$ is not empty because $u = -h$ gives $q \equiv 0$, therefore the problem $(\mathcal{P})$ always admits one solution and only one that is noted $\tilde{u}$. So there are still two problems to be solved

   - To calculate $\tilde{u}$,
   - Make sure that $\tilde{u} \neq -h$.

3. We use the penalty method to dispose of constraints, as well as to get the optimality system ("general case").

   (of theorem 4)

   Let q be a solution of system (5) and q_1 a solution of system (9).

   Then z is the solution of the following problem

$$\left\{ \begin{array}{rcll} -z' - \Delta z &=& u\chi_{\mathcal{O}} & in\mathcal{Q}, \\ z(T) &=& 0 & in\Omega, \\ z &=& 0 & on\Sigma. \end{array} \right. \tag{13}$$

We now introduce all the states that can be reached at time $t = 0$ defined by

$$F(0) = \{z(u,0); u \in U\}. \tag{14}$$

It is clear that $F(0)$ is a vector subspace of $L^2(\Omega)$.

According to **Hahn Banach's** theorem, it will be dense in $L^2(\Omega)$ if and only if its orthogonal in $L^2(\Omega)$ is reduced to zero.

As $\{0\} \subset F(0)^{\perp}$, it remains to be shown that $F(0)^{\perp} \subset \{0\}$.

Let $\rho^0 \in F(0)^{\perp}$, then

$$\left\langle \rho^0, z(0) \right\rangle_{L^2(\Omega)} = \int_{\Omega} \rho^0 z(0) dx = 0. \tag{15}$$

Where z is a solution of (13). It is therefore natural to defined the adjoint ρ of z, it's the solution of the following problem

$$\left\{ \begin{array}{rcll} \rho' - \Delta\rho &=& 0 & in\mathcal{Q}, \\ \rho(0) &=& \rho^0 & in\Omega, \\ \rho &=& 0 & on\Sigma. \end{array} \right. \tag{16}$$

The system (16) is a classic problem of the heat equation which has a unique solution ρ.

Now, multiply the first equation of the system (13) by ρ, and after integration by parts, we get

$\int_0^T \int_\Omega z'\rho dxdt - \int_0^T \int_\Omega \Delta z\rho dxdt = \int_0^T \int_{\mathcal{O}} u\rho dxdt - \int_\Omega \rho(T)z(T)dx + \int_\Omega \rho(0)z(0)dx + \int_0^T \int_\Omega \Delta\rho z dxdt - \int_0^T \int_\Omega \Delta z\rho dxdt = \int_0^T \int_{\mathcal{O}} u\rho dxdt.$

Applying **Green's formula**, we find

$-\int_\Omega \rho(T)z(T)dx + \int_\Omega \rho(0)z(0)dx - \int_0^T \int_\Omega \nabla\rho\nabla z dxdt + \int_0^T \int_{\partial\Omega} \frac{\partial\rho}{\partial n} z dxdt + \int_0^T \int_\Omega \nabla z\nabla\rho dxdt - \int_0^T \int_{\partial\Omega} \frac{\partial z}{\partial n}\rho dxdt = \int_0^T \int_{\mathcal{O}} u\rho dxdt.$

As z and ρ are solutions of (13) and (16) respectively, (17) becomes

$$\int_0^T \int_\Omega \rho u\chi_{\mathcal{O}} dxdt - \int_\Omega \rho^0 z(0)dx = 0. \tag{17}$$

This is equivalent to

$$\int_0^T \int_{\mathcal{O}} \rho u dxdt = 0. \tag{18}$$

Because, $\rho^0 \in F(0)^{\perp}$ and $z(0) \in F(0)$.

So,

$$\rho = 0 in \mathcal{O} \times (0,T). \tag{19}$$

According to a single continuation result of **S.MIZOHATA** [12], it follows that

$$\rho = 0 in \mathcal{Q}.$$

As a result, $\rho^0 = 0$ which shows that $F(0)^{\perp} = \{0\}$.

3.3 Optimal control characterization

In this section, we will characterize the optimal control using a result of duality of **Fenchel-Rockafellar.** (see [6]).

The system of optimality satisfied by $(\tilde{u}, \tilde{q})$ is established as follows.

Let $\rho^0 \in L^2(\Omega)$ and ρ the associated solution of (16). Now, we introduce the functional J_ε defined by $\mathrm{J}_\varepsilon = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |\rho|^2 \, dxdt + \varepsilon \left\|\rho^0\right\|_{L^2(\Omega)}$
$+ \int_0^T \int_\Omega \rho h \chi_{\mathcal{O}} dxdt.$

We consider the problem without constraints

$$(\mathcal{P}_\varepsilon) : \begin{cases} \min J_\varepsilon(\rho^0), \\ \rho^0 \in L^2(\Omega). \end{cases} \tag{20}$$

Then we have

Proposition 6 *The functional J_ε defined in (3.3) is coercive.*

To prove that J_ε is coercive [1], it suffices to show the following relation

$$\lim_{\left\|\rho^0\right\|_{L^2(\Omega)} \longrightarrow \infty} \frac{J_\varepsilon(\rho^0)}{\left\|\rho^0\right\|_{L^2(\Omega)}} \geq \varepsilon. \tag{21}$$

Let $\left(\rho_j^0\right) \subset L^2(\Omega)$ be a sequence of initial data for the adjoint system (16) with $\left\|\rho_j^0\right\|_{L^2(\Omega)} \longrightarrow \infty$. Let

$$\tilde{\rho}_j^0 = \frac{\rho_j^0}{\left\|\rho_j^0\right\|_{L^2(\Omega)}}. \tag{22}$$

So that $\left\|\tilde{\rho}_j^0\right\|_{L^2(\Omega)} = 1.$

On the other hand, let $\tilde{\rho}_j^0$ be the solution of (16) with initial data $\tilde{\rho}_j^0$. Then, we have

$$J_\varepsilon(\rho_j^0) = \Xi \tag{23}$$

And $\Xi = \frac{1}{2}\left\|\rho_j^0\right\|_{L^2(\Omega)} \left\|\rho_j^0\right\|_{L^2(\Omega)} \int_0^T \int_{\mathcal{O}} \left|\widetilde{\rho}_j\right|^2 dxdt$
$+\varepsilon \left\|\rho_j^0\right\|_{L^2(\Omega)} \left\|\tilde{\rho}_j^0\right\|_{L^2(\Omega)}$
$+ \left\|\rho_j^0\right\|_{L^2(\Omega)} \int_0^T \int_{\mathcal{O}} \tilde{\rho}_j h \chi_{\mathcal{O}} dxdt.$

The following two cases can occur

1. $\liminf \int_0^T \int_{\mathcal{O}} \left|\tilde{\rho}_j\right|^2 dxdt > 0$. In this case, we immediately obtain
$$\frac{J_\varepsilon(\rho_j^0)}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \longrightarrow +\infty. \tag{24}$$

2. $\liminf \int_0^T \int_{\mathcal{O}} \left|\tilde{\rho}_j\right|^2 dxdt = 0$. In this case, since $(\tilde{\rho}_j^0)_j$ is bounded in $L^2(\Omega)$, we can extract a subsequence $(\tilde{\rho}_j^0)_j$ such that
$$\tilde{\rho}_j^0 \rightharpoonup \psi^0 in L^2(\Omega), \tag{25}$$
and $\tilde{\rho}_j \rightharpoonup \psi$ in $L^2(0,T;H_0^1(\Omega)) \cap H^1\left(0,T;H^{-1}\left(\Omega\right)\right)$,
where ψ is solution of system (16) with initial data ψ^0. Moreover, by lower semi continuity of the norm, it comes
$$\int_0^T \int_{\mathcal{O}} |\psi|^2 \, dxdt \leq \liminf \int_0^T \int_{\mathcal{O}} \left|\tilde{\rho}_j\right|^2 dxdt = 0. \tag{26}$$
Therefore,
$$\psi = 0 in \mathcal{O} \times (0,T). \tag{27}$$
By applying the continuation property of **S.MIZOHATA** [12], one finds
$$\psi = 0 in \Omega \times (0,T).$$
Thus, $\tilde{\rho}_j \rightharpoonup 0 weakly converge in$
$L^2(0,T;H_0^1(\Omega)) \cap H^1\left(0,T;H^{-1}\left(\Omega\right)\right).$
But,
$$\liminf \frac{J_\varepsilon(\rho^0)}{\left\|\rho^0\right\|_{L^2(\Omega)}} \geq \liminf \left[\varepsilon + \int_0^T \int_\Omega \widetilde{\rho}_j h \chi_{\mathcal{O}} dxdt\right].$$
Thus,
$$\liminf \frac{J_\varepsilon(\rho^0)}{\left\|\rho^0\right\|_{L^2(\Omega)}} \geq \varepsilon.$$
Hence relation (21) is satisfied.

Theorem 7 *Problem (20) has a unique solution $\tilde{\rho}^0 \in L^2(\Omega)$. In addition if $\tilde{\rho}$ is the solution of (16) associated with $\tilde{\rho}^0$, then $(\tilde{u} = \tilde{\rho}\chi_{\mathcal{O}}, \tilde{q})$ such that (5), (7) and (8) are verified.*

[1] Let H be a real Hilbert space, has a bilinear form on H. We have a is coercive means $\exists \alpha > 0$ such that: $\forall u \in H, a(u,u) \geq \alpha \left\|u\right\|_H^2$.

As J_ε reaches its minimum value at $\tilde{\rho}^0 \in L^2(\Omega)$ and $s \in R$ we have

$$J_\varepsilon(\tilde{\rho}^0) \leq J_\varepsilon(\tilde{\rho}^0 + s\psi^0) \tag{28}$$

On the other hand,
$J_\varepsilon(\tilde{\rho}^0 + s\psi^0) = \frac{1}{2}\int_0^T\int_{\mathcal{O}} |\tilde{\rho} + s\psi|^2\, dxdt + \varepsilon\left\|\tilde{\rho}^0 + s\psi^0\right\|_{L^2(\Omega)} + \int_0^T\int_\Omega (\tilde{\rho} + s\psi) h\chi_{\mathcal{O}} dxdt.$

$$J_\varepsilon(\tilde{\rho}^0 + s\psi^0) = \Theta \tag{29}$$

and $\Theta = \frac{1}{2}\int_0^T\int_{\mathcal{O}} |\tilde{\rho}|^2\, dxdt + \frac{s^2}{2}\int_0^T\int_{\mathcal{O}} |\psi|^2\, dxdt + \varepsilon\left\|\tilde{\rho}^0 + s\psi^0\right\|_{L^2(\Omega)} + s\int_0^T\int_{\mathcal{O}} \tilde{\rho}.\psi dxdt + \int_0^T\int_{\mathcal{O}} (\tilde{\rho} + s\psi) h dxdt.$

Using (29) in (28), and after simplifications, we find

$$0 \leq \Upsilon \tag{30}$$

and $\Upsilon = \varepsilon\left[\left\|\tilde{\rho}^0 + s\psi^0\right\|_{L^2(\Omega)} - \left\|\rho_j^0\right\|_{L^2(\Omega)}\right] + \frac{s^2}{2}\int_0^T\int_{\mathcal{O}} |\psi|^2\, dxdt + s\int_0^T\int_{\mathcal{O}} \psi\,(\tilde{\rho} + h)\, dxdt.$

On the other hand,

$$\left\|\tilde{\rho}^0 + sc\right\|_{L^2(\Omega)} - \left\|\rho_j^0\right\|_{L^2(\Omega)} \leq |s|\left\|\psi^0\right\|_{L^2(\Omega)}. \tag{31}$$

From (30) and (31), we get for any $\psi^0 \in L^2(\Omega)$ and $s \in R$,
$0 \leq \varepsilon |s| \left\|\psi^0\right\|_{L^2(\Omega)} + \frac{s^2}{2}\int_0^T\int_{\mathcal{O}} |\psi|^2\, dxdt + s\int_0^T\int_{\mathcal{O}} \psi\,(\tilde{\rho} + h)\, dxdt.$

By dividing by $s > 0$ and passing to the limit $s \to 0$, we obtain
$0 \leq \varepsilon\left\|\psi^0\right\|_{L^2(\Omega)} + \int_0^T\int_{\mathcal{O}} \psi\,(\tilde{\rho} + h)\, dxdt.$

The same calculations with $s < 0$ give
$\left|\int_0^T\int_{\mathcal{O}} \psi\,(\tilde{\rho} + h)\, dxdt\right| \leq \varepsilon\left\|\psi^0\right\|_{L^2(\Omega)}, \forall\psi^0 \in L^2(\Omega).$

Also, if we take $\tilde{u} = \tilde{\rho}\chi_{\mathcal{O}}$ in (5) and multiply the first equation of system (5) by ψ solution of (16), we obtain after integration by parts on $\mathcal{Q}$, we find
$\int_\Omega q(0)\psi^0 dx = \int_0^T\int_{\mathcal{O}} \psi\,(\tilde{\rho} + h)\, dxdt.$

We get two last relations
$\left|\int_\Omega q(0)\psi^0 dx\right| \leq \varepsilon\left\|\psi^0\right\|_{L^2(\Omega)}, \forall\psi^0 \in L^2(\Omega).$

As a result,
$\left\|q(0)\right\|_{L^2(\Omega)} \leq \varepsilon.$

So, there is a unique couple $(\tilde{u}, \tilde{q})$ solution of the problem $(\mathcal{P})$, such that $\tilde{u} = \tilde{\rho}\chi_O$, where $\tilde{\rho}$ is the solution of system (16).

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