Numerical scheme for solving two point fractional Bagley-Torvik equation using Chebyshev collocation method

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Abstract: In this paper, we propose numerical scheme for solving two point fractional Bagley-Torvik equation (FBTE). The scheme is based on collocation and using shifted Chebyshev polynomials of the second kind (SCPSK) orthogonal basis functions. In this case, we replace an integer order derivatives by fractional order derivatives in Caputo sense. By using the properties of SCPSK to reduce fractional Bagley-Torvik equation into system of algebraic equations, which can be solved by iteration method. The error analysis and error bounds are discussed. The validation of the present algorithm is tested through number of examples. All computational results are done in Matlab.

Key–Words: Caputo fractional derivative, Chebyshev polynomials of the second kind, Collocation method, Fractional Bagley-Torvik equation, Convergence analysis

1 Introduction

We consider the two point fractional Bagley-Torvik equation in the following form

$$\left[a_0 D^2 + a_1 D^{\mu} + a_2 D\right] r(t) = p(t),\ t \in [0, L], \tag{1}$$

subject to boundary conditions

$$r(0) = \delta_1,\ \ r(L) = \delta_2, \tag{2}$$

where $a_0, a_1, a_2, \delta_1, \delta_2$ are constants with $a_0 \neq 0$, $p(t)$ is continuous on [0,L] and fractional differential operator D^{μ} in Caputo sense. In similar way we can study for the boundary conditions are

$$\alpha_0 r(0) + \gamma_0 r'(0) = \beta_0,\ \alpha_1 r(L) + \gamma_1 r'(L) = \beta_1, \tag{3}$$

where $\alpha_0, \alpha_1, \gamma_0, \gamma_1, \beta_0$ and β_1 are constants.

Initially, the FBTE have been invented during the work on behaviour of real material by use of fractional calculus [1,2]. The application of fractional Bagley-Torvik problem in various feilds like fluid mechanics, viscoelasticsity, digital control theory, bioengineering and biology. Many authors have been worked on fractional Bagley-Torvik equations[3-15].

We provide, some literature based on boundary value fractional Bagley-Torvik equations. Ray [16], used operational matrix based on Haar wavelet for solving FBTE where as the authors [17], developed the general solution for solving FBTE. In [18], Čermák and Kisela used Grnwald-Letnikov discretization to give exact and numerical solution for initial FBTE. In [19], Stanek talked over the solution of FBTE for existence and uniqueness. In [20], the authors studied on FBTE where fractional derivative is discretized by using finite difference method. In [21,22], Zahra and Elkholy, approached two schemes to solve FBTE with boundary and initial conditions. The first technique gives the approximate solution by using shooting method with cubic spline polynomials while the other technique based on approximation of fractional order term in the sense of Grnwald-Letnikov definition. Also in [23], Zahra and Elkholy, to find the numerical solution for fractional boundary value problem using quadratic spline polynomials.

In recent decades, the Chebyshev polynomials are most powerful polynomial approximation in numerical analysis we can see from theoretical as well as practical points of view. The Chebyshev polynomials have direct connections with Fourier and Laurent series, due to minimality properties in approximation theory and with orthogonality condition holds for both discrete and continuous in function spaces [24].

This paper is organized as follows. In Section 2, the idea of fractional derivative and some properties of Chebyshev polynomials. In Section 3, we utilize approximate formula for fractional derivative. In Section 4, error analysis and bound of error are given. In Section 5, the collocation method which is based on Chebyshev approximation. In Section 6, the error estimator of the present algorithm. In Section 7, numerical examples are presented for validation of the proposed method and finally in Section 8, conclusions are given.

2 Preliminaries

This section is devoted to relevant definition of Chebyshev polynomials and well known results of Caputo fractional derivative is considered for this work.

2.1 Fractional derivative

The definition of fractional derivatives are defined in many ways such as Riemann-Liouville, Grunwald-Letnikove and Caputo. In present work, Caputo fractional derivative is used for initial and boundary conditions for the formulation of the problem.

Definition 1 The fractional derivative of $\varphi(t)$ in the Caputo sense of order $\mu > 0$ is defined as [27,28]

$$D^{\mu}\varphi(t) = \frac{1}{\Gamma(s-\mu)}\int_0^t \frac{\varphi^{(s)}(y)}{(t-y)^{(\mu-s+1)}}dy, \quad (4)$$
$$s-1 < \mu \leqslant s,\ s \in \mathbb{N},\ t > 0.$$

The Caputo fractional derivative satisfies linearity properties in similar way to integer order differentiation:

$$D^{\mu}(\alpha_1 p(t)+\alpha_2 q(t)) = \alpha_1 D^{\mu}p(t)+\alpha_2 D^{\mu}q(t), \quad (5)$$

For Caputo fractional derivative we have

$$D^{\mu}K = 0, \quad \text{K is a constant}, \quad (6)$$

$$D^{\mu}t^{\nu} = \begin{cases} \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}t^{\nu-\alpha}, & \nu \in \mathbb{N}_0 \text{ and } \nu \geqslant \lceil\mu\rceil \\ 0, & \nu \in \mathbb{N}_0 \text{ and } \nu < \lceil\mu\rceil, \end{cases} \quad (7)$$

where $\mathbb{N} = \{1,2,\ldots\}$, $\mathbb{N}_0 = \{0,1,2,\ldots\}$ and the notation $\lceil\mu\rceil$ is ceiling function which means the smallest integer greater than or equal to μ.

2.2 Chebyshev polynomials of the second kind

The Chebyshev polynomials $U_i(t)$ are very well known polynomials which is defined on [-1,1] by the relation [29]

$$U_i(t) = \frac{\sin(i+1)\theta}{\sin\theta},$$

of degree i in t, where $t = \cos(\theta)$ and $\theta \in [0,\pi]$. The polynomials $U_i(t)$ is generated by the fundamental recurrence relations

$$U_{i+1}(t) = 2tU_i(t) - U_{i-1}(t),\ i = 1,2,\ldots$$

together with initial conditions

$$U_0(t) = 1,\ U_1(t) = 2t.$$

The Chebyshev polynomials $U_i(t)$ are orthogonal polynomials on [-1,1] with the weight function $w(t)$

$$\int_{-1}^{1} U_i(t)U_k(t)w(t)dt = \begin{cases} 0, & i \neq k, \\ \frac{\pi}{2}, & i = k, \end{cases} \quad (8)$$

where $w(t) = \sqrt{(1-t^2)}$. By using the properties of Gamma function, the analytical form of $U_i(x)$ is given

$$U_i(t) = \sum_{k=0}^{\lceil\frac{i}{2}\rceil}(-1)^k 2^{i-2k}\frac{\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(i-2k+1)}t^{i-2k}, i > 0, \quad (9)$$

where the notation $\lceil\frac{i}{2}\rceil$ is integer part of $i/2$.

For the numerical study it is convenient to use the range of interval on [0,1] instead of [-1,1]. The $U_i^*(t)$ is defined as

$$U_i^*(t) = U_i(2t-1).$$

The orthogonality conditions of polynomials $U_i^*(t)$ on [0,1] with the weight function $w^*(t)$

$$\int_0^1 U_i^*(t)U_k^*(t)w^*(t) = \begin{cases} 0 & i \neq k, \\ \frac{\pi}{8} & i = k, \end{cases} \quad (10)$$

where $w^*(t) = \sqrt{(t-t^2)}$. The fundamental recurrence relation of SCPSK is defined as

$$U_{i+1}^*(t) = 2(2t-1)U_i^*(t) - U_{i-1}^*(t),\ i = 1,2,\ldots,$$

together with the initial conditions

$$U_0^*(t) = 1,\ U_1^*(t) = 4t-2.$$

By using the properties of Gamma function, the analytical form of $U_i^*(t)$

$$U_i^*(t) = \sum_{k=0}^{i} (-1)^k 2^{2i-2k} \frac{\Gamma(2i-k+2)}{\Gamma(k+1)\Gamma(2i-2k+2)} t^{i-k}, \; i > 0. \tag{11}$$

The square integrable function $r(t)$ in the interval [0,1] can be expanded in terms of $U^*(t)$ as follows

$$r(t) = \sum_{i=0}^{\infty} c_i U_i^*(t), \tag{12}$$

where the expansion coefficients $c_i (i = 0, 1, 2, \dots)$ are unknown which is defined as

$$c_i = \frac{8}{\pi} \int_0^1 r(t) \sqrt{t - t^2} U_i^*(t) dt. \tag{13}$$

For practical purpose we take only first $(m+1)$-terms of $U_m^*(t)$ in approximation which is given

$$r_m(t) = \sum_{i=0}^{m} c_i U_i^*(t), \; i = 0, 1, 2, \dots, m. \tag{14}$$

3 Fractional derivative using Chebyshev expansion

In this section, we derive main approximate formula for $D^\mu r(t)$ which is given in Theorem 1.

Theorem 1 The approximation function $r(t)$ can be approximated by the second kind of Chebyshev polynomials which is defined in (14) and assume that $\mu > 0$ then

$$D^\mu(r_m(t)) = \sum_{i=\lceil\mu\rceil}^{m} \sum_{k=0}^{i-\lceil\mu\rceil} c_i w_{i,k}^{(\mu)} t^{i-k-\mu}, \tag{15}$$

where $w_{i,k}^{(\mu)}$ is given by

$$w_{i,k}^{(\mu)} = \frac{(-1)^k 2^{(2i-2k)} \Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\mu)}. \tag{16}$$

Proof. Since the Caputo's fractional derivative satisfies linear properties, we have

$$D^\mu(r_m(t)) = \sum_{i=0}^{m} c_i D^\mu(U_i^*(t)). \tag{17}$$

Applying Eqs. (6) and (7) we get

$$D^\mu(U_i^*(t)) = 0, \;\; i = 0, 1, \dots, \lceil\mu\rceil - 1, \; \mu > 0. \tag{18}$$

Also, for $i = \lceil\mu\rceil, \lceil\mu\rceil + 1, \dots, m$, and by using Eqs.(6) and (7), we get

$$\begin{aligned} D^\mu(U_i^*(t)) &= \sum_{k=0}^{i} (-1)^k 2^{(2i-2k)} \Gamma(2i-k+2) \\ &\quad \frac{1}{\Gamma(k+1)\Gamma(2i-2k+2)} D^\mu t^{i-k} \\ &= \sum_{k=0}^{i-\lceil\mu\rceil} \frac{(-1)^k 2^{(2i-2k)} \Gamma(2i-k+2)}{\Gamma(i+1-k-\mu)} . \\ &\quad \frac{\Gamma(i-k+1) t^{i-k-\mu}}{\Gamma(k+1)\Gamma(2i-2k+2)} \end{aligned} \tag{19}$$

A combination of Eqs. (17)-(19) we obtain

$$\begin{aligned} D^\alpha(r_m(t)) &= \sum_{i=\lceil\mu\rceil}^{m} \sum_{k=0}^{i-\lceil\mu\rceil} c_i (-1)^k \\ &\quad \frac{2^{(2i-2k)} \Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\mu)} t^{i-k-\mu}, \end{aligned} \tag{20}$$

the Eq. (20) can be rearranged in the following form

$$D^\mu(r_m(t)) = \sum_{i=\lceil\mu\rceil}^{m} \sum_{k=0}^{i-\lceil\mu\rceil} c_i w_{i,k}^{(\mu)} t^{i-k-\mu},$$

where $w_{i,k}^{(\mu)}$ is given by

$$w_{i,k}^{(\mu)} = \frac{(-1)^k 2^{(2i-2k)} \Gamma(2i-k+2)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i+1-k-\mu)}.$$

4 Error analysis

Theorem 2 (Chebyshev truncation theorem) The error in approximation $r(t)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If

$$r_m(t) = \sum_{i=0}^{m} c_i U_i^*(t) \tag{21}$$

then

$$E_T(m) \equiv \mid r(t) - r_m(t) \mid \leq \sum_{i=m+1}^{\infty} \mid c_i \mid, \tag{22}$$

for all $r(t)$, all m, and all $t \in [0, 1]$.

Proof. The maximum value of $U_i^*(t)$ is one, that is $|U_i^*(t)| \leq 1$ for all $t \in [0,1]$ and for all i. Therefore the approximating function of the ith term is bounded by coefficients c_i and subtracting the $(m+1)$-terms series from the infinite series, which gives the difference of each terms is bounded by the coefficients and summing the difference of bounding of each terms get the desired result.

Convergence analysis

Theorem 3 Let us assume $D^\mu r(t) \in S^2[0,1]$ and $D^\mu r_m(t)$ be the m^{th} approximation. Assuming that $|D^{\mu+2} r(t)| < M$ where M is constant, as $m \to \infty$ the approximate solution $D^\mu r_m(t)$ converges to $D^\mu r(t)$, i.e.,

$$|c_m| \leq \frac{M}{8m^2}.$$

Let,

$$D^\mu r(t) = \sum_{i=0}^{\infty} c_i U_i(t) \tag{23}$$

The m^{th} approximation of Eq. (23) is written as

$$D^\mu r_m(t) = \sum_{i=0}^{m} c_i U_i(t) \tag{24}$$

From Eqs. (23)-(24), we get

$$D^\mu r(t) - D^\mu r_m(t) = \sum_{i=m+1}^{\infty} c_i U_i(t) \tag{25}$$

Hence from Eq. (23), we obtain

$$\int_0^1 D^\mu r(t) T_m(t) w(t) dt = \int_0^1 \left(\sum_{i=0}^{\infty} c_i U_i(t) \right) T_m(t) w(t) dt \tag{26}$$

The relation between Chebyshev first kind and second kind polynomial is given by

$$U_i(T) = 2T_i(T) + \sum_{j<i} a_j T_j(t) \tag{27}$$

Now combing Eqs.(26)-(27), we obtain

$$c_m = \frac{1}{\pi} \int_0^1 D^\mu r(t) T_m(t) w(t) dt \tag{28}$$

Putting $2t - 1 = \cos(\theta)$, in Eq. (28), we obtain

$$c_m = \frac{1}{2\pi} \int_0^\pi \left(D^\mu r \left(\frac{1 + \cos(\theta)}{2} \right) \right) \cos(m\theta) d\theta \tag{29}$$

Now integration by parts two times in Eq. (29), we get

$$c_m = \frac{1}{16m\pi} \int_0^\pi \left(D^{\mu+2} r \left(\frac{1 + \cos(\theta)}{2} \right) \right) \delta_m d\theta, \tag{30}$$

where

$$\delta_m = \sin(\theta) \left[\frac{\sin(m-1)\theta}{m-1} - \frac{\sin(m+1)\theta}{m+1} \right].$$

Taking the modulus on both side of Eq. (30) and using $|D^{\mu+2} r(t)| < M$, we get

$$\begin{aligned}
|c_m| &= \left| \frac{1}{16m\pi} \int_0^\pi \left(D^{\mu+2} r \left(\frac{1 + \cos(\theta)}{2} \right) \right) \delta_m d\theta \right| \\
&\leq \frac{M}{16m\pi} \int_0^\pi |\delta_m| d\theta \\
&\leq \frac{M}{16m} \left[\frac{1}{m-1} - \frac{1}{m+1} \right] \\
&\leq \frac{M}{16m} \left[\frac{m+1-m+1}{(m-1)(m+1)} \right] \\
&\leq \frac{M}{8m(m-1)(m+1)}.
\end{aligned}$$

Hence, $m > 1$, and for large m,

$$|c_m| \leq \frac{M}{8m^2}. \tag{31}$$

Hence, $D^\mu r_m(t)$ converges to $D^\mu r(t)$.

Error bounds

Theorem 4 Suppose that $D^\mu r_m(t)$ be the m^{th} approximation of $D^\mu r(t) \in S^2[0,1]$, then following error estimate

$$e_m^2 D^\mu r(t) \leq \frac{\pi M^2}{3072} F_3(1+m),$$

where $F_m(z)$ is the Poly Gamma function defined by,

$$F_m(z) = (-1)^{m+1} \Gamma(m) \sum_{k=0}^{\infty} \frac{1}{(z+k)^{m+1}}.$$

Proof: The error term is given as

$$e_m^2 D^\mu r(t) = \left(\int_0^1 |D^\mu r(t) - D^\mu r_m(t)|^2 w(t)dt\right)^{\frac{1}{2}}$$
$$= \left(\int_0^1 \left|\sum_{i=0}^{\infty} c_i U(t) - \sum_{i=0}^{m} c_i U_m(t)\right|^2 w(t)dt\right)^{\frac{1}{2}}$$
$$= \left(\int_0^1 \left|\sum_{i=m+1}^{\infty} c_i U(t)\right|^2 w(t)dt\right)^{\frac{1}{2}}$$

Now using the orthogonality condition for second kind of Chebyshev polynomials, we obtain

$$e_m^2 D^\mu r(t) = \left(\frac{\pi}{8}\sum_{i=m+1}^{\infty} |c_i|^2\right) \tag{32}$$

Now using the results of Theorem 3, i.e., $|c_i| \leq \frac{M}{8i^2}$ in Eq. (32), we get

$$e_m^2 D^\mu r(t) = \frac{\pi M^2}{512}\sum_{i=m+1}^{\infty}\frac{1}{i^4} \tag{33}$$

Summing the series in Eq. (33), we get the following error estimate

$$e_m^2 D^\mu r(t) \leq \frac{\pi M^2}{3072} F_3(1+m).$$

Function approximation

Theorem 5 [30] Let us suppose a function $r(t) \in [0, L]$ be m times continuously differentiable. Let $r_m(t) = \sum_{i=0}^{m} c_i U_i^*(t) = \wedge^T \phi_m(t)$ be the best square approximation function of $r(t)$, where $\wedge = [c_0, c_1, \ldots, c_m]^T$, and $\phi_m(t) = [U_0^*(t), U_1^*(t), \ldots, U_m^*(t)]^T$, then

$$\|r(t) - r_m(t)\| \leqslant \frac{MS^{m+1}L}{(m+1)!}\sqrt{\frac{\pi}{8}}, \tag{34}$$

where $M = \max_{t\in[0,L]} r^{m+1}(t)$ and $S = \max[t_0, L - t_0]$.

Proof. We consider the following Taylor polynomial, we have

$$r(t) = r(t_0) + r'(t_0)\frac{(t-t_0)}{1!} + \cdots + r^m(t_0)\frac{(t-t_0)^m}{m!} + r^{m+1}(\xi)\frac{(t-t_0)^{m+1}}{(m+1)!},$$

where $t_0 \in [0, L]$ and $\xi \in [t_0, t]$.
Let

$$P_m(t) = r(t_0) + r'(t_0)\frac{(t-t_0)}{1!} + \cdots + \tag{35}$$
$$r^m(t_0)\frac{(t-t_0)^m}{m!}, \tag{36}$$

then

$$|r(t) - P_m(t)| = \left|r^{m+1}(\xi)\frac{(t-t_0)^{m+1}}{(m+1)!}\right|. \tag{37}$$

Since, $r_m(t) = \sum_{i=0}^{m} c_i U_i^*(t) = \wedge^T \phi_m(t)$, is the best square approximation function of $r(t)$, we obtain

$$\|r(t) - r_m(t)\|^2 \leqslant \|r(t) - P_m(t)\|^2$$
$$= \int_0^L w(t)[r(t) - P_m(t)]^2 dt$$
$$= \int_0^L w(t)[r^{m+1}(\xi)\frac{(t-t_0)^{m+1}}{(m+1)!}]^2 dt$$
$$\leqslant \frac{M^2}{[(m+1)!]^2}\int_0^L (t-t_0)^{2m+2} w(t)dt$$
$$= \frac{M^2}{[(m+1)!]^2}\int_0^L (t-t_0)^{2m+2}\sqrt{(Lt-t^2)}dt.$$

Since $S = max[t_0, L - t_0]$, we have

$$\|r(t) - r_m(t)\|^2 \leqslant \frac{M^2 S^{2m+2}}{[(m+1)!]^2}\int_0^L \sqrt{(Lt-t^2)}dt.$$
$$= \frac{M^2 S^{2m+2}}{[(m+1)!]^2}\frac{\pi L^2}{8}.$$

Taking square root both sides, we get

$$\|r(t) - r_m(t)\| \leqslant \frac{MS^{m+1}L}{(m+1)!}\sqrt{\frac{\pi}{8}}.$$

5 Collocation method

The Chebyshev collocation method is applied to solve fractional Bagley-Torvik boundary value problem

$$D^\mu r(t) = f(t, r(t)), \tag{38}$$

together with boundary conditions

$$r(0) = \delta_1, r(L) = \delta_2. \tag{39}$$

We assume that under certain condition on the function f, the fractional Bagley-Torvik boundary value problem (38)-(39) possesses unique solution in $r(t)$ in appropriate space of functions see [5]. The solution $r(t)$ is approximated by $\tilde{r}_m \in \mathbf{S}_{m,\mu}$ as the finite sum

$$\tilde{r}_m(t) = \sum_{i=0}^{m} c_i U_i^*(t;\mu), \tag{40}$$

where c_i are constants. If $\tilde{r}_m \in \mathbf{S}_{m,\mu}$, then $D^\mu \tilde{r}_m \in \mathbf{S}_{m,\mu}$, this key properties is crucial application for the collocation method to the fractional Bagley-Torvik boundary value problem (38)-(39). The unknown coefficients c_i in approximation (40) are obtained from boundary conditions

$$\tilde{r}_m(0) = \delta_1, \tag{41}$$

$$\tilde{r}_m(L) = \delta_2, \tag{42}$$

and the fact that $\tilde{r}_m(t)$ must satisfy the fractional differential equation with some appropriately chosen collocation points $\eta_i, i = 1, 2, \dots, m-1$, with the relations

$$D^\mu \tilde{r}_m(\eta_i) = f(\eta_i, \tilde{r}_m(\eta_i)),\ i = 1, 2, \dots, m-1. \tag{43}$$

The convergence of numerical solution and its computational stability gets affected by the particular choice of collocation points. In order to find the unknown coefficients, Chebyshev collocation method with collocation points $\eta_i = \frac{1}{2} + \frac{L}{2}\cos(i\frac{\pi}{m}),\ i = 1, 2, \dots, m-1$. Put Eq. (40) into (41) and (42), we obtain

$$g_0(c_0, c_1, \dots, c_m) = \sum_{i=0}^{m} c_i U_i^*(0;\mu) = \delta_1, \tag{44}$$

$$g_m(c_0, c_1, \dots, c_m) = \sum_{i=0}^{m} c_i U_i^*(m;\mu) = \delta_2. \tag{45}$$

Now from Eq. (43) we have $m-1$ algebraic equations

$$\begin{aligned} & g_i(c_0, c_1, \dots, c_m) \\ & = \sum_{i=0}^{m} c_i D^\mu U_i^*(\eta_i;\mu) - \\ & f\left(\eta_i, \sum_{i=0}^{m} c_i U_i^*(\eta_i;\mu)\right) = 0,\ i = 1, 2, \dots, m-1. \end{aligned} \tag{46}$$

Now combining Eqs. (44)-(46), we get $(m+1)$ system of algebraic equations for the unknown c_i which is written in the following form

$$G(c) = 0, \tag{47}$$

where $c = [c_0, c_1, c_2, \dots, c_m]^T$ and $G : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ is defined as

$$G(c) = \begin{pmatrix} g_0(c_0, c_1, \dots, c_m) \\ g_1(c_0, c_1, \dots, c_m) \\ \vdots \\ g_m(c_0, c_1, \dots, c_m) \end{pmatrix}.$$

Now put the Eq. (47) into the Eq. (40), we get the solution of fractional Bagley-Torvik problem of Eqs. (38)-(39).

6 Error estimator

The error estimator of the numerical solution which is defines in Section 5. The scheme is based on residual error estimation. Let $u(t)$ be exact solution and $\tilde{u}_m(t)$ be the numerical solution. Then we have

$$a_0 D^2 u(t) + a_1 D^\mu u(t) + a_2 D u(t) - p(t) = 0, \tag{48}$$

and

$$a_0 D^2 \tilde{u}_m(t) + a_1 D^\mu \tilde{u}_m(t) + a_2 D \tilde{u}_m(t) - p(t) = R, \tag{49}$$

where R is the residual function. Now from Eqs. (48)-(49), we obtain

$$a_0 D^2(u(t) - \tilde{u}_m(t)) + a_1 D^\mu(u(t) - \tilde{u}_m(t)) + a_2 D(u(t) - \tilde{u}_m(t)) = R. \tag{50}$$

Let $\xi_m(t) = u(t) - \tilde{u}_m(t)$ is the error function, then from Eq. (50)

$$a_0 D^2 \xi_m(t) + a_1 D^\mu \xi_m(t) + a_2 D \xi_m(t) = R. \tag{51}$$

Now Eq. (51) with boundary conditions $e_m(0) = 0$ and $e_m(L) = 0$, can be solved the Eqs. (38)-(39). Let

$$E = max\{|\xi_m| : \ 0 \le t \le 1\}.$$

The above equation is error estimation of the present method.

7 Numerical examples

In this section, we consider some numerical examples to check the accuracy and reliability of the proposed scheme for FBTE.

If u is exact solution of a given problem, the approximation errors on the discrimination parameter m is estimated in 2-norm

$$e_m = \sqrt{\sum_{i=0}^{m} \left(u(\xi_i) - \tilde{u}_m(\xi_i)\right)^2},$$

where $\tilde{u}_m$ is an approximated solution corresponding to discrimination parameter m.

Example 1 Consider the fractional order Bagley-Torvik equation [25,26]

$$D^2u(t) + D^{\frac{3}{2}}u(t) + u(t) = 1 + t, \quad t \in (0, 1), \tag{52}$$

together with the boundary conditions

$$u(0) = 1, \quad u(1) = 2, \tag{53}$$

where the exact solution is $u(t) = 1 + t$. Firstly, we approximate with $m = 2$,

$$u(t) = \sum_{i=0}^{2} c_i U^*(t) \tag{54}$$

$$u(t) = c_0U_0^*(t) + c_1U_1^*(t) + c_2U_2^*(t) \tag{55}$$

$$u(t) = c_0(1) + c_1(4t - 2) + c_2(16t^2 - 16t + 3) \tag{56}$$

Now from Eq. (53)

$$u(0) = c_0 - 2c_1 + 3c_2 = 1 \tag{57}$$

$$u(1) = c_0 + 2c_1 + 3c_2 = 2 \tag{58}$$

Now from Theorem 1 and Eqs. (43) and (52), we get

$$\begin{aligned} &32c_2 + 36.1088c_2t^{0.5} + c_0 + c_1(4t - 2)+ \\ &c_2(16t^2 - 16t + 3) = 1 + t \end{aligned} \tag{59}$$

A particular choice for the collocation points is $t = 0.5$ then Eq. (59) becomes

$$c_0 + 56.5323c_2 = 1.5 \tag{60}$$

Combining Eqs. (57)-(58) and (60), we get

$$u(t) = 1.5 + 0.25(4t - 2) = 1 + t. \tag{61}$$

The results obtained by the proposed method for $m = 2$ and $\alpha = 1.5$ of Eq. (52), get the exact solution where as in [25], solved this problem with $N = 9$ and $\alpha = 0.5$ using Bessel collocation method and get maximum absolute error $4.2834e - 015$, it seems that the proposed method needs only few terms of SCPSK and get exact solution. The comparison of exact and numerical solutions are given in Fig. 1.

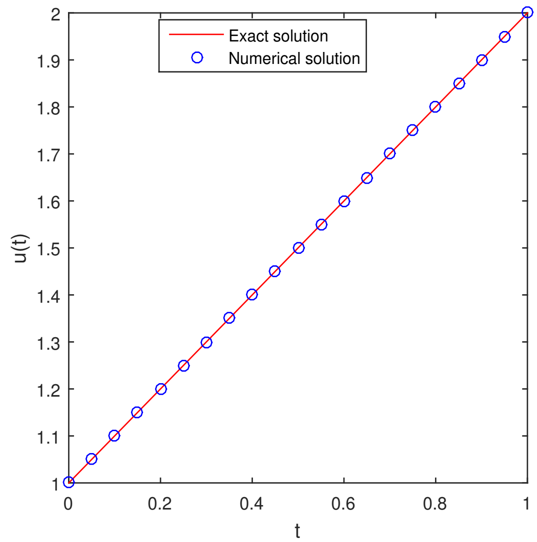

Figure 1: The comparison of the numerical solution with analytical solution with $m = 2$ for Example 1

Example 2 Consider the following fractional Bagely-Torvik equation [31]

$$D^2u(t) + D^{\frac{3}{2}}u(t) + u(t) = t^2 + 2 + 4\sqrt{\frac{t}{\pi}}, \; t \in (0, 1), \tag{62}$$

together with boundary conditions

$$u(0) = 0, \; u(1) = 1, \tag{63}$$

where the exact solution is $u(t) = t^2$. We apply the proposed method which is described in Section 5 with $m = 2$, in this case we get the exact solution $u(t) = t^2$. It is noticed that the proposed method is direct converge to the the exact solution and only take few terms of SCPSK see Fig. 2. Doha et al. [33] and Ramzi et al. [32] solved this problem whose exact and approximate solution are same.

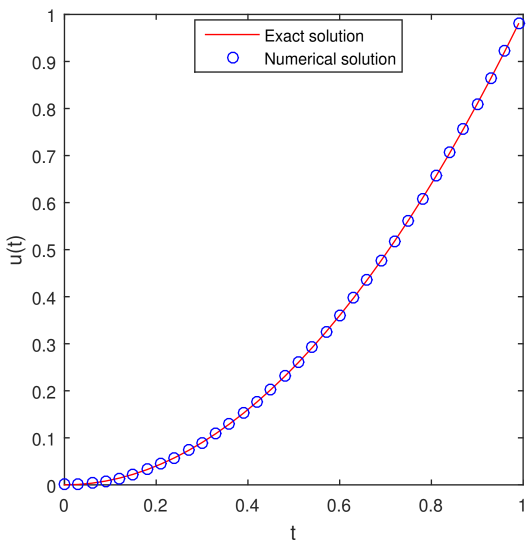

Figure 2: The comparison of the numerical solution with analytical solution $m = 2$ for Example 2

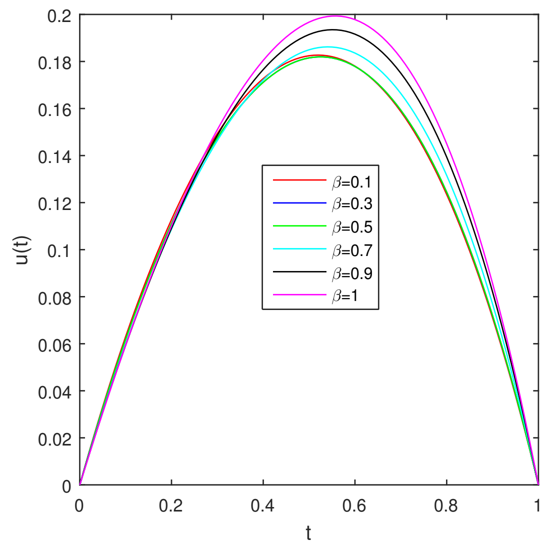

Figure 3: The behaviour of exact and numerical solution for different values of β and $\alpha = 2$ with $m = 9$ for Example 3

Example 3 Consider the following Baglet-Torvik equation [34,35]

$$D^{\alpha}u(t) - D^{\beta}u(t) = -1 - e^{(t-1)}, 1 < \alpha \leqslant 2, \quad 0 < \beta \leqslant 1, \tag{64}$$

together with boundary conditions

$$u(0) = u(1) = 0. \tag{65}$$

The exact solution for general values of α and β of Eqs. (64)-(65) is not known. However, $u(t) = t(1 - e^{(t-1)})$ is the exact solution for $\alpha = 2, \beta = 1$. The Eq. (64) is solved in [36] numerically for integer order case and applying HPM and Green function method. Also, in [34], solved this problem using Haar wavelet for integer order. The numerical results are given in Table 1. In Fig. 3, for $\alpha = 2$ and different values of β, which show that as β approaches to 1, numerical results approached to the numerical results for the integer order differential equations. The numerical results motivate that proposed method is practically when treating with two point fractional Bagley-Torvik equation. The numerical results in Table 1 reveals that the proposed method demonstrate the method [34] and [36].

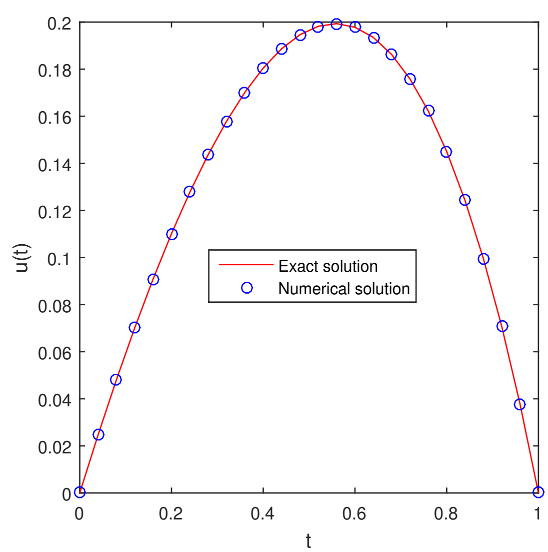

Figure 4: The comparison of numerical solution with exact solution with $m = 9$ and $\alpha = 2, \beta = 1$ for Example 3

Table 1: The comparison of present method with method [36] and [34] for $\alpha = 2, \beta = 1$ with $m = 9$ for Example 3

x	Exact solution	Method [36]	Method [34]	Present Method
0.0	0	0	0	0
0.1	0.05934303	0.05934820	0.05934300	0.05934303
0.2	0.11013421	0.11014318	0.11013418	0.11013421
0.3	0.15102441	0.15103441	0.15102438	0.15102441
0.4	0.18047535	0.18048329	0.18047531	0.18047535
0.5	0.19673467	0.19673826	0.19673463	0.19673467
0.6	0.19780797	0.19780653	0.19780792	0.19780797
0.7	0.18142725	0.18142196	0.18142718	0.18142725
0.8	0.14501540	0.14500893	0.14501532	0.14501540
0.9	0.08564632	0.08564186	0.08564623	0.08564632
1.0	0	0	0	0

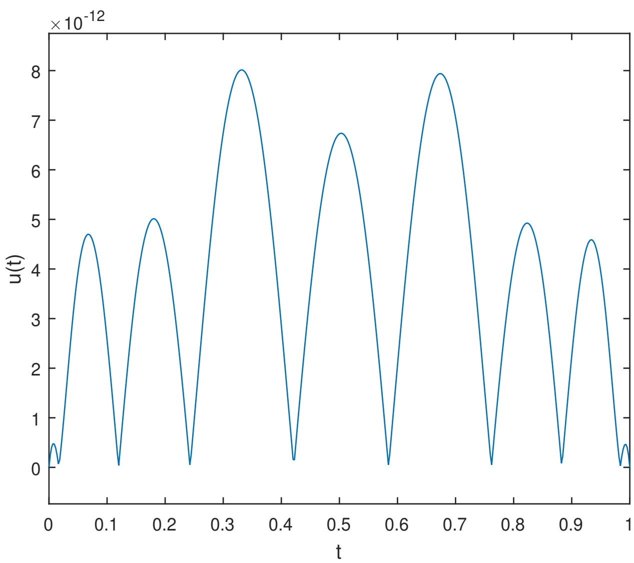

Figure 5: The graph of absolute error between exact solution and present method with $m = 9$ and $\alpha = 2, \beta = 1$ for Example 3

Example 4 Consider the fractional boundary value problem

$$D^{\frac{3}{2}}u(t)+u(t) = t^5-t^4+\frac{128}{7\sqrt{(\pi)}}t^{3.5}-\frac{64}{5\sqrt{(\pi)}}t^{2.5},\ t \in [0,1] \quad (66)$$

together with boundary conditions

$$u(0) = 0,\ u(1) = 0, \quad (67)$$

whose exact solution is $u(t) = t^5 - t^4$. The numerical solutions are presented in Table 2 and Fig. 6. In Fig. 6, the behaviour of exact and numerical solutions are reported for $m = 10$, which shows that the numerical results achieved good accuracy. In Table 2, the absolute error are presented for $m = 10, 15, 20$. In Fig. 7, the absolute error is given for $m = 15$.

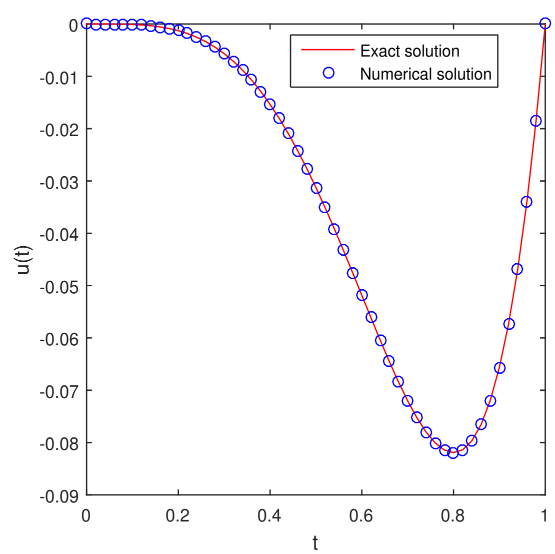

Figure 6: The comparison of the numerical solution with analytical solution $m = 10$ for Example 4

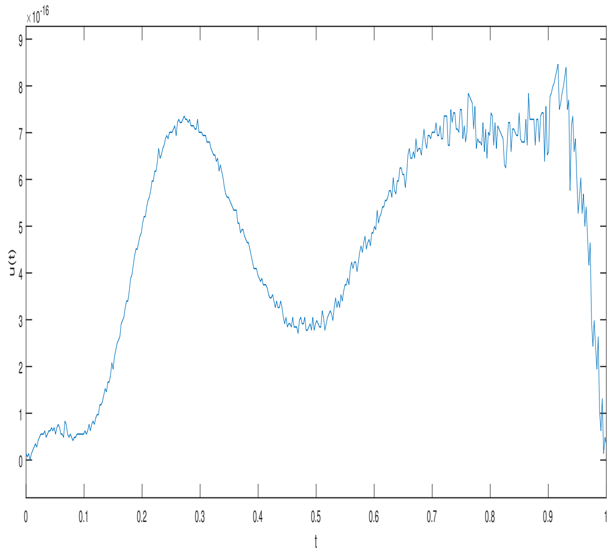

Figure 7: The graph of error function with $m = 15$ for Example 4

8 Conclusion

An efficient, Chebyshev collocation method is applied to solve FBTE with orthogonal Chebyshev polynomials. The properties of SCPSK are used to reduce fractional Bagley-Torvik equation into system of algebraic equations which can be solved numerically. The proposed method is characterise by its simplicity, efficiency and high accuracy. For validation of present scheme is tested through number of examples and compared with exiting methods. In Example 1, we compared present method with method [25], which reveals that the present method is high accuracy, for this we need few terms of Chebyshew expansion. In Example 2, the numerical result obtained by proposed method good agreement with method [32] and [33]. In Example 3, we compared present method with method [34] and [36], which show that the present method demonstrate the existing methods.

Acknowledgements: The authors are thankful to the Department of Applied Mathematics and Humanities, S.V. National Institute of Technology, Surat-395007, India, for providing senior research fellowship.

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Table 2: The absolute error are given with $m = 10, 15, 20$ for Example 4

x	Absolute error m=10	Absolute error m=15	Absolute error m=20
0.0	0	0	0
0.1	5.22482e-16	6.12645e-17	9.28406e-17
0.2	9.96516e-16	5.05718e-16	1.50435e-16
0.3	1.21694e-15	7.07840e-16	1.99252e-16
0.4	1.06858e-15	4.02671e-16	9.20731e-18
0.5	7.31709e-16	2.97424e-16	1.49626e-17
0.6	4.83479e-16	5.01111e-16	2.65018e-16
0.7	3.90560e-16	7.09809e-16	4.38472e-16
0.8	3.64167e-16	7.08777e-16	5.92636e-16
0.9	3.97972e-16	7.56103e-16	5.12409e-16
1.0	0	0	0

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