Stability of Linear Systems with the Generalized Lipschitz Property

MICHAEL GIL'
Ben Gurion University of the Negev
Department of Mathematics
Beer Sheva
ISRAEL
gilmi@bezeqint.net

Abstract: We consider non-autonomous multivariable linear systems governed by the equation $\dot{u} = A(t)u$ with the matrix $A(t)$ satisfying the generalized Lipschitz condition $\|A(t) - A(\tau)\| \leq a(|t - \tau|)$ $(t, \tau \geq 0)$, where $a(t)$ is a positive function. Explicit sharp stability conditions are derived. In the appropriate situations our results generalize and improve the traditional freezing method. An illustrative example is presented.

Key–Words: linear systems; stability; generalized Lipschitz conditions.

1 Introduction and statement of the main result

Let $\mathbf{C}^n$ be the complex n-dimensional Euclidean space with a scalar product $(.,.)$, the Euclidean norm $\|.\| = \sqrt{(.,.)}$ and the unit matrix I. For a linear operator A in $\mathbf{C}^n$ (matrix), $\|A\| = \sup_{x\in\mathbf{C}^n} \|Ax\|/\|x\|$ is the spectral (operator) norm.

The purpose of this note is to suggest new sufficient stability conditions for a slowly varying in time system described by the equation

$$\dot{u}(t) = A(t)u(t) \ \ (t \geq 0;\ \dot{u}(t) = du(t)/dt), \quad (1.1)$$

where $A(t)$ is a variable $n\times n$ matrix $[0, \infty)$ satisfying the generalized Lipschitz condition

$$\|A(t) - A(\tau)\| \leq a(|t - \tau|) \quad (t, \tau \geq 0). \quad (1.2)$$

where $a(t)$ is a positive piece-wise continuous function defined on $[0, \infty)$.

The problem of stability analysis of linear systems continues to attract the attention of many specialists despite its long history. It is still one of the most burning problems of control theory, because of the absence of its complete solution. One of the main methods for the stability analysis of systems with slowly varying matrices is the freezing method [1, 3], [6]-[9], [11, 12, 4]. In particular, in the interesting recent paper [7] a numerical method is suggested.

The main features of the present note are the following: in the framework of the traditional freezing approach it is assumed that $A(t)$ either differentiable with small derivative or satisfies the Lipschitz condition

$$\|A(t)-A(\tau)\| \leq q_0|t-\tau| \ (q_0 = const \geq 0;\ t, \tau \geq 0).$$

So condition (1.2) holds in the special case $a(t) := q_0|t|$. Thus condition (1.2) enables us to generalize the traditional freezing method and improve it in the appropriate situation.

A solution to (1.1) for a given $u_0 \in \mathbf{C}^n$ is a function $u : [0, \infty) \to \mathbf{C}^n$ having at each point $t \geq 0$ a bounded derivative and satisfying (1.1) for all $t > 0$ and $u(0) = u_0$. The existence and uniqueness of solutions under consideration are obvious. Equation (1.1) is said to be exponentially stable, if there are positive constants M and ϵ, such that $\|u(t)\| \leq M exp\ [-\epsilon t]\|u(0)\|$ $(t \geq 0)$ for any solution $u(t)$ of (1.1).

In addition to (1.2) suppose that there is a positive integrable on $[0, \infty)$ function $p(t)$ independent of s integrable and uniformly bounded on $[0, \infty)$, such that

$$\|\exp[A(s)t]\| \leq p(t) \ \ (t, s \geq 0). \quad (1.3)$$

Now we are in a position to formulate the main result of the paper.

Theorem 1 *Let the conditions (1.2),(1.3) and*

$$\zeta_0 := \int_0^\infty a(s)p(s)ds < 1 \quad (1.4)$$

hold. Then equation (1.1) is exponentially stable.

This theorem is proved in the next section. Theorem 1.1 is sharp in the following sense: if $A(t)$ is constant, then $a(t) = 0$ and condition (1.4) automatically holds for any exponentially stable equation.

To apply Theorem 1.1 to concrete systems introduce the quantity

$$g(A) = (N_2^2(A) - \sum_{k=1}^n |\lambda_k(A)|^2)^{1/2} \quad (A \in \mathbf{C}^{n\times n}),$$

where $N_2(A)$ is the Hilbert-Schmidt (Frobenius) norm of A: $N_2(A) = \sqrt{trace\ AA^*}$, A^* is the adjoint operator, $\lambda_k(A)$ $(k = 1, ..., n)$ are the eigenvalues with their multiplicities. The following relations are checked in cf. [5, Section 3.1]:

$$g^2(A) \le N_2^2(A) - |Trace\ A^2|, g(A)$$

$$\le \frac{1}{\sqrt{2}} N_2(A - A^*)$$

and $g(e^{ia}A + zI_H) = g(A)$ $(a \in \mathbf{R}, z \in \mathbf{C})$; if A is a normal matrix: $A^*A = AA^*$, then $g(A) = 0$. If A_1 and A_2 are commuting matrices, then $g(A_1 + A_2) \le g(A_1) + g(A_2)$. In addition, by the inequality between the geometric and arithmetic mean values,

$$(\frac{1}{n}\sum_{k=1}^n |\lambda_k(A)|^2)^n \ge (\prod_{k=1}^n |\lambda_k(A)|)^2.$$

Hence, $g^2(A) \le N_2^2(A) - n(det\ A)^{2/n}$. Due to Example 3.2 [5]

$$\|e^{At}\| \le e^{\alpha(A)t} \sum_{k=0}^{n-1} \frac{t^k g^k(A)}{(k!)^{3/2}} \quad (t \ge 0),$$

where $\alpha(A) = \max_k Re\ \lambda_k(A)$. Assume that

$$g_0 := \sup g(A(t)) < \infty \tag{1.5}$$

and

$$\alpha_0 := \sup \alpha(A(t)) < 0. \tag{1.6}$$

Then (1.3) holds with $p(t) = \hat{p}(t)$, where

$$\hat{p}(t) := e^{\alpha_0 t} \sum_{k=0}^{n-1} \frac{t^k g_0^k}{(k!)^{3/2}} \quad (t \ge 0).$$

Now Theorem 1.1 implies

Corollary 2 *Let the conditions (1.2),(1.5), (1.6) and*

$$\hat{\zeta} := \int_0^\infty a(s)\hat{p}(s)ds < 1 \tag{1.7}$$

hold. Then (1.1) is exponentially stable.

2 Proof of Theorem 1.1

We need the following result.

Lemma 3 *Let conditions (1.2)-(1.4) hold. Then any solution $u(t)$ of (1.1) satisfies the inequality*

$$\sup_{t\ge 0} \|u(t)\| \le \frac{p_M}{1-\zeta_0}\|u(0)\|,$$

where $p_M = \sup_t p(t)$.

Proof: Rewrite (1.1) as

$$\frac{du(t)}{dt} = A(\tau)u(t) + [A(t) - A(\tau)]u(t)$$

with an arbitrary fixed $\tau \ge 0$. So (1.1) is equivalent to the equation

$$u(t) = \exp[A(\tau)t]u(0)$$

$$+ \int_0^t \exp[A(\tau)(t-s)][A(s) - A(\tau)]u(s)ds.$$

Hence,

$$\|u(t)\| \le \|\exp[A(\tau)t]\|\|u(0)\|$$

$$+ \int_0^t \|\exp[A(\tau)(t-s)]\|\|A(s) - A(\tau)\|\|u(s)\|ds.$$

According to (1.2) and (1.3),

$$\|u(t)\| \le p(t)\|u(0)\| + \int_0^t p(t-s)a(s-\tau)\|u(s)\|ds.$$

Taking $\tau = t$, we obtain

$$\|u(t)\| \le p(t)\|u(0)\| + \int_0^t p(t-s)a(t-s)\|u(s)\|ds$$

and therefore,

$$\|u(t)\| \le p(t)\|u(0)\| + \int_0^t p(t_1)a(t_1)\|u(t-t_1)\|dt_1.$$

Hence for any positive finite T,

$$\sup_{t\le T}\|u(t)\| \le p_M\|u(0)\| + \sup_{t\le T}\|u(t)\| \int_0^T p(t_1)a(t_1)dt_1$$

$$\le p_M\|u(0)\| + \sup_{t\le T}\|u(t)\| \int_0^\infty p(t_1)a(t_1)dt_1$$

$$= p_M\|u(0)\| + \sup_{t\le T}\|u(t)\|\zeta_0.$$

According to (1.4) we get

$$\sup_{t\le T}\|u(t)\| \le p_M\|u(0)\|(1-\zeta_0)^{-1}$$

Extending this result to all $T \geq 0$ we prove the lemma. Q.E.D.

Proof of Theorem 1.1: By the substitution

$$u(t) = u_\epsilon(t)e^{-\epsilon t} \tag{2.1}$$

with an $\epsilon > 0$ into (1.1), we obtain the equation

$$du_\epsilon(t)/dt = (\epsilon I + A(t))u_\epsilon(t). \tag{2.2}$$

Taking ϵ small enough and applying Lemma 2.1 to equation (2.2) we can assert that $\|u_\epsilon(t)\| \leq const\|u(0)\|$. Hence due to (2.1) we arrive at the required result. Q.E.D.

3 Example

Consider equation (1.1), taking

$$A(t) = \begin{pmatrix} -1 & b\ sin(\omega t) \\ -b\ sin(\omega t) & -1 \end{pmatrix} \tag{3.1}$$

with positive constants b and ω. In this case one can apply various methods, for example the Wazewsky inequality, but to compare our results with [7] we apply Theorem 1.1. We have

$$\|A(t) - A(s)\| \leq b|\sin(\omega t) - \sin(\omega s)| \quad (t, s \geq 0). \tag{3.2}$$

Since

$$\sin x - \sin y = 2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y) \quad (x, y \in \mathbf{R}),$$

we obtain

$$\|A(t) - A(s)\| \leq 2b|\sin(\frac{\omega(t-s)}{2})|.$$

So $a(t) = 2b|\sin(\omega t/2)|$. Simple calculations show that

$$\lambda_{1,2}(A(t)) = -1 \pm ib\ sin(\omega t)$$

and therefore $\alpha(A(t)) \equiv -1$. In addition, $g(A(t)) \equiv 0$, since $A(t)$ is normal. So $p(t) = e^{-t}$ and

$$\hat{\zeta} = 2b\int_0^\infty e^{-t}|\sin(\omega t/2)|dt$$

$$= (4b/\omega)\int_0^\infty e^{-2x/\omega}|\sin x|dx.$$

Taking $c = 2/\omega$ we can write

$$\int_0^\infty e^{-cx}|\sin x|dx = \sum_{k=0}^\infty \int_{2\pi k}^{\pi 2(k+1)} e^{-cx}|\sin x|dx.$$

But

$$\int_{2\pi k}^{\pi 2(k+1)} e^{-cx}|\sin x|dx = e^{-c2\pi k}\int_0^{2\pi} e^{-cy}|\sin y|dy$$

$$= e^{-c2\pi k}(1 + e^{-c\pi})\int_0^\pi e^{-cy}\sin y dy$$

$$= -e^{-c2\pi k}(1 + e^{-c\pi})\frac{1}{(1+c^2)}e^{-cx}\cos x|_{x=0}^\pi$$

$$= e^{-c2\pi k}\frac{1}{(1+c^2)}(e^{-c\pi} + 1)$$

$$\frac{1}{(1+c^2)}(e^{-c\pi} + 1).$$

Hence,

$$\int_0^\infty e^{-cx}|\sin x|dx = \frac{1}{(1+c^2)}(e^{-c\pi} + 1)\sum_{k=0}^\infty e^{-c2\pi k}$$

$$= \frac{(e^{-c\pi} + 1)}{(1 - e^{-2c\pi})(1+c^2)}.$$

Due to Corollary 1.2 the considered equation is exponentially stable, provided

$$\hat{\zeta} = \frac{4b\omega(e^{-2\pi/\omega} + 1)}{(1 - e^{-4\pi/\omega})(4 + \omega^2)} < 1. \tag{3.3}$$

For instance take $\omega = 1$. Then (3.3) holds, provided

$$b < 2.4 < \frac{5(1 - e^{-4\pi})}{e^{-2\pi} + 1} \tag{3.4}$$

The traditional freezing method can be applied if instead of (3.2) we take into account that under consideration

$$\|A(t) - A(s)\| \leq b\omega|t - s|$$

So $a(t) = b\omega|t|$ and according to Corollary 1.2 in this case with $\omega = 1$ the stability condition is provided by the inequality

$$b\int_0^\infty e^{-t}tdt = b < 1.$$

So (3.4) is considerably better than this condition.

This example shows that the application of Theorem 1.1 to equations with matrices containing two and more parameters requires simpler calculations than the method suggested in [7] but Theorem 1.1, in contrast to [7], requires the point-wise Hurwitzness of matrix $A(t)$.

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