Application of a Stochastic-Fuzzy Approach to Modeling Optimal Discrete Time Dynamical Systems by Using Large Scale Data Processing

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Abstract: - In the paper the proposed dynamical model of the system is defined by fuzzy states and a transition function. The transition function is represented by conditional probabilities of respective fuzzy events. The criteria of optimization, as well as constraints, constitute fuzzy sets. The dynamical model has a form of a stochastic-fuzzy knowledge base, where the rules and weights of rules have been built by using large-scale data sets.

Key-Words: - Fuzzy control, Knowledge base, Stochastic system with fuzzy states

1 Introduction

In the revolutionary contribution entitled *Decision making in a fuzzy environment* by R.E. Bellman and L.A. Zadeh a very important mathematical tool to optimal control of dynamic systems has been formulated [1].

According to that work a fuzzy environment represents fuzzy constraints and fuzzy optimization criteria. During next years, many works on the field of fuzzy decision making and fuzzy control has been initiated. In the 1980s, many of J. Kacprzyk's works have been dedicated to multistage fuzzy control problems concerning deterministic, stochastic or fuzzy dynamic objects [2],[3],[4]. The class of control models presented in the mentioned above works, is known as prescriptive in opposite to descriptive models, given by e. g. linguistic rules.

In the works by Mamdani and Assilian, also Tong and many other authors, the control models were called linguistic, to accentuate relationships between descriptive models and the linguistic way of formulating problem and finding solutions, specific for human beings [5],[6].

The structure of this paper is as follows: the fuzzy optimal control problem of the discrete time stochastic system is formulated according to the object literature, in section 2. In section 3, the proposition of the stochastic-fuzzy control model in a form of the knowledge base is presented. The weights of rules represent empirical conditional probabilities of the state transition.

2 Fuzzy Multistage Control Models

For the transparency, let us first consider the fuzzy control problem, where an object, an objective function and constraints are very well determined. Optimal solution can be find on the way of mathematical transformations. In this chapter an example concerning the multistage fuzzy control of a stochastic dynamic object is presented.

A dynamic system determined in the discrete domains of its variables is considered. Let the state space be a finite set, $\Xi = \{x_1, x_2, ..., x_M\}$; let $s_t, s_{t+1} \in \Xi$ mean the states at moments t, $t+1$, respectively. The state variable is identified with the output variable. Also, let the space of a control (input) variable be a finite set, $\Lambda = \{u_1, u_2, ..., u_N\}$; let $c_t \in \Lambda$ denotes the control variable applied at moment t. A dynamics of the system is described by a transition law, conditional probability function

$$p(s_{t+1} / s_t, c_t) \in [0,1], \quad t = 0,1,2,... \qquad (1)$$

Transition law (1) determines a probability of the occurrence the state $s_{t+1} \in \Xi$ at moment $t+1$, if at time t the system state was $s_t \in \Xi$ and control $c_t \in \Lambda$ was applied. The state $s_{t+1} \in \Xi$ does not depend on the earlier states (Markov property).

Transition law (1) satisfies the conditions of the distribution functions:

$$\sum_{s_{t+1}\in\Xi} p(s_{t+1}/s_t, c_t) = 1, \qquad (2)$$

$$s_t = const, \quad c_t = const.$$

Conditional probability functions (1) realize a mapping the cartesian product of the corresponding spaces into the unitary interval of real numbers, $[0,1]\subset R$:

$$p: \Xi\times\Lambda\times\Xi \to [0,1]. \qquad (3)$$

Let us note, that the transition law (1), as conditional probability functions of discrete variables, can be presented in a form of a matrix (table).

A formalization of a multistage fuzzy control task, when an object is a Markov stochastic chain given by a transition function in a form of a conditional probability $p(s_{t+1}/s_t, c_t)$, $t = 0,1,2,...$ of the occurrence the state $s_{t+1}\in\Xi$ at moment t+1, providing $s_t\in\Xi$ and $c_t\in\Lambda$ at time t, must be completed as follows:

- the number of stages is given, t=0,1,.., N;
- there are constraints K^t, t=0,1,…,N-1 on the control variable $c_t\in\Lambda$, in a form of fuzzy subsets of a control space Λ; constraints are determined by their membership functions $\mu_{\mathrm{K}^t}(c_t)$, such that $\mu_{\mathrm{K}^t}(c_t)$: $\Lambda\to[0,1]\subset R$; t=0,1,…,N-1;
- there is a fuzzy objective function G^N on the final state $s_N\in\Xi$; G^N is a fuzzy subset of the state space Ξ, given by its membership function $\mu_{G^N}(s_N)$, such that $\mu_{G^N}(s_N): \Xi \to [0,1]\subset R$.

Hence, the states $\{s_t\}$ represent a realization of a stochastic process, the fuzzy objective function G^N stands for a fuzzy event. Probability of any fuzzy event, according to Zadeh's definition, is equal to the expected value of its membership function, $P(G^N) = E[\mu_{G^N}(s_N)]\in[0,1]$, and can be written as follows [8]:

$$P(G^N) = \sum_{x_i\in\Xi}\mu_{G^N}(s_N = x_i)\,p[(s_N = x_i)/s_{N-1}, c_{N-1}]$$
(4)

The sequence of optimal decisions $c_0^\bullet,...,c_{N-1}^\bullet$ (optimal control) at stages t=0,1,…,N-1 maximizes the probabilities of reaching the goal and (in the sense of fuzzy logic) fulfils the constraints [1]. This optimization criterion can be written by using dynamic programming in the recurrent form [1], [7]:

$$\mu_{G^{N-i}}(s_{N-i}) = \max_{c_{N-i}}[\mu_{K^{N-i}}(c_{N-i}) \wedge E[\mu_{G^{N-i+1}}(s_{N-i+1})]]$$
(5)

Example. This simple example shows particular steps of the optimal fuzzy control task, according to Bellman and Zadeh's approach, given above.

Suppose the stochastic Markov object determined by: state space $\Xi = \{x_1, x_2, x_3\}$, control space $\Lambda = \{u_1, u_2\}$, conditional probability functions of a state transition $p(s_{t+1}/s_t, c_t)\in[0,1]$, $s_t, s_{t+1}\in\Xi$, $c_t\in\Lambda$, given in table 1. Also suppose two stages (N=2) for achieving the fuzzy objective function

$$G^2(s_2) = 0.8/x_1 + 1/x_2 + 0.5/x_3\ . \qquad (6)$$

Fuzzy constraints at stages t=0 and t=1 are given as follows:

$$\mathrm{K}^0 = 1/u_1 + 0.5/u_2, \qquad (7)$$

$$\mathrm{K}^1 = 0.5/u_1 + 1/u_2. \qquad (8)$$

Tab.1. The conditional probability functions $p(s_{t+1}/s_t, c_t)$ of the state transition

s_t, c_t	s_{t+1}		
	x_1	x_2	x_3
x_1, u_1	1	0	0
x_2, u_1	0.2	0.6	0.2
x_3, u_1	0.1	0.2	0.7
x_1, u_2	0	1	0
x_2, u_2	0	0.2	0.8
x_3, u_2	0.1	0.7	0.2

Let us start from the final stage, N=2, to compute the conditional probabilities of the achieving the fuzzy goal G^2, depend on the state s_1 and control c_1 at the previous stage. According to (4) and (6) also taking into account the first raw in table 1, we have

$$P(G^2(s_2/x_1, u_1)) = 1\bullet 0.8 + 0\bullet 1 + 0\bullet 0.5 = 0.8$$

Similarly, taking into account the second raw in table 1, we obtain:

$$P(G^2(s_2/x_2, u_1)) = 0.2\bullet 0.8 + 0.6\bullet 1 + 0.2\bullet 0.5 = 0.86$$

The possible six values of conditional probabilities $P(G^2(s_2/s_1,c_1))$ have been presented in table 2.

Taking into account the results in table 1 and constraint (8), the following rules of control at stage t=1 can be formulated:

-starting from the state s_1=x_1 apply $c_1^\bullet$=u_2,

- starting from the state s_1=x_2 apply $c_1^\bullet$=u_2,

- starting from the state $s_1=x_3$ apply $c_1^{\bullet}=u_2$.

Tab. 2. The conditional probabilities of achieving the fuzzy goal at t=2, $P(G^2(s_2/s_1,c_1))$

c_1	s_1		
	x_1	x_2	x_3
u_1	0.8	0.86	0.63
u_2	1	0.6	0.88

Continuing calculations at stage t=1, assuming iteration i=1 in (5), we derive the relationship for the fuzzy objective function at t=1:

$$\mu_{G^1}(s_1) = \max_{c_1}[\mu_{K^1}(c_1) \wedge E[\mu_{G^2}(s_2)]]. \quad (9)$$

It is easy to note, that (9) expresses a fuzzy composition of the *max-min* type, and can be written in the form:

$$[0.5 \quad 1] \circ \begin{bmatrix} 0.8 & 0.86 & 0.63 \\ 1 & 0.6 & 0.88 \end{bmatrix} = [1 \quad 0.6 \quad 0.88]. \quad (10)$$

The right side of (10) represents the membership coefficients of the fuzzy objective function G^1, that is:

$$G^1(s_1) = 1/x_1 + 0.6/x_2 + 0.88/x_3. \quad (11)$$

Hence, to optimize control at stage t=1 the conditional probabilities $P(G^1(s_1/s_0, c_0))$ should be calculated. The fuzzy event G^1 determined by (11) and the conditional probabilities given in table 1 serve as data to calculation according to (4). Results of calculation are presented in table 3.

Tab. 3. The conditional probabilities of achieving the fuzzy goal at t=1, $P(G^1(s_1/s_0,c_0))$

c_0	s_0		
	x_1	x_2	x_3
u_1	1	0.74	0.84
u_2	0.6	0.84	0.70

The fuzzy objective function at stage t=0 can be determined on the basis of (5), for i=2, as the result of a fuzzy *max-min* composition :

$$\mu_{G^0}(s_0) = \max_{c_0}[\mu_{K^0}(c_0) \wedge E[\mu_{G^1}(s_1)]], \quad (12)$$

$$[1 \quad 0.5] \circ \begin{bmatrix} 1 & 0.74 & 0.84 \\ 0.6 & 0.84 & 0.7 \end{bmatrix} = [1 \quad 0.74 \quad 0.84]$$

That is

$$G^0(s_0) = 1/x_1 + 0.74/x_2 + 0.84/x_3. \quad (13)$$

Thus, there are the following rules of control $c_0^{\bullet}$ at stage t=0:

- starting from the state $s_0=x_1$ apply $c_0^{\bullet}=u_1$,
- starting from the state $s_0=x_2$ apply $c_0^{\bullet}=u_1$,
- starting from the state $s_0=x_3$ apply $c_0^{\bullet}=u_1$.

The sequence of optimal decisions (optimal control) $c_0^{\bullet}, c_1^{\bullet}$ at stages t=0 and t=1, respectively, maximizes the probability of reaching the fuzzy objective function at final stage and (in the sense of fuzzy logic) fulfils the constraints at stages t=0 and t=1.

3 Large Scale Data and Knowledge-based Models

Big data repositories and specialized methods and algorithms of knowledge discovery became sources of creating more meaningful knowledge bases.

A knowledge base as a part of computer expert systems usually has a form of logic conditional rules *IF…THEN* given by experts. Such set of rules can represent knowledge for e.g. diagnostic task, prediction problem or control problem, considered in this paper. According to mathematics and logic operations applied in creating the knowledge model we can think of deterministic, stochastic, fuzzy or stochastic-fuzzy knowledge bases.

Consider control of a stochastic dynamic system with fuzzy states. Assume discrete state space $\Xi = \{x_1, x_2, ..., x_M\}$, and distinguished fuzzy states $X_1, X_2, ..., X_K$ of the system. States are defined by respective membership functions $\mu_{X_1}(x), \mu_{X_2}(x), ..., \mu_{X_K}(x), x \in \Xi$, Borel measurable and fulfilling relationship [9]:

$$\sum_{k=1}^{K} \mu_{X_k}(x) = 1, \forall x \in \Xi \quad (14)$$

Moreover, it is assumed the existing probabilities of non-fuzzy elementary events, $x_1, x_2, ..., x_M \in \Xi$, equal to $p(x_1), p(x_2), ..., p(x_M) \in [0,1]$, and fulfilling the condition of distribution function

$$\sum_{m=1}^{M} p(x_m) = 1.$$

Stochastic dynamic system with fuzzy states is determined by the following conditional probabilities of the transition of fuzzy states:

- conditional probability functions of the occurrence the fuzzy state $S_{t+1} \in \{X_1, X_2,..., X_K\}$ at moment t+1, providing fuzzy state $S_t \in \{X_1, X_2,..., X_K\}$ and control $c_t \in \Lambda$ at moment t

$$P(S_{t+1} / S_t, c_t) \in [0,1], \quad t = 0,1,2,... \tag{15}$$

such that

$$\sum_{S_{t+1} \in \{X_1,...,X_K\}} P(S_{t+1} / S_t, c_t) = 1, \tag{16}$$

$$S_t = const, \quad c_t = const.$$

Also an initial probability distribution function of fuzzy events exists

$$P(S_0) = \{P(X_k)\}, k = 1,2,..., K$$

where

$$P(X_k) \in [0,1],$$

$$P(X_k) = \sum_{x_m \in \Xi} \mu_{X_k}(x_m) p(x_m), k = 1,2,..., K, \tag{17}$$

$$\sum_{k=1}^{K} P(X_k) = 1.$$

Moreover, conditional probabilities $P(S_{t+1}/S_t,c_t)$ do not depend on earlier states (Markov property of stochastic process with fuzzy states) [9].

Example. Let the space of numerical values of the tested object be $\Xi = \{x_1, x_2,..., x_5\}$. Let the probability $p(x)$=0.2 be the same for every element of the space. Suppose, that three fuzzy states have been determined by experts:

$X_1 = 1/x_1 + 0.5/x_2$, $X_2 = 0.5/x_2 + 1/x_3 + 0.5/x_4$, $X_3 = 0.5/x_4 + 1/x_5$.

Fuzzy states can have some linguistic meaning, as e.g. *low*, *middle*, *high*, etc.

Initial probability distribution of fuzzy states calculated according to (17) is :

$$P(S_0) = \{P(X_k)\}_{k=1,2,3} = \{0.3, 0.4, 0.3\}.$$

The values of conditional probabilities $P(S_{t+1}/S_t, c_t)$ can be calculated on the basis of empirical data from data repositories, as it is shown in [9], [10]. In table 4 there are some exemplary values.

Tab. 4. Exemplary values of conditional probabilities of the fuzzy state transition $P(S_{t+1}/S_t, c_t)$

S_t, c_t	S_{t+1}		
	X_1	X_2	X_3
X_1, u_1	1	0	0
X_2, u_1	0.2	0.6	0.2
X_3, u_1	0.1	0.2	0.7
X_1, u_2	0	1	0
X_2, u_2	0	0.2	0.8
X_3, u_2	0.1	0.7	0.2

Table 4 constitute the stochastic-fuzzy control model of the tested process. This model can be written in a form of conditional weighted rules in fuzzy logic, as follows:

R$_1$: *If* (S_t *is* X_1) *And* (c_t is u_1) *Then*
(S_{t+1} *is* X_1)(P=1);

R$_2$: *If* (S_t is X_2) *And* (c_t is u_1) *Then*
(S_{t+1} *is* X_1)(P=0.2),
Also (S_{t+1} *is* X_2)(P=0.6),
Also (S_{t+1} *is* X_3)(P=0.2);

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R$_6$: *If* (S_t is X_3) *And* (c_t is u_2) *Then*
(S_{t+1} *is* X_1)(P=0.1),
Also (S_{t+1} *is* X_2)(P=0.7),
Also (S_{t+1} *is* X_3)(P=0.2).

To make decision concerning c_t at moment t, experts can use some measure determining a coefficient of similarity between fuzzy sets: certain fuzzy goal G^N and fuzzy states S_{t+1}=S^N included into antecedent of the given rule.

Let the goal be given as:

$$G^N = 0.1/x_1 + 0.3/x_2 + 1/x_3 + 0.2/x_4 + 0/x_5.$$

Taking normalized Hamming's measure as the similarity coefficient

$$\eta(G^N, S_N) = 1 - \frac{1}{I}\sum_{i=1}^{I}\left|\mu_{S_N}(x_i) - \mu_{G^N}(x_i)\right| \quad (18)$$

we have:

$\eta(G^N,X_1)$=0.54, $\eta(G^N,X_2)$=0.64, $\eta(G^N,X_3)$=0.46.

Consider fuzzy state $S_{t+1}=X_2$ which is rather close to S^N. Probability of the achieving this state at moment t+1 is equal to 0.7 if $S_t=X_3$ (rule R_6) and the control value $c_t=u_2$ was applied. Also, probability of $S_{t+1}=X_2$ is equal to 0.6 if at moment t, $S_t=X_2$ (rule R_2) and the control value $c_t=u_1$ was applied. In the other rules probabilities of transitions to $S_{t+1}=X_2$ are very low.

4 Conclusion

Two presented approaches to fuzzy control can be summarized as follows:

-when the model of a stochastic object is given in the precise mathematical form, the sequence of optimal decisions (optimal control) at particular stages maximizes the probability of reaching the fuzzy objective function at final stage and (in the sense of fuzzy logic) fulfils the constraints at the stages;

- when the model of the object has a form of a stochastic-fuzzy knowledge base, the choice of control is rather suboptimal, because the model depends on the historical data, concerning observed situations. When the collection of data constitutes a large scale set and concerns many situations, thus the selection of the control value can be optimal.

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