Complete Results on Control and Filtering of Discrete Systems with Time Scales

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Abstract: The paper provides complete results of the feedback control design problem for a wide class of discrete-time systems possessing fast and slow modes. The mode-separation is expressed in terms of an inequality relating norms of system sub-matrices. The slow and fast subsystems are considered to be completely controllable and observable. A systematic two-stage procedure is developed which enables designing separate gain matrices for the fast and slow subsystems based on $\mathbb{H}_\infty$ and $\mathbb{H}_2$ optimization criteria and using linear matrix inequalities. It is established that the composite control yields first-order approximations to the behavior of the discrete system. The theoretical analysis is extended to designing of Kalman filters and linear quadratic Gaussian controllers. It is shown that the design procedure eventually reduces to solving pure-slow and pure-fast reduced-order Kalman filters followed by pure-slow and pure-fast reduced-order discrete-time algebraic Riccati equations. Typical applications are considered to illustrate the design procedure.

Key–Words: Time-scale modeling; Composite control; Linear quadratic Gaussian; Kalman filter; Slow subsystem; Fast subsystem

1 Introduction

The usefulness of time-scale modeling approach for the control analysis and design of dynamical systems with fast and slow modes has been widely recognized as a powerful tool for over three decades [1, 2]. In addition, $\mathbb{H}_\infty$-control [3] has been an active topic of research for almost three decades and has received the attention of researchers in the theory of dynamical systems with time-scales [4, 5, 11]. A salient feature of the available results is that the control analysis and design are implemented in two stages, such that an appropriate reduced-order model is handled at each stage. Extension of the time-scale approach to the analysis and control design of discrete systems has been developed [7] using explicitly invertible linear transformations and a quasi-steady-state assumption [6]. It has been shown that, when an inequality relating the norms of subsystem matrices is satisfied, the discrete model can be approximated by (a) a slow submodel with large eigenvalues distributed near the unit circle and (b) a fast submodel with small eigenvalues centered around the origin in the complex plane. This allows feedback control to be implemented using separate gain matrices.

In this paper, we build on the theory in [7, 6] and extend it further to provide a state-space solution for $\mathbb{H}_\infty$ and $\mathbb{H}_2$ composite state-feedback controls of a two-time-scale discrete system. The results are expressed in terms of two independent linear matrix inequalities (LMIs). It is shown that the new feedback design yields first-order perturbation in the behavior of the discrete system. Moreover, we show that the results of [6]–[14] can be extended to Kalman filtering using slow-fast separation. These results are used to build up reduced-order slow and fast Kalman filters as well Kalman full-order filters and to establish the conditions under which reduced-order filters could be designed. After that, we compare the results of the approximated filtered system with the actual filtered system.

The contributions of this paper are

1. it complements the results obtained in [6]–[14] on structural properties of discrete systems with fast-slow separation;

2. it establishes the conditions under which full- and reduced-order Kalman filters can be designed to reconstruct the fast and slow states; and

3. it provides a two-stage procedure to compute the gain matrices of based on $\mathbb{H}_\infty$ and $\mathbb{H}_2$ optimization criteria.

The developed methods are implemented on typical application models to illustrate the theoretical analysis.

Notations: We use W^T, W^{-1} and $||W||$ to denote the transpose, the inverse and induced-norm of any square matrix W, respectively. We use $W < 0$ to denote a symmetric negative definite matrix W and I to denote the $n \times n$ identity matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. In symmetric block matrices or complex matrix expressions, we use the symbol $\bullet$ to represent a term that is induced by symmetry. Sometimes, the arguments of a function will be omitted when no confusion can arise.

2 Discrete Systems with Time-Scales

Consider a class of linear shift-invariant systems described by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \Gamma w(k) \\ y(k) &= Cx(k) + \Phi w(k) \end{aligned} \quad (1)$$

where the vectors $x(k) \in \Re^n,\ u(k) \in \Re^m,\ y(k) \in \Re^q,\ w(k) \in \Re^s$ are the state, control input, output and disturbance input, respectively. We assume that system (1) is asymptotically stable, the pair $(A,\ B)$ is completely reachable and the pair $(A,\ C)$ is completely observable. In the literature, there are two categories of modeling to exhibit the time-scale phenomena in discrete Systems. One category can be termed "explicit" since a particular time-scale parameter appears explicitly in the model [10, 12, 13, 14]. A member class of this category is given by

$$\begin{aligned} x_1(k+1) &= A_{11}x_1(k) + \varepsilon^{1-j}A_{12}x_2(k) \\ &+ B_1u(k) + \Gamma_1 w(k) \\ \varepsilon^{2i}x_2(k+1) &= \varepsilon^i A_{21}x_1(k) + \varepsilon A_{22}x_2(k) \\ &+ \varepsilon^i B_2u(k) + \Gamma_2 w(k) \end{aligned} \quad (2)$$

where $i,\ j\ \in \{0,1\},\ x_1(k) \in \Re^{n_1},\ x_2(k) \in \Re^{n_2}$ are the state components, $u(k) \in \Re^m$ is the control input. Three limiting cases are of interest: $\{i = 0,\ j = 0\}$ in which the time-scale parameter is retained in the column blocks, $\{i = 0,\ j = 1\}$ in which the time-scale parameter is retained in the row blocks and $\{i = 1,\ j = 1\}$ in which the time-scale parameter is retained in the diagonal blocks. Additional classes arise as a result of numerical solution or sampling continuous-time systems with time-scales and using appropriate block diagonal transformation scheme. In case of fast sampling of $T_f = \varepsilon$, a class of discrete-time systems with two-time scales is given be:

$$\begin{aligned} x_1(n+1) &= [I + \varepsilon D_{11}]x_1(n) + \varepsilon D_{12}x_2(n) \\ &+ \varepsilon E_1u(k) + \Gamma_1 w(k) \\ x_2(n+1) &= D_{21}x_1(n) + D_{22}x_2(k) \\ &+ E_2u(k) + \Gamma_2 w(k) \end{aligned} \quad (3)$$

where n is the fast sampling instant. However, if the sampling is slow $T_s = 1$, we could have the discrete systems will be

$$\begin{aligned} x_1(p+1) &= D_{11}x_1(p) + \varepsilon E_{12}x_2(p) \\ &+ E_1u(p) + \Gamma_1 w(k) \\ x_2(p+1) &= D_{21}x_1(p) + \varepsilon E_{22}x_2(p) \\ &+ E_2u(p) + \Gamma_2 w(k) \end{aligned} \quad (4)$$

where p denotes the slow sampling instant, $n = p[1/\varepsilon$. It is significant to note that the analysis and design of systems (2)-(4) requires the identification of the scalar $\varepsilon > 0$ a priori.

The other modeling category can be called "implicit" since there is no time-scale parameter in the model. In this category, to exhibit the behavior of fast-slow modes of system (1), only suitable arrangement of system matrices is often required, which can be attained via permutation and scaling of states. This leads to the model

$$\begin{aligned} x_1(k+1) &= A_1x_1(k) + A_2x_2(k) \\ &+ B_1u(k) + \Gamma_1 w(k) \\ x_2(k+1) &= A_3x_1(k) + A_4x_2(k) \\ &+ B_2u(k) + \Gamma_2 w(k) \\ y(k) &= C_1x_1(k) + C_2x_2(k) + \Phi w(k) \end{aligned} \quad (5)$$

which possesses the time-scale property [1]. The output matrices are $C_1 \in \Re^{q \times n_1},\ C_2 \in \Re^{q \times n_2}$ and the disturbance weighting matrices are $\Gamma_1 \in \Re^{n_1 \times s},\ \Gamma_2 \in \Re^{q \times s},\ \Phi \in \Re^{q \times s}$. This implies that the eigen-spectrum $\lambda(A)$ of system (1) consists of a cluster of n_1 large eigenvalues, distributed near the unit circle, separated from a cluster of n_2 small eigenvalues centered around the origin in the complex plane. The first n_1 eigenvalues designate the slow modes of the system (1) because their response is slower than that of the fast modes represented by the remaining n_2 eigenvalues.

3 Discrete Systems with Time-Scales

It can be readily established, following a discrete quasi-steady-state analysis [7], that system (5) can be

[1] It has been shown in , see [6, 7] that a sufficient condition for mode-separation in discrete systems is $||A_o^{-1}|| << (||A_4|| + ||(I - A_4)^{-1}A_3||||A_2||)^{-1}$ where $A_o = A_1 + A_2(I - A_4)^{-1}A_3$ and $A_1 \in \Re^{n_1 \times n_1},\ A_2 \in \Re^{n_1 \times n_2},\ A_3 \in \Re^{n_2 \times n_1}$ and $A_4 \in \Re^{n_2 \times n_2}$

decomposed into a slow subsystem

$$\begin{aligned} x_s(k+1) &= A_o x_s(k) + B_o u_s(k) + \Gamma_o w(k) \\ y_s(k) &= C_o x_s(k) + D_o u_s(k) + \Psi_o w(k) \\ A_o &= A_1 + A_2(I - A_4)^{-1}B_2, \\ B_o &= A_1 + A_2(I - A_4)^{-1}A_3 \\ C_o &= C_1 + C_2(I - A_4)^{-1}A_3, \\ D_o &= C_2(I - A_4)^{-1}B_2, \\ \Gamma_o &= \Gamma_1 + A_2(I - A_4)^{-1}\Gamma_2, \\ \Psi_o &= C_2(I - A_4)^{-1}\Gamma_2 \end{aligned} \quad (6)$$

of order n_1, and a fast subsystem

$$\begin{aligned} x_f(k+1) &= A_4 x_f(k) + B_2 u_f(k) + \Gamma_2 w(k) \\ y_f(k) &= C_2 x_f(k) \end{aligned} \quad (7)$$

of order n_2. As shown in [9] the slow control $u_s(k)$ and the fast control $u_f(k)$ produce a composite control $u_c(k)$ according to $u_c(k) = u_s(k) + u_f(k)$. Suppose that a linear feedback scheme of the type $u_s(k) = G_s x_s(k),\ u_f(k) = G_f x_f(k)$ with independent gains $G_o,\ G_f$ has been designed for slow and fast subsystems subject to prescribed specifications. In view of the mode-separation [6], the following result is established:

Lemma 1 *The composite control*

$$\begin{aligned} &u_c(k) = \\ &[(I - G_f(I - A_4)^{-1}B_2)^{-1}G_o - G_f(I - A_4)^{-1}A_3]x_1(k) \\ &+G_f x_2(k) \end{aligned} \quad (8)$$

yields a first-order approximation to the state trajectories of system (5).

The objective now is to design $\mathcal{H}_\infty$ and $\mathcal{H}_2$ controllers to guarantee stabilizing system (5) with prescribed performance.

4 $\mathbb{H}_\infty$ Control Design

Instead of designing a full-order $\mathcal{H}_\infty$, we decompose it into two separate slow and fast $\mathcal{H}_\infty$ controllers and later on we recompose them in the manner of **Lemma** 1.

4.1 Slow $\mathbb{H}_\infty$ controller

Let $V_s = x_s^t(k) P_s x_s(k),\ \ P_s > 0$ be a Lyapunov function associated with the slow subsystem (6). The objective of slow $\mathcal{H}_\infty$ controller can then be phrased as: *Given a scalar $\gamma_f > 0$, determine the controller $u_s(k) = G_s x_s(k)$ that stabilizes system (6) and ensuring that $||y_s(k)||_2^2 < \gamma_s^2\ ||w(k)||_2^2$.* The design result is provided by the following theorem:

Theorem 2 *: System (6) is stabilizable by the controller $u_s(k) = G_s x_s(k)$ and $||y_s(k)||_2^2 < \gamma_s^2\ ||w(k)||_2^2$ if there exist matrices $X_s > 0,\ Y_s$ and a scalar $\gamma_s > 0$ such that the following LMI is feasible*

$$\begin{bmatrix} -X_s & 0 & X_s A_o^t + Y_s^t B_o^t & X_s C_o^t + Y_s^t D_o^t \\ \bullet & -\gamma_s^2 I & \Gamma_o^t & 0 \\ \bullet & \bullet & -X_s & X_s \Phi_o^t \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \quad (9)$$

The $\mathcal{H}_\infty$ slow gain is given by $G_s = Y_s X_s^{-1}$.

Proof: It follows from robust control theory [3] that the solution of the slow $\mathcal{H}_\infty$ control problem corresponds to determining the controller gain G_s that guarantees the feasibility of

$$\Pi_s = \Delta V_s + y_s^t(k) y_s(k)\ -\ \gamma_s^2\ w^t(k) w(k) < 0 \quad (10)$$

Evaluation of the first-forward difference ΔV_s along the solutions of (6) with $u_s(k) = G_s x_s(k)$, we express inequality (10) in the form

$$\begin{aligned} \Pi_s &= \begin{bmatrix} x_s \\ w_s \end{bmatrix}^t \Xi_s \begin{bmatrix} x_s \\ w_s \end{bmatrix} < 0 \quad (11) \\ \Xi_s &= \begin{bmatrix} \Xi s1 & \Xi_{s2} \\ \bullet & -\Xi_{s3} \end{bmatrix} \\ \Xi_{s1} &= -P_s + (A_o^t + G_s^t B_o^t) P_s (A_o + B_o G_s) \\ &+ (C_o^t + G_s^t D_o^t)(C_o + D_o G_s) \\ \Xi_{s2} &= (A_o^t + G_s^t B_o^t) P_s \Gamma_o + (C_o^t + G_s^t D_o^t)\Phi_o \\ \Xi_{s3} &= \gamma^2 I - \Phi_o^t \Phi_o - \Gamma_o^t P_s \Gamma_o \end{aligned} \quad (12)$$

Inequality (11) implies that $\Xi_s < 0$. Employing Schur complements to $\Xi_s < 0$ and applying the congruent transformation $X_s,\ \ I,\ \ X_s,\ I$ with $X_s = P_s^{-1},\ G_s X_s = Y_s,$ we readily obtain inequality (9). □

4.2 Fast $\mathbb{H}_\infty$ controller

Similarly, let $V_f = x_f^t(k) P_f x_f(k),\ \ P_f > 0$ be a Lyapunov function associated with the fast subsystem (7). The objective of fast $\mathcal{H}_\infty$ controller can then be phrased as: *Given a scalar $\gamma_s > 0$, determine the controller $u_f(k) = G_f x_f(k)$ that stabilizes system (7) and ensuring that $||y_f(k)||_2^2 < \gamma_f^2\ ||w(k)||_2^2$.* The corresponding design result is provided by the following theorem:

Theorem 3 *: System (7) is stabilizable by the controller $u_f(k) = G_f x_f(k)$ and $||y_f(k)||_2^2 < \gamma_f^2\ ||w(k)||_2^2$ if there exist matrices $X_f > 0,\ Y_f$ and a* *scalar $\gamma_f > 0$ such that such that the following LMI is feasible*

$$\begin{bmatrix} -X_f & 0 & X_f A_4^t + Y_f^t B_2^t & X_f C_2^t \\ \bullet & -\gamma_s^2 I & \Gamma_2^t & 0 \\ \bullet & \bullet & -X_f & 0 \\ \bullet & \bullet & \bullet & -I \end{bmatrix} < 0 \quad (13)$$

The $\mathcal{H}_\infty$ fast gain is given by $G_f = Y_f X_f^{-1}$.

Proof: Follows by parallel development to **Theorem** 2. □

By combining **Lemma** 1, **Theorems** 2 and 3, the composite $\mathcal{H}_\infty$ control is derived by the following lemma:

Lemma 4 *Consider system (5) and let $X_s > 0,\ Y_s$ and $X_f > 0,\ Y_f$ be the feasible solutions of the LMIs (9) and (13), respectively. Then the $\mathcal{H}_\infty$ composite control*

$$\begin{aligned} u_c(k) &= [(I - Y_f X_f^{-1}(I - A_4)^{-1} B_2)^{-1} Y_s X_s^{-1} \\ &- Y_f X_f^{-1}(I - A_4)^{-1} A_3] x_1(k) \\ &+ Y_f X_f^{-1} x_2(k) \end{aligned} \quad (14)$$

guarantees that $||y(k)||_2^2 < \gamma^2\ ||w(k)||_2^2$ with $\gamma \in [\gamma_s,\ \gamma_f]$. Moreover, it yields a first-order approximation to the state trajectories of the original system (5).

In case that the fast subsystem is asymptotically stable, a reduced-order $\mathcal{H}_\infty$ control can be derived in the following lemma:

Lemma 5 *Consider system (5) and let $X_s > 0,\ Y_s$ be the feasible solution of the LMI (9). Then the $\mathcal{H}_\infty$ reduced-order control*

$$u_c(k) = Y_s X_s^{-1} x_1(k) \quad (15)$$

guarantees that $||y(k)||_2^2 < \gamma^2\ ||w(k)||_2^2$ with $\gamma \in [\gamma_s,\ \gamma_f]$. Moreover, it yields a first-order approximation to the state trajectories of the original system (5).

Proof: Follows by parallel development to [6, 7]. □

Remark 6 *It is significant to note that the results of Theorems 2 and 3 and Lemmas 4 and 5 are new in the field of discrete systems with time scales. It further strengthen the fact that system (5) is a good representative model of two-time-scale discrete-time systems with implicit characterization of the mode-separation property.*

We next direct attention to the design of $\mathbb{H}_2$ composite control design.

5 $\mathbb{H}_2$ Control Design

Similarly, instead of designing a full-order $\mathbb{H}_2$, we decompose it into two separate slow and fast $\mathbb{H}_2$ controllers and later on we recompose them in the manner of **Lemma** 1.

5.1 Slow $\mathbb{H}_2$ controller

Let $V_s = x_s^t(k) P_s x_s(k),\ P_s > 0$ be a Lyapunov function associated with the slow subsystem (6). The objective of slow $\mathbb{H}_2$ controller is to ensure the stability of closed-loop slow subsystem and to keep the $\mathbb{H}_2$-norm of the transfer function $H_{y_s w}(s)$ from w to y_s as small as possible.

Given the slow control $u_s(k) = G_s x_s(k)$ into (6), the closed-loop slow subsystem becomes

$$\begin{aligned} x_s(k+1) &= A_{co} x_s(k) + \Gamma_o w(k) \\ y_s(k) &= C_{co} x_s(k) + \Psi_o w(k) \\ A_{co} &= A_o + B_o G_s, \\ C_{co} &= C_o + D_o G_s \end{aligned} \quad (16)$$

From the Lyapunov theorem given G_s, the closed-loop system (16) is internally asymptotically stable $w(k) \equiv 0$ if

$$\mathcal{P} - A_{co}^t \mathcal{P} A_{co} > 0 \quad (17)$$

Then the square of the $\mathbb{H}_2$-norm of the transfer function $H_{zw}(s)$ can be expressed in terms of the solution of a Lyapunov equation (controllability Grammian) such that the corresponding minimization problem with respect to the controller gain G_s is given by

$$\begin{aligned} &\min\ Tr[C_{co} \mathcal{P}_s C_{co}^t] \\ &subject\ to \\ &\left\{ \mathcal{P}_s - A_{co}^t \mathcal{P}_s A_{co} + \Gamma_o \Gamma_o^t = 0 \right\} \end{aligned} \quad (18)$$

where $Tr[.]$ denotes the trace operator. Since $\mathcal{P}_s < \mathcal{P}$ for any $\mathcal{P}$ satisfying

$$\mathcal{P} - A_{co}^t \mathcal{P} A_{co} + \Gamma_o \Gamma_o^t < 0 \quad (19)$$

it is readily verified that $||H_{zw}(s)||_2^2 = Tr[C_{co} \mathcal{P}_s C_{co}^t] < \nu$ with $\Psi_o \equiv 0$ if and only if there exists $\mathcal{P} > 0$ satisfying (19) and $Tr[C_{co} \mathcal{P} C_{co}^t] < \nu$.

Introducing an auxiliary parameter $\mathcal{Z}$, the following design result is obtained:

Theorem 7 *: System (6) is stabilizable by the controller $u_s(k) = G_s x_s(k)$ and $||H_{zw}(s)||_2^2 < \nu$ for a* *prescribed ν if and only if there exist matrices $\mathcal{P} > 0,\ \mathcal{Q},\ \mathcal{Z} > 0$ such that*

$$Tr(\mathcal{Z}) \ < \ \nu, \ \begin{bmatrix} \mathcal{Z} & C_o\mathcal{P} + D_o\mathcal{Q} \\ \bullet & \mathcal{P} \end{bmatrix} \ > \ 0,$$
$$\begin{bmatrix} \mathcal{P} & A_o\mathcal{P} + B_o\mathcal{Q} & \Gamma_o \\ \bullet & \mathcal{P} & 0 \\ \bullet & \bullet & I \end{bmatrix} \ > \ 0 \qquad (20)$$

Moreover, the slow gain is given by $G_s = \mathcal{Q}\mathcal{P}^{-1}$

Proof: It follows from standard convex analysis similar to [17, 18]. □

5.2 Fast $\mathbb{H}_2$ controller

Similarly, let $V_f = x_f^t(k)P_f x_f(k),\ P_f > 0$ be a Lyapunov function associated with the fast subsystem (7). The objective of fast $\mathcal{H}_2$ controller is to ensure the stability of closed-loop fast subsystem and to keep the $\mathcal{H}_2$-norm of the transfer function $H_{y_f w}(s)$ from w to y_f as small as possible. The corresponding design result is provided by the following theorem in a parallel development to Theorem 7:

Theorem 8 *: System (7) is stabilizable by the controller $u_f(k) = G_f x_f(k)$ and $||H_{y_f w}(s)||_2^2 < \nu$ for a prescribed ν if and only if there exist matrices $\mathcal{R} > 0,\ \mathcal{S},\ \mathcal{W} > 0$ such that*

$$Tr(\mathcal{W}) \ < \ \nu, \ \begin{bmatrix} \mathcal{W} & C_2\mathcal{R} \\ \bullet & \mathcal{R} \end{bmatrix} \ > \ 0,$$
$$\begin{bmatrix} \mathcal{R} & A_4\mathcal{P} + B_o\mathcal{S} & \Gamma_o \\ \bullet & \mathcal{R} & 0 \\ \bullet & \bullet & I \end{bmatrix} \ > \ 0 \qquad (21)$$

Moreover, the slow gain is given by $G_f = \mathcal{S}\mathcal{R}^{-1}$

By combining **Lemma** 1, **Theorems** 7 and 8, the composite $\mathcal{H}_2$ control is derived by the following lemma:

Lemma 9 *Consider system (5) and let $\mathcal{P} > 0,\ \mathcal{Q},\ \mathcal{Z} > 0$ and $\mathcal{R} > 0,\ \mathcal{S},\ \mathcal{W} > 0$ be the feasible solutions of the LMIs (20) and (21), respectively. Then the $\mathcal{H}_2$ composite control*

$$\begin{aligned} u_c(k) &= [(I - \mathcal{S}\mathcal{R}^{-1}(I - A_4)^{-1}B_2)^{-1}\mathcal{Q}\mathcal{P}^{-1} \\ &- \mathcal{S}\mathcal{R}^{-1}(I - A_4)^{-1}A_3]x_1(k) \\ &+ \mathcal{S}\mathcal{R}^{-1}x_2(k) \qquad (22) \end{aligned}$$

guarantees that the stability of closed-loop system while keeping the $\mathbb{H}_2$-norm of the transfer function $H_{yw}(s)$ from w to y_s as small as possible. Moreover, it yields a first-order approximation to the state trajectories of the original system (5).

Remark 10 *In a similar way, the results of* **Theorems** *7 and 8 and* **Lemmas** *9 and 5 are new in the field of discrete systems with time scales. It assert that the operations of permutation ans/or scaling are essential in casting discrete-time systems of the type (5) in the form of two-time-scale discrete-time systems with implicit characterization of the mode-separation property.*

Remark 11
- *The stabilizability-detectability conditions of the triples (A_0, B_0, C_0) and (A_4, B_4, C_4) are eventually independent. More importantly, it has been established [5], [6] that they are equivalent to the stabilizability-detectability of the triple (A, B, C) of the original system (l), where $B^T = [B_1^T B_2^T]$.*
- *The control laws $u_s(k)$ and $u_f(k)$ given by (25) and (29) are only subsystem optimal; that is, with respect to the slow and fast subsystem variables. However, it is much easier and computationally simpler to determine them than the optimal control for the overall system (1).*

6 Simulation Example I

Now, we apply the foregoing results to an engine and dynamometer test rig for which a linearized model was developed in [15, 16]. It has the dynamometer rotor speed, shaft torque, engine speed and current amplifier states as the state variables. The input variables are the throttle servo voltage and dynamometer source current. It can easily checked that the model exhibits a mode-separation with two slow states ($n_1 = 2$) and three fast states ($n_2 = 2$). Given the data from [15, 16], the slow model (6) is described by

$$A_o = \begin{bmatrix} 0.762 & 0 \\ -0.029 & 0.689 \end{bmatrix},$$
$$B_o = \begin{bmatrix} 0 & 1.049 \\ 0.090 & -0.018 \end{bmatrix}$$
$$C_o = \begin{bmatrix} 0 & 1 \\ -0.221 & 8.191 \end{bmatrix},$$
$$D_o = \begin{bmatrix} 0 & 0 \\ 0.765 & -0.144 \end{bmatrix}$$

whereas the fast model (7) is given by

$$A_4 = \begin{bmatrix} 0.160 & -0.002 & -0.258 \\ 0 & -0.038 & 0 \\ 0.231 & 0 & -0.381 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} 0.702 & -0.083 \\ 0 & 22.400 \\ 0.142 & 0.026 \end{bmatrix}, \ C_2^t = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Application of **Theorems** 2 and 3 gives the $\mathbb{H}_\infty$ slow and fast gains as

$$\begin{aligned} G_s &= \begin{bmatrix} 0.008 & -0.094 \\ 0.007 & 0.089 \end{bmatrix}, \gamma_s = 0.453, \\ G_f &= \begin{bmatrix} -0.286 & -0.001 & -0.079 \\ -0.277 & -0.011 & -0.084 \end{bmatrix}, \gamma_f = 0.629 \end{aligned}$$

This yields the $\mathbb{H}_\infty$ composite control as

$$\begin{aligned} G_c &= \begin{bmatrix} 0.054 & 0.030 & -0.288 & 0.012 & -0.078 \\ 0.051 & 0.114 & -0.269 & 0.078 & -0.103 \end{bmatrix} \\ \gamma_c &\in [0.453,\ 0.629] \end{aligned}$$

On the other hand, application of **Theorems** 7 and 8 with $\nu = 1.245$ yields the $\mathbb{H}_2$ slow and fast gains as

$$\begin{aligned} G_s &= \begin{bmatrix} 0.016 & -0.085 \\ 0.002 & 0.097 \end{bmatrix}, \\ G_f &= \begin{bmatrix} -0.305 & -0.013 & -0.044 \\ -0.225 & -0.001 & -0.103 \end{bmatrix} \end{aligned}$$

From which, the $\mathbb{H}_2$ composite control takes the form

$$G_c = \begin{bmatrix} 0.047 & 0.028 & -0.309 & 0.104 & -0.036 \\ 0.062 & 0.209 & -0.283 & 0.055 & -0.201 \end{bmatrix}$$

According to **Lemmas** 4 and 9, the ensuing composite gains guarantee close approximation to the closed-loop state-trajectories.

7 Kalman Filter Design

Based on the foregoing results, this section investigates a reduced discrete Kalman filter design to estimate the slow and fast states.

7.1 Slow filter design

We consider the slow model (6). Subject to the detectability of (A_0, C_0), the observer form of the Kalman filter for the slow subsystem (6) is given as:

$$\begin{aligned} \hat{x}_s(k+1) &= A_0\hat{x}_s(k) + B_0u_s(k) \\ &+ K_s[y_s(k) - \hat{y}_s(k)] \\ &= (A_0 - K_sC_0)\hat{x}_s(k) + K_sC_0x_s(k) \\ &+ B_0u_s(k) \\ &+ [K_s\Psi_0w(k) + K_s]w(k) \qquad (23) \end{aligned}$$

It follows from (6) and (23), that the slow estimation error has the form:

$$\begin{aligned} e_s(k+1) &= x_s(k+1) - \hat{x}_s(k+1) \\ &= [A_0 - K_sC_0]e_s(k) \\ &+ [\Gamma_0 - K_s\Psi_0 - K_s]w(k) \end{aligned}$$

The associated estimation error covariance is:

$$\begin{aligned} \Sigma_s(k+1) &= \mathbb{E}[e_s(k+1)e_s^T(k+1)] \\ &= [A_0 - K_sC_0]\Sigma_s(k)[A_0 - K_sC_0]^T \\ &+ [\Gamma_0 - K_s\Psi_0]W[\Gamma_0 - K_s\Psi_0 - K_s]^T \\ &+ K_s]WK_s^T \qquad (24) \end{aligned}$$

Evaluating the prediction update, we deduce that

$$\begin{aligned} e_s(k+1) &= [I - K_sC_0]e_s(k) \\ &- [K_s\Psi_0 + K_s]w(k) \qquad (25) \end{aligned}$$

Hence, the estimation error covariance is updated as:

$$\begin{aligned} \Sigma_s(k+1) &= [I - K_sC_0]\Sigma_s(k)[I - K_sC_0]^T \\ &- [K_s\Psi_0 + K_s]W[K_s\Psi_0 + K_s]^T \end{aligned}$$

Following standard procedure, we can get the slow Kalman gain by differentiating the trace of the estimation error covariance matrix and setting it equals to zero. This procedure yields

$$\begin{aligned} \bar{W} &= E_0WE_0^T \\ K_s &= \Sigma_s(k)C_0^T[C_0\Sigma_s(k)C_0^T + \bar{W}]^{-1} \qquad (26) \end{aligned}$$

7.2 Fast filter design

In a similar fashion, the observer form of the Kalman filter for the fast subsystem is given by:

$$\begin{aligned} \hat{x}_f(k+1) &= A_4\hat{x}_f(k) + B_2u_f(k) \\ &+ K_f[y_f(k) - \hat{y}_f(k)] \\ &= (A_4 - K_fC_2)\hat{x}_f(k) + K_fC_2x_f(k) \\ &+ B_2u_f(k) + K_fw(k) \qquad (27) \end{aligned}$$

Likewise, the estimation error is

$$\begin{aligned} e_f(k+1) &= x_f(k+1) - \hat{x}_f(k+1) \\ &= [A_4 - K_fC_2]e_f(k) + G_2w(k) \\ &- K_fw(k) \end{aligned}$$

The estimation error covariance is expressed as

$$\begin{aligned} \Sigma_f(k+1) &= E[e_f(k+1)e_f^T(k+1)] \\ &= [A_4 - K_fC_2]\Sigma_f(k)[A_4 - K_fC_2]^T \\ &+ \Psi_2W\Psi_2^T - K_fWK_f^T \qquad (28) \end{aligned}$$

and the updated estimation error is represented by

$$e_f(k+1) = [I - K_fC_2]e_f(k) - K_fv(k) \qquad (29)$$

Hence, the estimation error covariance updating becomes

$$\begin{aligned} \Sigma_f(k+1) &= \mathbb{E}[e_f(k+1)e_f^T(k+1)] \\ &= [I - K_fC_2]\Sigma_f(k)[I - K_fC_2]^T \\ &- K_fVK_f^T \end{aligned}$$

In a similar way, we can get the Kalman gain by differentiating the trace of the estimation error covariance matrix and setting it equal to zero:

$$K_f = \Sigma_f(k)C_2^T[C_2\Sigma_f(k)C_2^T + V]^{-1} \quad (30)$$

7.3 Full order Kalman filter

Had we considered the full system (5), we would have derived the Kalman filter equation in the form:

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + K[y(k) - \hat{y}(k)] \\ &= (A - KC)\hat{x}(k) + KCx(k) \\ &+ Bu(k) + Kw(k) \end{aligned} \quad (31)$$

The estimation error is

$$\begin{aligned} e(k) &= x(k+1) - \hat{x}(k+1) \\ &= Ae(k) + Gw(k) \end{aligned}$$

then the estimation error covariance defines as

$$\begin{aligned} \Sigma(k) &= E[e(k)e^T(k)] \\ &= A\Sigma(k)A^T + GWG^T \end{aligned} \quad (32)$$

The updated estimation error is represented as

$$e(k) = [I - KC]e(k) - Kv(k)$$

and the updating the estimation error covariance is

$$\begin{aligned} \Sigma(k) &= E[e(k)e^T(k)] \\ &= [I - KC]\Sigma(k)[I - KC]^T - KWK^T \end{aligned}$$

The Kalman filter gain is

$$K = \Sigma C^T[C\Sigma C^T + W]^{-1} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \quad (33)$$

where K_1 and K_2 are the subsystem Kalman gain matrices. Considering the full-order Kalman filter for (31) along with the full-order system (5), we obtain the corresponding filtered states as:

$$\begin{aligned} &\begin{bmatrix} \hat{x}_1(k+1) \\ \hat{x}_2(k+1) \end{bmatrix} = \\ &\begin{bmatrix} A_1 - K_1C_1 & A_2 - K_1C_2 \\ A_3 - K_2C_1 & A_4 - K_2C_2 \end{bmatrix} \\ &\times \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} \\ &+ \begin{bmatrix} K_1C_1 & K_1C_2 \\ K_2C_1 & K_2C_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \\ &+ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} w(k) \\ &\hat{y}(k) = C_1\hat{x}_1(k) + C_2\hat{x}_2(k) \end{aligned} \quad (34)$$

7.4 Degree of approximation

To assess the value of the developed results, it is desired to find the degree of approximation between the state-estimate generated by the approximated slow-fast subsystems:

$$\begin{aligned} &\begin{bmatrix} \hat{x}_s(k+1) \\ \hat{x}_f(k+1) \end{bmatrix} = \begin{bmatrix} A_0 - K_sC_0 & 0 \\ 0 & A_4 - K_fC_2 \end{bmatrix} \\ &\begin{bmatrix} \hat{x}_s(k) \\ \hat{x}_f(k) \end{bmatrix} \\ &+ \begin{bmatrix} K_sC_0 & 0 \\ 0 & K_fC_2 \end{bmatrix} \begin{bmatrix} x_s(k) \\ x_f(k) \end{bmatrix} + \begin{bmatrix} B_0 \\ B_2 \end{bmatrix} u(k) \\ &+ \begin{bmatrix} K_sE_0 + K_s \\ K_f \end{bmatrix} w(k) \end{aligned} \quad (35)$$

and those derived by the original subsystems (34). The end result is summarized by the following theorem:

Theorem 12 *The developed slow and fast estimators are first-order approximation to the original subsystem estimators in the sense that*

$$\begin{aligned} K_s &\approx K_1 + \Delta_1 \\ K_f &\approx K_2 + \Delta_4 \end{aligned} \quad (36)$$

where Δ_1 and Δ_4 are $\bigcirc(\sigma)$ where σ signifies the eigenvalue separation.[2] *This enures that*

$$\hat{x}_s(k) \approx \hat{x}_1(k), \;\; \hat{x}_f(k) \approx \hat{x}_2(k)$$

Proof: Applying the similarity transformation

$$\begin{aligned} T &= \begin{bmatrix} I_1 + ML & M \\ L & I_2 \end{bmatrix}; \\ T^{-1} &= \begin{bmatrix} I_1 & -M \\ -L & I_2 + LM \end{bmatrix} \end{aligned}$$

to system (34) where matrices L and M are real roots of

$$\begin{aligned} &[A_4 - K_2C_2]L - L[A_1 - K_1C_1] + \\ &L[A_2 - K_1C_2]L - [A_3 - K_2C_1] = 0, \\ &M[(A_4 - K_2C_2) + L(A_2 - K_1C_2)] - \\ &[(A_1 - K_1C_1) - (A_2 - K_1C_2)L]M \\ &+(A_2 - K_1C_2) = 0 \end{aligned} \quad (37)$$

For the free system (5), a first-order approximation to the L and M matrices are given by

$$\begin{aligned} L_o &= -(I - A_4)^{-1}A_3, \\ M_o &= [A_1 + A_2(I - A_4)^{-1}A_3]^{-1}A_2 \end{aligned}$$

[2] A vector or matrix $\pi(\sigma)$ of a positive scalar σ is said to be $\bigcirc(\sigma)$ if there exists positive constants d and σ^* such that $||\pi(\sigma)|| \leq d\sigma^*$ for all $\sigma \leq \sigma^*$.

Algebraic manipulation using results of [6], it follows that the gain expressions (36) hold where

$$\begin{aligned}
\Delta_1 &= MLA_1 - MLK_1C_1 + MA_3 - MK_2C_1 \\
&- MLA_2L + MLK_1C_2L - MA_4L \\
&+ MK_2C_2L, \\
\Delta_2 &= A_2 - K_1C_2 + MLA_2 - MLK_1C_2 + MA_4 \\
&- MK_2C_2 - A_1M + K_1C_1M - MLA_1M \\
&+ MLK_1C_1M - MA_3M + MK_2C_1M \\
&+ A_2LM - K_1C_2LM + MLA_2LM \\
&- MLK_1C_2LM + MA_4LM - MK_2C_2LM, \\
\Delta_3 &= -LA_2L + LK_1C_2L - A_4L + K_2C_2L. \\
\Delta_4 &= LA_2LM - LK_1C_2LM + A_4LM \\
&- LK_1C_2 - LA_1M + LK_1C_1M - A_3M \\
&+ K_2C_1M + LA_2 - K_2C_2LM
\end{aligned}$$

These values guarantee that Δ_1 and Δ_4 are of order '$\bigcirc(\sigma)$'. $\square$

Remark 13 *One of the salient features of the forgoing design is that the computational load of computing a Kalman filter of order $n = n_1 + n_2$ is now replaced to a first-order approximation by computing two Kalman filters of order n_1 and n_2.*

8 Simulation Example II

A fourth-order discrete two-time-scale system will be shown to demonstrate the main objective of this paper [23]. The system is arranged as follows:

$$A = \begin{bmatrix} 0.9 & 0 & 0 & 0.1 \\ 0.1 & 0.8 & 0.05 & -0.1 \\ -0.1 & 0 & 0.15 & 0 \\ 0.12 & 0.03 & 0 & 0.1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0.5 \\ 0.5 & 0 \end{bmatrix}, \Gamma = \begin{bmatrix} 0.1 & 0 \\ 0.9 & 0.6 \\ 0 & 0.1 \\ 0.3 & 0.1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.1 & 0 & 0 & 0.1 \\ 0 & 0.1 & 0.2 & 1 \end{bmatrix},$$

the Kalman filter gains are resulting as

$$K_s = \begin{bmatrix} 11.1691 & -10.9447 \\ 5.4686 & -5.1510 \end{bmatrix}$$

$$K_f = \begin{bmatrix} -17.8942 & 18.8445 \\ -12.2918 & 13.2971 \end{bmatrix}$$

The presented approach will be used to design a Kalman filter based on the Kalman filters of the reduced-order systems. The Kalman filter design of the reduced-order system resulted in the following gain:

$$K_c = \begin{bmatrix} 11.1691 & -10.9447 \\ 5.4686 & -5.1510 \\ -17.8942 & 18.8445 \\ -12.2918 & 13.2971 \end{bmatrix}$$

And the state-feedback regulator design of the reduced-order system resulted in the following gain:

$$G_c = \begin{bmatrix} -0.3521 & -0.0604 & -0.0353 & -0.0557 \\ 0.2786 & -0.2519 & -0.0939 & 0.0748 \end{bmatrix}$$

While the Kalman filter design of the exact system resulted in the following gain:

$$K = \begin{bmatrix} 5.3173 & -4.3839 \\ 2.1899 & -1.3527 \\ -2.2945 & 2.4683 \\ 2.2836 & -2.0322 \end{bmatrix}$$

The state-feedback regulator design of the exact system resulted in the following gain:

$$G = \begin{bmatrix} -0.6259 & -0.0279 & -0.0078 & -0.1193 \\ 0.6581 & -0.3648 & -0.1627 & 0.1649 \end{bmatrix}$$

Two different test-input signals were used to check the response of the exact system based on the exact design of the LQG controller and based on the reduced-order one. Results were very good, a matter that reflects the potential this approach has. Fig. (1) shows the response of the system due to sine wave input subjected to process and measurement noises. It can be seen that the Kalman filter is doing a great job in isolating the noisy measurements from the controller as can be seen from Fig. (1). To give a clearer image

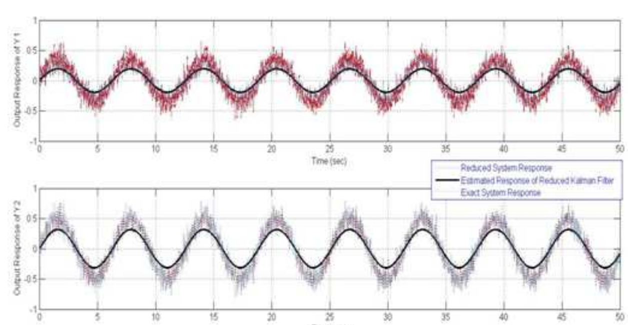

Figure 1: System response due to noisy sine input.

about the performance of the LQG controller based on the reduced-order approach presented here, it was compared to the response of the Exact LQG controller as shown in Fig. (2). when subjected to no noise. It is obvious that the performances of both schemes are very close. Fig. (3). shows the control signals for

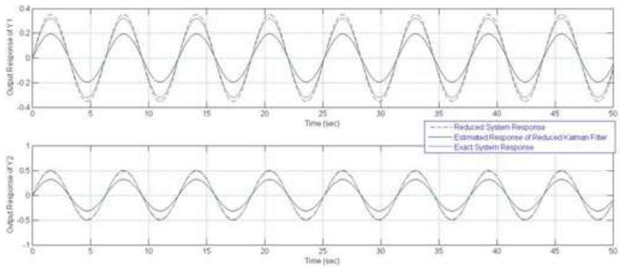

Figure 2: System response due to sine input without noise.

control design schemes which are very close too. It is worth mentioning that the control signal is affected by the noise, but in this example, a successful choice of the weighing matrices was a key issue in obtaining the results presented here. The states have something

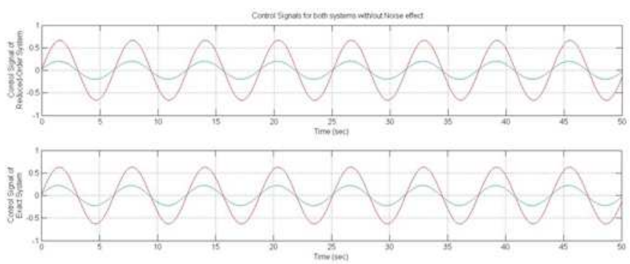

Figure 3: Control signal of both subsystems without noise.

to tell as well, Fig. (4). shows the states'responses to the input and it is also clear that the states are close. Another thing that may help in confirming the

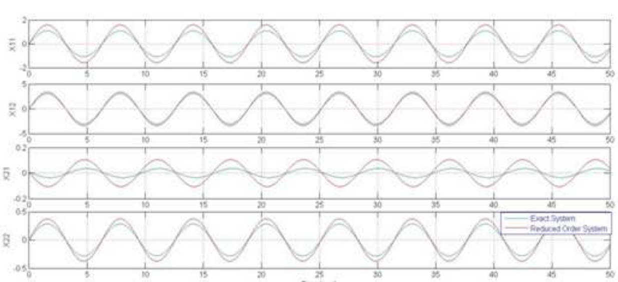

Figure 4: States response due to a sine input.

results claimed here is to consider the error in estimation based on both the exact and the reduced-order Kalman filters. Fig. 5. shows the estimation error of the system outputs as found by the two filters. The error is small and the estimation is also close. Follow-

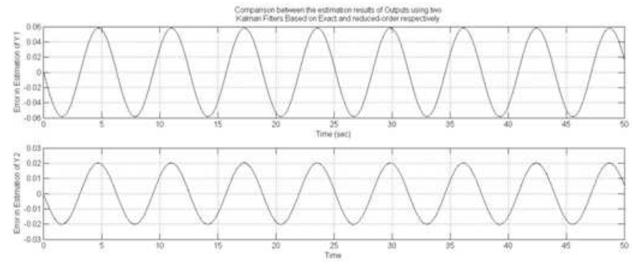

Figure 5: Estimation error between the two filters.

ing, additional simulation results of the same system due to a step input can be seen as a confirmation for the validity of the approach stated here.

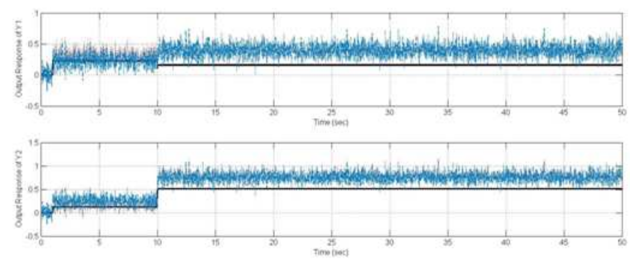

Figure 6: Simulation results of the same system due to a step input with noise.

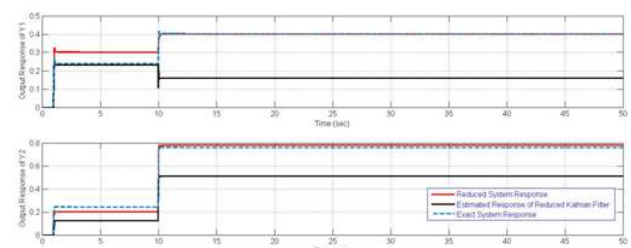

Figure 7: Simulation results of the same system due to a step input without noise.

9 Conclusion

This paper has been concerned with the feedback control design problem for a wide class of discrete-time systems possessing fast and slow modes. The slow and fast subsystems are considered to be completely controllable and observable, which is a less restrictive condition than the complete controllability and observability of the original system. Adopting either the $\mathbb{H}_\infty$ or $\mathbb{H}_\infty$ optimization criteria, a two-stage design procedure has been developed using separate gain

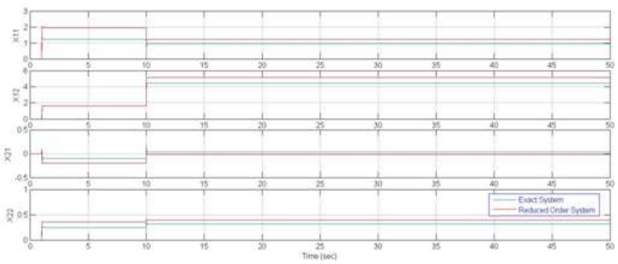

Figure 8: system states of the same system due to a step input.

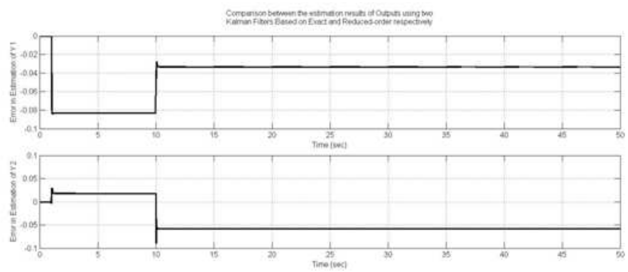

Figure 9: Estimation error of the same system due to a step input.

matrices for the fast and slow subsystems. A composite control has been constructed to yield first-order approximations to the behavior of the discrete system. By parallel development, we developed Kalman filters for the slow and fast subsystems and assessed the degree of approximation. The ensuing results have reflected the potential of this approach.

Acknowledgments: The author would like to thank the deanship for scientific research (DSR) at KFUPM for support through distinguished professorship award project (no. IN 161065).

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