On the Observability of Multiagent Neural Networks

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Abstract: We obtain a characterization of observability for a class of linear systems which appear in multiagent neural networks research. Due to the connection between mathematical concept of control dynamical systems and cognitive control, there has been growing interest in the descriptive analysis of complex networks with linear dynamics obtaining considerable advances in the description of the properties both structural and dynamical about many aspects from everyday life. Notwithstanding, much less effort has been devoted to studying the observability of the dynamics taking place on them. In this work, a review of observability concepts is presented and provides conditions for observability of the multiagent systems.

Key–Words: Multiagent neural network, eigenstructure, observability, linear systems

1 Introduction

As it is well known, many linear and time-invariant dynamic systems can be conveniently expressed with state space equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ y(t) &= Cx(t) \\ x(0) &= x_0 \in \mathbb{R}^n \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$, and $y(t) \in \mathbb{R}^p$ are the system state, control input and measurement output, respectively. $(A, C) \in M_n(\mathbb{R}) \times M_{p\times n}(\mathbb{R})$ are real constant matrices.

Observability is a fundamental property for linear time-invariant systems. The observability allows us to determine if the initial states can be observed or not from the exit, [19]. Observability character is an important concept for neural networks systems.

Observability can be studied under different points of view, for example structural observability, on the other hand, analyze the property of a system with its structure only, without the specific knowledge of the values of its elements. The structural analysis examines the connection between the structure of a model and the functional dependence among its elements.

Another and not less important point of view to analyze observability is the exact observability. The exact observability reflects the same physical sense that state $x(t)$ is uniquely determined by output $y(t)$ (almost surely).

The brain can be structured is a deep recurrent complex neuronal network. The term neural network refers to a particular model for understanding brain function, in which neurons are the basic computational units, and computation is interpreted in terms of network interactions, (see [5]).

The brain neural systems permit humans to perform the multiple complex cognitive functions necessary for daily life and these can alter their dynamics to meet the demands of tasks, [5]. These aptitudes are known as power control. The concept of cognitive control is analogous to the mathematical concept of control of dynamic systems used in engineering, where the state of a complex system can be modulated by the energy input. Neural network systems are very attractive systems due to their structure predisposes certain components to specific control actions. The neuronal sets of the brain can be designed as the nodes of a complex system and the anatomical cables of interconnection as the axes, this system exerts an impact on the neural function. It is therefore plausible that the brain regulates cognitive function through a process of transient network level control similar to technological systems modelled mathematically as complex systems, [10], [14]. Although the complete understanding of the relationship between mathematical control measures and the notions of cognitive control of neuroscience are difficult to achieve, small advances in the study can favour the study and action against learning difficulties such as dyscalculia or other disturbances like the phenomena of forgetting, (see [10], [9], for example).

The study of the control of complex networks with linear dynamics has gained importance in both science and engineering. Observability of a dynamical system has being largely studied by several authors and under many different points of view, (see [1], [2], [5], [20], [3], [12], [15], [7], [8], [16] and [21], for example).

Another important aspect of control is the notion of input observability that describes the ability of an external data to move the output from any initial condition to any final in a finite time. Some results about can be found in [7].

In this paper we analyze observability properties for multiagent neural network to be applied to neuroscience problems.

2 Preliminaries

2.1 Algebraic Graph theory

We consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of order N with the set of vertices $\mathcal{V} = \{1, \dots, N\}$ and the set of edges $\mathcal{E} = \{(i,j) \mid i,j \in \mathcal{V}\} \subset \mathcal{V} \times \mathcal{V}$.

Given an edge (i,j) i is called the parent node and j is called the child node and j is in the neighbor of i, concretely we define the neighbor of i and we denote it by $\mathcal{N}_i$ to the set $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i,j) \in \mathcal{E}\}$.

The graph is called undirected if verifies that $(i,j) \in \mathcal{E}$ if and only if $(j,i) \in \mathcal{E}$. The graph is called connected if there exists a path between any two vertices, otherwise is called disconnected.

Associated to the graph we can consider a matrix $G = (g_{ij})$ called unweighted adjacency matrix defined as follows $a_{ii} = 0$, $g_{ij} = 1$ if $(i,j) \in \mathcal{E}$, and $g_{ij} = 0$ otherwise.

(In a more general case we can consider a weighted adjacency matrix is $A = (a_{ij})$ with $a_{ii} = 0$, $a_{ij} > 0$ if $(i,j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise).

The adjacency matrix corresponding to the graph given in figure 1,

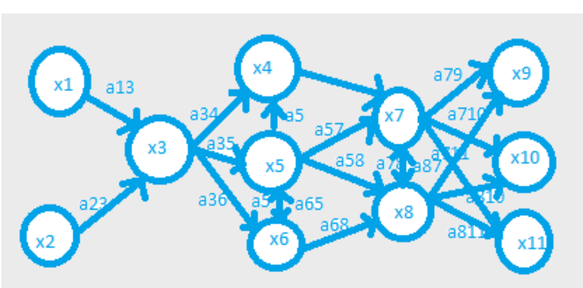

Figure 1: Neural Network.

is as follows

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{13} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{34} & 0 & a_{54} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{35} & 0 & 0 & a_{65} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{36} & 0 & a_{56} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{47} & a_{57} & 0 & 0 & a_{87} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{58} & a_{68} & a_{78} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{79} & a_{89} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{710} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{711} & a_{811} & 0 & 0 \end{pmatrix}$$

The Laplacian matrix of the graph is

$$\mathcal{L} = (l_{ij}) = \begin{cases} |\mathcal{N}_i| & \text{if } i = j \\ -1 & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases}$$

For more details about graph theory see [23].

A possible manner to study the control of the neural networks can be associating a dynamical system to graph:

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ y(t) &= Cx(t) \end{aligned} \qquad (2)$$

where

$x = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^t$ stands for the states nodes, $A = (a_{ij})$ is the adjacency matrix to the graph where a_{ij} represents the weight of a directed link from node i to j and C is the $p \times n$ outputs matrix.

2.2 Observability Properties

The observability character can be computed by means of the well-known Kalman's rank condition

Proposition 1 ([13]) *The system 2 is controllable if and only if:*

$$\operatorname{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n \qquad (3)$$

or by means the Hautus Test for observability of linear dynamical systems.

Proposition 2 ([11]) *The system 2 is controllable if and only if:*

$$\operatorname{rank} \begin{pmatrix} sI - A \\ C \end{pmatrix} = n, \ \forall s \in \mathbb{C} \Big\}. \qquad (4)$$

To solve this challenging task of to ensure controllability and observability, Y. Y. Liu et al. [18] proposed the maximum matching algorithm based on the network representation of the A matrix to select the control and observer nodes that ensure controllable and observable systems.

We recall now the concept of structural controllability. Structural controllability is a generalization of the controllability concept. It is of great interest because many times we know the entries of the matrices only approximately. Roughly speaking, a linear system is said to be structurally controllable if one can find a set of values for the parameters in the matrices such that the corresponding system is controllable. More concretely, the definition is as follows.

Definition 3 *The system $\dot{x}(t) = Ax(t), y(t) = Cx(t)$ is structurally observable if and only if $\forall \varepsilon > 0$, there exists a completely observable system $\dot{x}(t) = A_1x(t), y_1(t) = C_1x(t)$ of the same structure as $\dot{x}(t) = Ax(t), y(t) = Cx(t)$ such that $\|A_1 - A\| < \varepsilon$ and $\|C_1 - C\| < \varepsilon$.*

Recall that, a dynamic system $\dot{x}(t) = Ax(t), y(t) = Cx(t)$ is has the same structure as another system $\dot{x}(t) = A_1x(t), y_1(t) = C_1x(t)$, of the same dimensions, if for every fixed zero entry of the matrices A and C, the corresponding entry of the matrices A_1 and C_1 is fixed zero and vice versa.

A dual concept is the structural controllability concept, [17].

Definition 4 *The linear system $\dot{x}(t) = Ax(t)+Bu(t)$ is structurally controllable if and only if $\forall \varepsilon > 0$, there exists a completely controllable linear system $\dot{x}(t) = \overline{A}x(t) + \overline{B}u(t)$, of the same structure as $\dot{x}(t) = Ax(t) + Bu(t)$ such that $\|\overline{A} - A\| < \varepsilon$ and $\|\overline{B} - B\| < \varepsilon$.*

Analogously, a linear dynamic system $\dot{x}(t) = Ax(t)+Bu(t)$ has the same structure as another linear dynamical system $\dot{x}(t) = \overline{A}x(t)+\overline{B}u(t)$, of the same dimensions, if for every fixed zero entry of the pair of matrices (A, B), the corresponding entry of the pair of matrices $(\overline{A}, \overline{B})$ is fixed zero and vice versa.

Definition 5 *The system $x(k + 1) = Ax(k), y(k) = Cx(k)$ is called exactly observable, if for any $T > 0$, it is verified that if $y(t) \equiv 0, \forall t \in [0, T]$, then $x_0 = 0$.*

A dual concept is the exact controllability asking for the minimum number of controls are needed to make the system controllable. More concretely, given a state space representation of a homogeneous linear dynamical system $\dot{x}(t) = Ax(t)$.

Definition 6 *The exact controllability $n_D(A)$ is the minimum of the rank of all possible matrices B making the system 2 controllable.*

$$
\begin{array}{l}
n_D(A) = \\
\min\{\text{rank}\, B, \forall B \in M_{n\times i}\ 1 \leq i \leq n\ (A, B) \\
\qquad \text{controllable}\}.
\end{array}
$$

Proposition 7 ([24])

$$n_D = \max_i \{\mu(\lambda_i)\}$$

where $\mu(\lambda_i) = \dim \text{Ker}\,(A - \lambda_i I)$ is the geometric multiplicity of the eigenvalue λ_i.

3 Observability of multiagent neural networks

The complexity of the brain drives that in order to study control problems, the global model is divided into several local submodels, each with its complex and interrelated network structure. Structuring, in this way, the brain as a neuronal multi-network with a common goal.

Let us consider a group of k agents. The dynamic of each agent is given by the following linear dynamical systems

$$
\begin{array}{ll}
\dot{x}^i(t) & = A_i x^i(t) \\
y^i(t) & = C_i x^i(t),
\end{array} \tag{5}
$$

$x^i(t) \in \mathbb{R}^n$, $y^i(t) \in \mathbb{R}^p$, $1 \leq i \leq k$.

$$z^i(t) = \sum_{j\in\mathcal{N}_i} C(x^i - x^j).$$

Writing

$$
\mathcal{X}(t) = \begin{pmatrix} x^1(t) \\ \vdots \\ x^k(t) \end{pmatrix}, \quad \dot{\mathcal{X}}(t) = \begin{pmatrix} \dot{x}^1(t) \\ \vdots \\ \dot{x}^k(t) \end{pmatrix},
$$

$$
\mathcal{A} = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}, \mathcal{C} = \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_k \end{pmatrix}.
$$

Following this notation, we can describe the multisystem as a system:

$$
\begin{array}{ll}
\dot{\mathcal{X}}(t) & = \mathcal{A}\mathcal{X}(t) \\
\mathcal{Y}(t) & = \mathcal{C}\mathcal{X}(t).
\end{array} \tag{6}
$$

Clearly, this system is observable if and only if each subsystem is observable, and the condition can be written as

$$\text{rank}\begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_k \\ C_1A_1 & & \\ & \ddots & \\ & & C_kA_k \\ \vdots & & \\ C_1A_1^{n-1} & & \\ & \ddots & \\ & & C_kA_k^{n-1} \end{pmatrix} = \text{rank}\begin{pmatrix} \mathcal{C} \\ \mathcal{C}\mathcal{A} \\ \vdots \\ \mathcal{C}\mathcal{A}^{n-1} \end{pmatrix} = n \cdot k,$$

but we are interested in the case where the agents of the system are interrelated by the communication topology among agents.

Then, we consider the undirected graph $\mathcal{G}$ with

i) Vertex set: $V = \{1, \dots, k\}$

ii) Edge set: $\mathcal{E} = \{(i, j) \mid i, j \in V\} \subset V \times V$

defining the communication topology, and considering it, the outputs are modified as $z_i = C_i \sum_{j\in\mathcal{N}_i}(x^i(t) - x^j(t))$ for all $1 \leq i \leq k$ and defining

$$\mathcal{Z}(t) = \begin{pmatrix} \sum_{j\in\mathcal{N}_1} x^1(t) - x^j(t) \\ \vdots \\ \sum_{j\in\mathcal{N}_k} x^k(t) - x^j(t) \end{pmatrix}$$

the output of the system is rewritten as

$$\mathcal{Y}(t) = \mathcal{C}\mathcal{Z}(t).$$

That using Kronecker product is expressed in the form

$$\mathcal{Y}(t) = \mathcal{C}(\mathcal{L} \otimes I_n)\mathcal{X}(t).$$

Theorem 8 *The system $\dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t), \mathcal{Y}(t) = \mathcal{C}(\mathcal{L} \otimes I_n)\mathcal{X}(t)$ is observable if and only if*

$$\text{rank}\begin{pmatrix} \mathcal{C}(\mathcal{L}\otimes I_n) \\ \mathcal{C}(\mathcal{L}\otimes I_n)\mathcal{A} \\ \vdots \\ \mathcal{C}(\mathcal{L}\otimes I_n)\mathcal{A}^{n-1} \end{pmatrix} = n \cdot k.$$

Proof: The system $\dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t), \mathcal{Y}(t) = \mathcal{C}(\mathcal{L} \otimes I_n)\mathcal{X}(t)$ is observable if and only if

$$\text{rank}\begin{pmatrix} \mathcal{C}(\mathcal{L}\otimes I_n) \\ \mathcal{C}(\mathcal{L}\otimes I_n)\mathcal{A} \\ \vdots \\ \mathcal{C}(\mathcal{L}\otimes I_n)\mathcal{A}^{kn-1} \end{pmatrix} = n \cdot k.$$

But taking into account Cayley-Hamilton theorem, $\mathcal{A}^n = \sum_{i=0}^{n-1} \mathcal{A}^i$. Then

$$\text{rank}\begin{pmatrix} \mathcal{C}(\mathcal{L}\otimes I_n) \\ \mathcal{C}(\mathcal{L}\otimes I_n)\mathcal{A} \\ \vdots \\ \mathcal{C}(\mathcal{L}\otimes I_n)\mathcal{A}^{kn-1} \end{pmatrix} = \text{rank}\begin{pmatrix} \mathcal{C}(\mathcal{L}\otimes I_n) \\ \mathcal{C}(\mathcal{L}\otimes I_n)\mathcal{A} \\ \vdots \\ \mathcal{C}(\mathcal{L}\otimes I_n)\mathcal{A}^{n-1} \end{pmatrix}$$

□

Corollary 9 *Equivalently, the system $\dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t), \mathcal{Y}(t) = \mathcal{C}(\mathcal{L} \otimes I_n)\mathcal{X}(t)$ is observable if and only if*

$$\text{rank}\begin{pmatrix} l_{11}C_1 & \dots & l_{1k}C_1 \\ \vdots & & \vdots \\ l_{k1}C_k & \dots & l_{kk}C_k \\ l_{11}C_1A_1 & \dots & l_{1k}C_1A_k \\ \vdots & & \vdots \\ l_{k1}C_kA_1 & \dots & l_{kk}C_kA_k \\ \vdots & & \vdots \\ l_{11}C_1A_1^{n-1} & \dots & l_{1k}C_1A_k^{n-1} \\ \vdots & & \vdots \\ l_{k1}C_kA_1^{n-1} & \dots & l_{kk}C_kA_k^{n-1} \end{pmatrix} = n \cdot k.$$

Example 10 *We consider 3 agents with the following dynamics of each agent*

$$\begin{aligned} \dot{x}^1 &= A_1x^1 \\ \dot{x}^2 &= A_2x^2 \\ \dot{x}^3 &= A_3x^3 \end{aligned} \tag{7}$$

with $A_1 = A_2 = A_3 = \begin{pmatrix} 0 & -0.1 \\ 1 & -0.5 \end{pmatrix}$, 0 , and $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$.

The communication topology is defined by the graph $(\mathcal{V}, \mathcal{E})$:

$V = \{1,2,3\}$

$\mathcal{E} = \{(i,j) \mid i,j \in V\} = \{(1,2),(1,3)\} \subset V \times V$

The neighbors of the parent nodes are $\mathcal{N}_1 = \{2,3\}$, $\mathcal{N}_2 = \{1\}$, $\mathcal{N}_3 = \{1\}$.

The Laplacian matrix of the graph is

$$\mathcal{L} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

We have that

$$\mathcal{C}(\mathcal{L}\otimes I_n) = \begin{pmatrix} 0 & 2 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{A} = \begin{pmatrix} 0 & -0.1 & 0 & 0 & 0 & 0 \\ 1 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 0 & 0 \\ 0 & 0 & 1 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 0 & 1 & -0.5 \end{pmatrix}$$

$$\text{rank}\begin{pmatrix} \mathcal{C}(\mathcal{L}\otimes I_n) \\ \mathcal{C}(\mathcal{L}\otimes I_n)\mathcal{A} \end{pmatrix} = \text{rank}\begin{pmatrix} 0 & 2 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 2 & -1 & -1 & 0.5 & -1 & 0.5 \\ -1 & 0.5 & 1 & -0.5 & 0 & 0 \\ -1 & 0.5 & 0 & 0 & 1 & -0.5 \end{pmatrix} = 4$$

Then, the system is not observable.

Remark 11 *Notice that, if we consider the multisystem without communication topology, then the system is observable:*

$$\text{rank}\begin{pmatrix} \mathcal{C} \\ \mathcal{C}\mathcal{A} \end{pmatrix} = \text{rank}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.5 \end{pmatrix} = 6.$$

We are involved in possible output injections in such a way that the new matrix of the system has arbitrary spectrum, that is to say, we are interested in estimators

$$WC_i \sum_{j\in\mathcal{N}_i} (x^i - x^j)$$

such that the system has prescribed eigenvalues.

Proposition 12 *Considering the estimator* $WC_i \sum_{j\in\mathcal{N}_i}(x^i(t) - x^j(t))$, *for all* $1 \leq i \leq k$ *the closed-loop system can be described as*

$$\dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t) + \mathcal{W}\mathcal{Z}(t).$$

where $\mathcal{W} = \begin{pmatrix} W & & \\ & \ddots & \\ & & W \end{pmatrix}$.

Computing the matrix $\mathcal{A}+\mathcal{W}\mathcal{C}(\mathcal{L}\otimes I_n)$ we obtain

$$\begin{pmatrix} A_1 + l_{11}WC_1 & l_{12}WC_1 & \dots & l_{1k}WC_1 \\ l_{21}WC_2 & A_2 + l_{22}WC_2 & \dots & l_{2k}WC_2 \\ \vdots & \vdots & \ddots & \vdots \\ l_{k1}WC_k & l_{k2}WC_k & \dots & l_{kk}WC_k \end{pmatrix}$$

Example 13 *Taking the multiagent system considered in example 10 we have that, t aking* $K = \begin{pmatrix} k & \ell \end{pmatrix}$

The matrix of the system is

$$\begin{pmatrix} 0 & -0.1+2v & 0 & -v & 0 & -v \\ 1 & -0.5+2w & 0 & -w & 0 & -w \\ 0 & -v & 0 & -0.1+v & 0 & 0 \\ 0 & -w & 1 & -0.5+w & 0 & 0 \\ 0 & v & 0 & 0 & 0 & -0.1+v \\ 0 & -w & 0 & 0 & 1 & -0.5+w \end{pmatrix}$$

Taking $W = \begin{pmatrix} -0.5 \\ -0.2 \end{pmatrix}$ *the eigenvalues are* $-0.5500 + 1.1391i$, $-0.5500 - 1.1391i$, $-0.2500 + 0.1936i$, $-0.2500 - 0.1936i$, $-0.3500 + 0.6910i$, $-0.3500 - 0.6910i$, *then the system has a stable solution.*

As we can see in the example, all agents on the multi-agent system, have an identical linear dynamic mode. In this particular case proposition 12 can be rewritten in the following manner (see [4], [22]).

Proposition 14 *Taking the control* $u^i(t) = K\sum_{j\in\mathcal{N}_i}(x^i(t) - x^j(t))$, $1 \leq i \leq k$ *the closed-loop system for a multiagents having identical linear dynamical mode, can be described as*

$$\begin{aligned} \dot{\mathcal{X}}(t) &= ((I_k \otimes A) + (I_k \otimes CW)(\mathcal{L}\otimes I_n))\mathcal{X}(t) \\ \mathcal{Y}(t) &= \mathcal{C}(\mathcal{L}\otimes I_n)\mathcal{X}(t). \end{aligned}$$

4 Conclusions

The observability for multiagent systems with communication topology are analyzed. We shown that the observability of each subsystem is not sufficient condition for observability of multiagent system with communication topology.

In this paper, a criterium for checking observability is presented.

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