Bifurcation Phenomenon in An Oscillator Model for EI Niño and Southern Oscillation

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Abstract: In this paper, a recharge-discharge oscillator model for the EI Niño and southern oscillation with different delays is investigated. The conditions which ensure the local stability and the existence of Hopf bifurcation at the zero equilibrium of the model are obtained. It shows that the two different time delays have different effect on the dynamical behavior of the model. An example together with its numerical simulations shows the feasibility of the main results. Finally, main conclusions are included.

Key–Words: EI Niño and southern oscillation model, Hopf bifurcation, Stability, Periodic solution, Delay

1 Introduction

El $Ni\tilde{n}$-southern Oscillation (ENSO) is an inter annual phenomenon involved in the tropical Pacific ocean atmosphere interactions. It is one of strongest signals in inter-annual change in the present whole world climate system. Its occurrence will leads to serious dry waterlogging disasters which bring for the global general areas and has seriously effect on climate and ecology changes all over the world. Thus the investigation on the law and prevention of ENSO has received great attention from domestic and foreign academic circles [1-10]. Recently, numerous excellent results on the ENSO models have been reported. For example, Mo et al. [11] discussed a class of homotopic solving method for ENSO model, Fedorov and Philander [10] investigated the stability of tropical ocean-atmosphere interactions for El Niño, Mo and Lin [11] analyzed the perturbed solution of a ENSO nonlinear model. Zhu et al. [12] focused on the perturbed solution of a class of ENSO delayed sea-air oscillator, Mo et al. [13] consider a delayed sea-air oscillator coupling model for the ENSO. Feng et al. [14] made a theoretical analysis on the dynamical behavior and instability evolution of air-sea oscillator, Zhao et al. [15] studied the recharge-discharge oscillator model for El Nino-Southern Oscillation (ENSO). In 2001, Neelin et al. [16] investigated the nonlinear delayed model on Equator Pacific Ocean:

$$\begin{cases} \frac{dT}{dt} = a\tau_1 - b_1\tau_1(t-\eta) - \varepsilon T^3, \\ \frac{d\tau_1}{dt} = dT - R_{\tau_1}\tau_1, \end{cases} \tag{1}$$

where T is is the $Ni\tilde{n}o{-}3$ anomaly, τ_1 $Ni\tilde{n}o{-}4$ zonal wind stress anomaly,η is a time delay for waves to travel to the western boundary and return to the eastern Pacific, $a$is a coefficient representing the positive feedback between T, b_1 is a coefficient representing the negative feedback due to waves reflection at the western boundary, d is a positive coefficient that relates the $Ni\tilde{n}o - 3$ anomalies to the $Ni\tilde{n}o - 4$ zonal wind stress anomalies, R_{τ_1} is a damping coefficient, ε is a cubic damping coefficient, where ε is a small positive constant. a, b_1, d, R_{τ_1} are all positive constants.

In real natural world, Considering that the coefficients of model often change with time, Wang [1] investigated the periodic solution of the following unified oscillator model for the El Niño-Southern oscillation

$$\begin{cases} \frac{dT}{dt} = a(t)\tau_1 - b_1(t)\tau_1(t-\eta) - \varepsilon T^3, \\ \frac{d\tau_1}{dt} = d(t)T - R_{\tau_1}(t)\tau_1, \end{cases} \tag{2}$$

where $a(t), b_1(t), d(t)$ and R_{τ_1} are all continuous ω-periodic functions. By means of the coincidence degree theory, Wang [1] obtained the sufficient condition which ensures the existence of periodic solution of system (2). In order to reveal what the time delay has effect on the dynamical behavior, Cao et al. [17]

investigated the stability and Hopf bifurcation nature of system (1) by regarding the time delay η bifurcation parameter.

In 2013, Li et al [20] considered the Hopf bifurcation and periodic solutions of the following delayed sea-air oscillator coupling model for the ENSO

$$\begin{cases} \frac{dT}{dt} = \frac{ad}{R_{\tau_1}}T - \frac{b_1 d}{R_{\tau_1}}T(t-\eta) \\ \qquad + \frac{b_2 e}{R_{\tau_2}}h(t-\delta) - \varepsilon T^3, \\ \frac{dh}{dt} = -\frac{cd}{R_{\tau_1}}T(t-\lambda) - R_h h. \end{cases} \tag{3}$$

By assuming that $\eta = \lambda = 0$ and regarding time delay δ as bifurcation parameter,Li et al [20] obtained the sufficient condition which ensures the existence of Hopf bifurcation of model (3). In addition, by the coincidence degree theory, the sufficient condition which ensures the existence of periodic solution of system (3) is established. Here we would like to point out that although Li et al [20] discussed the effect of time delay on the dynamical behavior of model (3), the different time delays have different effect on the dynamical behavior of system, Li et al [20] did not consider this aspect. Thus we think that it is necessary to investigate this topic, i.e, what effect different time delays have on the dynamical behavior on system? For simplification, we assume that $\eta = \lambda$, then model (3) becomes

$$\begin{cases} \frac{dT}{dt} = \frac{ad}{R_{\tau_1}}T - \frac{b_1 d}{R_{\tau_1}}T(t-\eta) \\ \qquad + \frac{b_2 e}{R_{\tau_2}}h(t-\delta) - \varepsilon T^3, \\ \frac{dh}{dt} = -\frac{cd}{R_{\tau_1}}T(t-\eta) - R_h h, \end{cases} \tag{4}$$

Let $a_1 = \frac{ad}{R_{\tau_1}}, a_2 = \frac{b_1 d}{R_{\tau_1}}, a_3 = \frac{b_2 e}{R_{\tau_2}}, c_1 = \frac{cd}{R_{\tau_1}}$, then system (4) can be written as

$$\begin{cases} \frac{dT}{dt} = a_1 T - a_2 T(t-\eta) \\ \qquad + a_3 h(t-\delta) - \varepsilon T^3, \\ \frac{dh}{dt} = -c_1 T(t-\eta) - R_h h. \end{cases} \tag{5}$$

In this paper, choosing time delays η and δ as bifurcation parameters,respectively, we will make a detailed analysis on the Hopf bifurcation of system (5). The sufficient conditions which ensure the stability of the equilibrium and the existence of Hopf bifurcation for system (5) are obtained. This reveals that the time delays have important effect on the dynamical behavior of system (5).

2 Stability of Equilibrium and Existence of Hopf Bifurcation

IIf the condition

$$(\mathrm{H1}): (a_1 - a_2)R_h < a_3 c_1$$

holds, then system (5)has a unique equilibrium $E(0,0)$. The linearized equation of system (5) near $E(0,0)$ is given by

$$\begin{cases} \frac{dT}{dt} = a_1 T - a_2 T(t-\eta) + a_3 h(t-\delta), \\ \frac{dh}{dt} = -c_1 T(t-\eta) - R_h h. \end{cases} \tag{6}$$

Then characteristic equation of (6) takes the form

$$\det \begin{bmatrix} \lambda - a_1 + a_2 e^{-\lambda\eta} & -a_3 e^{-\lambda\delta} \\ c_1 e^{-\lambda\delta} & \lambda + R_h \end{bmatrix} = 0.$$

Namely

$$\lambda^2 + (R_h - a_1)\lambda - a_1 R_h + (a_2\lambda + a_2 R_h)$$
$$\times e^{-\lambda\eta} + a_3 c_1 e^{-\lambda(\delta+\eta)} = 0. \tag{7}$$

In the following, we consider six cases.

Case 1 When $\eta = \delta = 0$, then (7) becomes

$$\lambda^2 + (R_h - a_1 + a_2)\lambda + (a_2 - a_1)R_h + a_3 c_1 = 0. \tag{8}$$

If the following condition

$$(\mathrm{H2}): \quad R_h - a_1 + a_2 > 0, (a_2 - a_1)R_h + a_3 c_1 > 0$$

is satisfied, then all the roots of Eq. (8)have negative real part. Thus the equilibrium $E(0,0)$ of system (5) is local asymptotically stable if the conditions (H1) and (H2) hold.

Case 2 When $\eta = 0, \delta > 0$, then Eq. (7) becomes

$$\lambda^2 + (R_h - a_1 + a_2)\lambda + (a_2 - a_1)R_h + a_3 c_1 e^{-\lambda\delta} = 0. \tag{9}$$

Let $\lambda = i\omega$ be a root of (9), then

$$-\omega^2 + (R_h - a_1 + a_2)i\omega + (a_2 - a_1)R_h + a_3 c_1 e^{-i\omega\delta} = 0.$$

Separating the real and imaginary part, we get

$$\begin{cases} a_3 c_1 \cos\omega\delta = \omega^2 - (a_2 - a_1)R_h, \\ a_3 c_1 \sin\omega\delta = (R_h - a_1 + a_2)\omega. \end{cases} \tag{10}$$

Then

$$\omega^4 + r_1\omega^2 + r_2 = 0, \tag{11}$$

where

$$r_1 = (R_h - a_1 + a_2)^2 - 2(a_2 - a_1)R_h,$$
$$r_2 = [(a_2 - a_1)R_h]^2 - (a_3 c_1)^2.$$

Define $\Delta_1 = r_1^2 - 4r_2$. In view of Theorem 2.1 in [21] and [22], we have the following result.

Lemma 1 Under the conditions (H1) and (H2), (i) if $r_1 < 0, \Delta_1 = 0$, then when $\delta = \delta_n^+$, Eq. (9) has a pair of pure imaginary roots $\pm i\omega_+$; (ii) if $r_1 < 0, \Delta_1 > 0$, then when $\delta = \delta_n^{\pm}$, Eq. (9) has two pairs of pure imaginary roots $\pm i\omega_+$ and $\pm i\omega_-$, where

$$\delta_n^{\pm} = \frac{1}{\omega_{\pm}} \arccos\left[\frac{\omega_{\pm}^2 - (a_2 - a_1)R_h}{a_3 c_1}\right] + \frac{2n\pi}{\omega_{\pm}}, \tag{12}$$

where $n = 0, 1, 2, \ldots, \omega_{\pm}$ satisfies

$$\omega_+^2 = \frac{-r_1 + \sqrt{\Delta_1}}{2}, \omega_-^2 = \frac{-r_1 - \sqrt{\Delta_1}}{2}; \tag{13}$$

(iii) if $r_1 > 0$ or $\Delta_1 > 0$, then for any $\delta \geq 0$, all the roots of Eq. (9) have negative real part.

Proof. It is easy to see that (11) has positive root

$$\omega_+ = \left[\frac{-r_1 + \sqrt{\triangle_1}}{2}\right]^{\frac{1}{2}}.$$

Thus (i) holds. If $r_1 < 0, \triangle_1 > 0$, then it follows from (11) that

$$\omega_+^2 = \frac{-r_1 + \sqrt{\triangle_1}}{2}, \omega_-^2 = \frac{-r_1 - \sqrt{\triangle_1}}{2}.$$

Then (ii) holds. If $r_1 > 0$ or $\triangle_1 > 0$, Then we know that (11) has no solution, thus all the roots of Eq. (9) have negative real part. So (iii) holds true.

In view of Lemma 1, we have the following theorem.

Theorem 1 Let $\delta_n^{\pm}$ be defined by (12) and $\delta_0 = \min\{\delta_n^+\}(n = 0, 1, 2, \cdots)$. Under the conditions (H1) and (H2), (i) if $r_1 < 0, \Delta_1 = 0$, then when $\delta \in [0, \delta_0)$, the equilibrium $E(0, 0)$ of system (5) is asymptotically stable, when $\delta > \delta_0$, the equilibrium $E(0, 0)$ of system (5) is unstable. When $\delta = \delta_0$, Hopf bifurcation occurs; (ii) if $r_1 < 0, \Delta_1 > 0$, when $\delta = \delta_n^+$ and $\delta = \delta_n^-$, Hopf bifurcation occurs.

Proof. It follows from (H1) and (H2) that all the roots of Eq. (9) have a negative real part. If $r_1 < 0, \Delta_1 = 0$, then when $\delta = \delta_n^+$, Eq. (9) has a pair of pure imaginary roots $\pm i\omega_+$, then when $\delta \in [0, \delta_0)$, the equilibrium $E(0, 0)$ of system (5) is asymptotically stable. Then the front section of (1) holds. Let $\lambda(\delta) = \alpha(\delta) + i\omega(\delta), \alpha(\delta) = 0, \omega(\delta) = \delta_0$. It follows from (9) that

$$\left[\frac{d\lambda}{d\delta}\right]^{-1} = \frac{(2\lambda + R_h - a_1 + a_2)e^{\lambda\delta}}{a_3 b_1} - \frac{\delta}{\lambda}.$$

Then

$$\mathrm{Re}\left[\frac{d\lambda}{d\delta}\right]_{\delta=\delta_0}^{-1} = \frac{-4(a_2 - a_1)R_h}{(a_3 b_1)^2} > \frac{4a_3 c_1}{(a_3 b_1)^2} > 0.$$

Thus the second part of (i) holds. If $r_1 < 0, \Delta_1 > 0$, in view of (10), we have

$$\mathrm{Re}\left[\frac{d\lambda}{d\delta}\right]_{\delta=\delta_n^{\pm}}^{-1} = \mathrm{Re}\left[\frac{\pm\sqrt{\triangle_1}}{(a_3 b_1)^2}\right].$$

Then

$$\mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\delta}\right]_{\delta=\delta_n^{\pm}}\right\} = \mathrm{sign}\left\{\mathrm{Re}\left[\frac{\pm\sqrt{\triangle_1}}{(a_3 b_1)^2}\right]\right\}.$$

Hence

$$\mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\delta}\right]_{\delta=\delta_n^+}\right\} = 1,$$

$$\mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\delta}\right]_{\delta=\delta_n^-}\right\} = -1.$$

Thus

$$\mathrm{Re}\left[\frac{d\lambda}{d\delta}\right]_{\delta=\delta_n^+} > 0, \mathrm{Re}\left[\frac{d\lambda}{d\delta}\right]_{\delta=\delta_n^-} < 0.$$

So (ii) holds.

Case 3 When $\eta > 0, \delta = 0$, Eq. (7) becomes

$$\lambda^2 + (R_h - a_1)\lambda - a_1 R_h + (a_2\lambda + a_2 R_h + a_3 c_1)e^{-\lambda\eta} = 0. \tag{14}$$

Let $\lambda = i\psi$ be the root of Eq. (14), then

$$-\psi^2 + (R_h - a_1)i\psi - a_1 R_h + (a_2 i\psi + a_2 R_h + a_3 c_1)e^{-i\psi\eta} = 0.$$

Separating the real and imaginary part, we derive

$$\begin{cases} (a_2 R_h + a_3 c_1)\cos\psi\eta + a_2\psi\sin\psi\eta \\ = \psi^2 + a_1 R_h, \\ a_2\psi\cos\psi\eta - (a_2 R_h + a_3 c_1)\sin\psi\eta \\ = -(R_h - a_1)\psi. \end{cases} \tag{15}$$

Then

$$\eta^4 + s_1\eta^2 + s_2 = 0, \tag{16}$$

where

$$s_1 = R_h^2 + a_1^2 - a_2^2, s_2 = (a_1 R_h)^2 - (a_2 R_h + a_3 c_1)^2.$$

Define $\Delta_2 = s_1 - 4s_2$. According to Theorem 2.1 in [21] and [22], we have the following results.

Lemma 2 Under the conditions (H1) and (H2), (i) if $s_1 < 0, \Delta_2 = 0$, then when $\eta = \eta_n^+$, Eq. (14) has a pair of pure imaginary roots $\pm i\psi_+$; (ii) if $s_1 < 0, \Delta_2 > 0$, then when $\eta = \eta_n^+$, Eq. (14) has two pairs of pure imaginary roots $\pm i\psi_+, \pm i\psi_-$, where

$$\eta_n^{\pm} = \frac{1}{\psi_{\pm}} \arccos$$

$$\left[\frac{(\psi_\pm^2+a_1R_h)(a_2R_h+a_3c_1)-(R_h-a_1)\psi_\pm^2 a_2}{(a_2R_h+a_3c_1)^2+(a_2\psi_\pm^2)^2}\right] + \frac{2n\pi}{\psi_\pm}(n=0,1,2,\dots) \quad (17)$$

$\psi_\pm$ satisfies

$$\psi_+^2=\frac{-s_1+\sqrt{\Delta_2}}{2}, \psi_-^2=\frac{-s_1-\sqrt{\Delta_2}}{2}; \quad (18)$$

(iii) if $s_1>0$ or $\Delta_2>0$, then for any $\eta\geq 0$, all the roots of Eq. (14) have a negative real part.

From Lemma 2, we have the following theorem.

Theorem 2 Let $\eta_n^\pm$ be defined by (17). Under the conditions (H1) and (H2), (i) if $s_1<0, \Delta_2=0$, then when $\eta\in[0,\eta_0)$. When $\eta>\eta_0$, the equilibrium $E(0,0)$ of system (5) is unstable. When $\eta=\eta_0$, Hopf bifurcation occurs; (ii) If $s_1>0$ or $\Delta_2<0$, when $\eta=\eta_n^+$ and $\eta=\eta_n^-$, Hopf bifurcation occurs.

Case 4 When $\eta>0,\delta>0$ and δ is in its stable interval. By regarding η as bifurcation parameter. Without loss of generality, we consider system (4) under the assumptions (H1)and (H2). Let $\lambda=i\psi^*(\psi^*>0)$ be a root of (7), then

$$-\psi^{*2}+(R_h-a_1)i\psi^*-a_1R_h+(a_2i\psi^* +a_2R_h)e^{-i\psi^*\eta}+a_3c_1e^{-i\psi^*(\delta+\eta)}=0. \quad (19).$$

Separating the real and imaginary part, we have

$$\begin{cases}(a_2R_h+a_3c_1\cos\psi^*\delta)\cos\psi^*\eta\\ +(a_2\psi^*-a_3c_1\sin\psi^*\delta)\sin\psi^*\eta\\ =\psi^{*2}-a_1R_h,\\ (a_2\psi^*-a_3c_1\sin\psi^*\delta)\cos\psi^*\eta\\ -(a_2R_h+a_3c_1\cos\psi^*\delta)\sin\psi^*\eta\\ =-(R_h-a_1)\psi^*.\end{cases} \quad (20)$$

Eliminating η from (20)

$$\psi^{*4}+k_1\psi^{*2}+k_2\psi^*+k_3=0, \quad (21)$$

where

$$\begin{aligned}k_1 &= (R_h-a_1)^2-2a_1R_h-a_2^2,\\ k_2 &= 2a_2a_3c_1\sin\psi^*\delta,\\ k_3 &= (a_1R_h)^2-(a_2R_h)^2-(a_3c_1\cos\psi^*\delta)^2\\ &\quad -2a_2a_3c_1\cos\psi^*\delta-(a_3c_1\sin\psi^*\delta)^2.\end{aligned}$$

Denote

$$h(\psi^*)=\psi^{*4}+k_1\psi^{*2}+k_2\psi^*+k_3=0. \quad (22)$$

Assume that

$$(\mathrm{H3}): k_3<0.$$

If (H3) holds, then $h(0)<0$, Since $\lim_{\psi^*\to+\infty}h(\psi^*)=+\infty$. Thus (21) has finite positive roots $\psi_1^*,\psi_2^*,\dots,\psi_n^*$. for every $\psi_i^*, i=1,2,\dots,k$, there exists a sequence $\{\eta_i^j|j=1,2,\dots,\}$ such that (21) holds. Let

$$\eta_0=\min\{\eta_i^j|i=1,2,\dots,k;j=1,2,\dots\}. \quad (23)$$

Then when $\eta=\eta_0$, $\delta\in[0,\delta_0)$, (7) has a pair of pure imaginary roots $\pm i\tilde{\eta}$. Next, we assume that

$$(\mathrm{H4}): \left[\frac{d(\mathrm{Re}\lambda)}{d\eta}\right]_{\lambda=i\tilde{\eta}}\neq 0.$$

According to the general Hopf bifurcation theorem for FDES in Hale [23], we have the following result on the stability and Hopf bifurcation in system (5).

Theorem 3 For system (5), Assume that (H1)–(H4)hold and $\delta\in[0,\delta_0)$, then when $\eta\in[0,\eta_0)$, system (5) is asymptotically stable; When $\eta=\eta_0$, Hopf bifurcation of system (5) occurs around the equilibrium $E(0,0)$.

Case 5 When $\eta>0,\delta>0$ and η is in its stable interval. By choosing δ as bifurcation parameter. With loss of generality, we consider system (4) under the assumptions (H1)and (H2). Let $\lambda=i\omega^*(\omega^*>0)$ be the root of (7), then

$$-\omega^{*2}+(R_h-a_1)i\omega^*-a_1R_h+(a_2i\omega^* +a_2R_h)e^{-i\omega^*\eta}+a_3c_1e^{-i\omega^*(\delta+\eta)}=0. \quad (24).$$

Separating the real and imaginary part, we have

$$\begin{cases}a_3c_1\cos\omega^*\eta\cos\omega^*\delta+a_3c_1\sin\omega^*\eta\sin\omega^*\delta\\ =\omega^{*2}+a_1R_h-a_2R_h\cos\omega^*\eta-a_2\omega^*\sin\omega^*\eta,\\ a_3c_1\cos\omega^*\eta\sin\omega^*\delta-a_3c_1\sin\omega^*\eta\cos\omega^*\delta\\ =(R_h-a_1)\omega^*+a_2\omega^*\cos\omega^*\eta-a_2R_h\sin\omega^*\eta.\end{cases} \quad (25)$$

Eliminating δ from (25)

$$\omega^{*4}+m_1\omega^{*3}+m_2\omega^{*2}+m_3\omega^*+m_4=0, \quad (26)$$

where

$$\begin{aligned}m_1 &= -2a_2\sin\omega^*\eta,\\ m_2 &= (a_2\sin\omega^*\eta)^2+(R_h-a_1)^2\\ &\quad +a_2^2+2a_2(R_h-a_1)\cos\omega^*\eta\\ &\quad +2(a_1R_h-a_2R_h\cos\omega^*\eta),\\ m_3 &= 2a_2^2R_h\cos\omega^*\eta\sin\omega^*\eta\\ &\quad -2a_2R_h(R_h-a_1)\sin\omega^*\eta\\ &\quad -2a_2(a_1R_h-a_2R_h\cos\omega^*\eta)\sin\omega^*\eta,\\ m_4 &= (a_1R_h-a_2R_h\cos\omega^*\eta)^2\\ &\quad +(a_2R_h\sin\omega^*\eta)^2-(a_3c_1)^2.\end{aligned}$$

Denote

$$f(\omega^*)=\omega^{*4}+m_1\omega^{*3}+m_2\omega^{*2}$$

$$+m_3\omega^*+m_4=0. \quad (27)$$

Assume that

$$(\mathrm{H5}): m_4<0.$$

If (H5) holds, then $f(0)<0$, Since $\lim_{\omega^*\to+\infty} f(\omega^*)=+\infty$. Thus (26) has finite positive roots $\omega_1^*,\omega_2^*,\dots,\omega_n^*$. For every $\omega_i^*, i=1,2,\dots,k$, there exists a sequence $\{\delta_i^j|j=1,2,\dots,\}$ such that (26) holds. Let

$$\delta_0=\min\{\delta_i^j|i=1,2,\dots,k;j=1,2,\dots\}. \quad (28)$$

Then when $\delta=\delta_0$, for $\eta\in[0,\eta_0)$, (7) has a pair of pure imaginary roots $\pm i\tilde{\delta}$. In the following, we assume that

$$(\mathrm{H6}): \left[\frac{d(\mathrm{Re}\lambda)}{d\delta}\right]_{\lambda=i\tilde{\delta}}\neq 0.$$

According to the general Hopf bifurcation theorem for FDES in Hale [23], we have the following result on the stability and Hopf bifurcation in system (5).

Theorem 4 For system (5), assume that (H1), (H2),(H5) and (H6) hold and $\eta\in[0,\eta_0)$, then when $\delta\in[0,\delta_0)$, system (5) is asymptotically stable; When $\delta=\delta_0$, Hopf bifurcation of system (5) occurs around the equilibrium $E(0,0)$.

Case 6 When $\eta=\delta$, Eq. (7) becomes

$$\lambda^2+(R_h-a_1)\lambda-a_1R_h+(a_2\lambda+a_2R_h)e^{-\lambda\eta}$$

$$+a_3c_1e^{-2\lambda\eta}=0. \quad (29)$$

Multiplying $e^{-\lambda\eta}$ on both sides of (29), it is easy to obtain

$$[\lambda^2+(R_h-a_1)\lambda-a_1R_h]e^{\lambda\eta}+a_2\lambda+a_2R_h$$

$$+a_3c_1e^{-\lambda\eta}=0. \quad (30)$$

When $\eta=0$, (30) becomes

$$\lambda^2+(R_h-a_1+a_2)\lambda-a_1R_h+a_2R_h+a_3c_1=0. \quad (31)$$

Obviously, if the condition (H2) is satisfied, then all the roots of (31) have a negative real part. Thus if the conditions (H1) and (H2) are satisfied, then the equilibrium $E(0,0)$ of system (5) is local asymptotically stable.

Let $\lambda=i\theta(\theta>0)$ be the root of (30), then

$$[-\theta^2+(R_h-a_1)i\theta-a_1R_h]e^{i\theta\eta}$$

$$+a_2i\theta+a_2R_h+a_3c_1e^{-i\theta\eta}=0. \quad (32)$$

Separating the real and imaginary part, we obtain

$$\begin{cases}(a_1R_h-\theta^2+a_3c_1)\cos\theta\eta\\ +(R_h-a_1)\theta\sin\theta\eta=-a_2R_h,\\ (R_h-a_1)\theta\cos\theta\eta\\ +(a_1R_h-\theta^2-a_3c_1)\sin\theta\eta=-a_2\theta.\end{cases} \quad (33)$$

then $\sin\theta\eta=$

$$\frac{a_2R_h(R_h-a_1)\theta-a_2\theta(a_1R_h-\theta^2+a_3c_1)}{(a_1R_h-\theta^2)^2-(a_3c_1)^2-(R_h-a_1)^2\theta^2}, \quad (33)$$

$\cos\theta\eta=$

$$\frac{a_2\theta^2(R_h-a_1)-a_2R_h(a_1R_h-\theta^2-a_3c_1)}{(a_1R_h-\theta^2)^2-(a_3c_1)^2-(R_h-a_1)^2\theta^2}. \quad (34)$$

In view of $\sin^2\theta\eta+\cos^2\theta\eta=1$, we get

$$\theta^8+u_3\theta^6+u_2\theta^4+u_1\theta^2+u_0=0, \quad (35)$$

where

$$\begin{aligned}u_0 &= [(a_1R_h)^2-(a_3b_1)^2]^2\\ &\quad -[a_2R_h(a_1R_h-a_3c_1)]^2,\\ u_1 &= 2[2a_1R_h-(R_h-a_1)^2]\\ &\quad \times[(a_1R_h)^2-(a_3c_1)^2]\\ &\quad -(a_2R_h^2-2a_1a_2R_h-a_2a_3c_1)^2\\ &\quad +2a_2R_h(a_2R_h-a_1a_2)(a_1R_h-a_3c_1),\\ u_2 &= [(R_h-a_1)^2-2a_1R_h]^2\\ &\quad +2[(a_1R_h)^2-(a_3c_1)^2]\\ &\quad -2a_2(a_2R_h^2-2a_1a_2R_h-a_2a_3c_1)\\ &\quad -(2a_2R_h-a_1a_2)^2,\\ u_3 &= 2[2a_1R_h-(R_h-a_1)^2-a_2^2].\end{aligned}$$

Let $z=\theta^2$, then (35) becomes

$$z^4+u_3z^3+u_2z^2+u_1z+u_0=0, \quad (36)$$

Denote

$$h(z)=z^4+u_3z^3+u_2z^2+u_1z+u_0. \quad (37)$$

then

$$h'(z)=4z^3+3u_3z^2+2u_2z+u_1. \quad (38)$$

Suppose that

$$4z^3+3u_3z^2+2u_2z+u_1=0. \quad (39)$$

Let $y=z+\frac{u_3}{4}$. Then (39) can be written as

$$y^3+p_1y+q_1=0, \quad (40)$$

where

$$p_1 = \frac{u_2}{2} - \frac{3}{16}u_3^2, q_1 = \frac{u_3^3}{32} - \frac{u_3 u_2}{8} + \frac{u_1}{4}.$$

Define

$$\begin{aligned} D &= \left(\frac{q_1}{2}\right)^2 + \left(\frac{p_1}{3}\right)^3, \sigma = \frac{-1+\sqrt{3}}{2}, \\ y_1 &= \sqrt[3]{-\frac{q_1}{2} + \sqrt{D}} + \sqrt[3]{-\frac{q_1}{2} - \sqrt{D}}, \\ y_2 &= \sqrt[3]{-\frac{q_1}{2} + \sqrt{D}\sigma} + \sqrt[3]{-\frac{q_1}{2} - \sqrt{D}\sigma^2}, \\ y_3 &= \sqrt[3]{-\frac{q_1}{2} + \sqrt{D}\sigma^2} + \sqrt[3]{-\frac{q_1}{2} - \sqrt{D}\sigma}, \\ z_i &= y_i - \frac{p_1}{4}, i = 1, 2, 3. \end{aligned}$$

In view of [24-25], we have the following results.

Lemma 3 If $u_0 < 0$, then Eq. (39) has at least one positive root.

Lemma 4 Suppose that $u_0 \geq 0$, then the following conclusions are true. (i) If $D \geq 0$, then Eq. (39) has positive root if and only if $z_1 > 0$ and $h'(z_1) < 0$;
(ii) If $D < 0$, then Eq. (39) has positive root if and only if there exists at least one $z^* \in \{z_1, z_2, z_3\}$ such that $z^* > 0$ and $h'(z^*) \leq 0$.

Without loss of generality, we assume that (39) has four positive roots, denoted by z_1, z_2, z_3, z_4, respectively, then (38) has four positive roots

$$\theta_1 = \sqrt{z_1},\ \theta_2 = \sqrt{z_2},\ \theta_3 = \sqrt{z_3},\ \theta_4 = \sqrt{z_4}.$$

By (29), if we denote

$$\eta_k^{(j)} = \frac{1}{\theta_k}\Bigg\{ \arccos \left[\frac{a_2\theta_k^2(R_h - a_1) - a_2 R_h(a_1 R_h - \theta_k^2 - a_3 c_1)}{(a_1 R_h - \theta_k^2)^2 - (a_3 c_1)^2 - (R_h - a_1)^2\theta_k^2}\right] + 2j\pi \Bigg\}, \tag{41}$$

where $k = 1, 2, 3, 4; j = 0, 1, \cdots$, then when $\eta = \eta_k^{(j)}$, $\pm i\theta_k$ are a pair of pure imaginary roots of Eq. (30). Define

$$\eta_0 = \eta_{k_0}^{(0)} = \min_{k\in\{1,2,3,4\}}\{\eta_k^{(0)}\},\ \eta_0 = \eta_{k_0}. \tag{42}$$

Based on the analysis above, we have the following results.

Lemma 5 For $\eta = \delta$. If (H1) and (H2) hold, then when $\eta \in [0, \eta_0)$, all the roots of (5) have a negative real part. When $\eta = \eta_k^{(j)}$ $(k = 1, 2, 3, 4; j = 0, 1, 2, \cdots)$, (5) has a pair of pure imaginary roots $\pm i\theta_k$.

Let $\lambda(\eta) = \alpha(\eta) + i\theta(\eta)$ be the root of (30) with $\eta = \eta_k^{(j)}$, where $\alpha(\eta_k^{(j)}) = 0, \theta(\eta_k^{(j)}) = \theta_k$. According to the theory of functional differential equations, for every $\eta_k^{(j)}, k = 1, 2, 3, 4; j = 0, 1, 2, \cdots$., there exists an $\varepsilon > 0$ such that $\lambda(\eta)$ is continuously differential in η for $|\eta - \eta_k^{(j)}| < \varepsilon$. Substituting $\lambda(\eta)$ into the left-hand side of (30) and taking the derivative with respect to η, we have

$$\left[\frac{d\lambda}{d\eta}\right]^{-1} = \frac{(2\lambda + R_h - a_1)e^{\lambda\eta} + a_2}{\lambda(a_3 c_1 e^{-\lambda\eta} - e^{\lambda\eta})} - \frac{\eta}{\lambda}. \tag{43}$$

Then

$$\begin{aligned} &\left[\frac{d(\mathrm{Re}\lambda(\eta))}{d\eta}\right]^{-1}_{\eta=\eta_k^{(j)}} \\ &= \mathrm{Re}\left\{\frac{(2\lambda + R_h - a_1)e^{\lambda\eta} + a_2}{\lambda(a_3 c_1 e^{-\lambda\eta} - e^{\lambda\eta})}\right\}_{\eta=\eta_k^{(j)}} \\ &= \mathrm{Re}\left\{\frac{\gamma_1 + \gamma_2 i}{\gamma_3 + \gamma_4 i}\right\} = \frac{\gamma_1\gamma_3 + \gamma_2\gamma_4}{\gamma_3^2 + \gamma_4^2}, \end{aligned}$$

where

$$\begin{aligned} \gamma_1 &= (R_h - a_1)\cos\theta_k\eta_k^{(j)} \\ &\quad -2\theta_k\sin\theta_k\eta_k^{(j)} + a_2, \\ \gamma_2 &= (R_h - a_1)\sin\theta_k\eta_k^{(j)} - 2\theta_k\cos\theta_k\eta_k^{(j)}, \\ \gamma_3 &= \theta_k(a_3 c_1 - 1)\sin\theta_k\eta_k^{(j)}, \\ \gamma_4 &= \theta_k(a_3 b c_1 - 1)\cos\theta_k\eta_k^{(j)}. \end{aligned}$$

We assume that

$$(\mathrm{H7}) : \gamma_1\gamma_3 + \gamma_2\gamma_4 \neq 0.$$

Based on the analysis above, we have the following results.

Theorem 5 For $\eta = \delta = 0$, if (H1) and (H2) hold, then for $\eta \in [0, \eta_0)$, the equilibrium $E(0, 0)$ of system (5) is asymptotically stable. Under the conditions (H1) and (H2), suppose that (H7) holds, then when $\eta = \eta_k^{(j)}(k = 1, 2, 3, 4; j = 0, 1, 2, \cdots)$, Hopf bifurcation of system (5) occurs near the equilibrium $E(0, 0)$.

3 Computer Simulations

In this section, we will carry out some numerical simulations to verify the theoretical findings obtained in the previous sections. Consider the following system

$$\begin{cases} \frac{dT}{dt} = 0.6T - 0.4T(t-\eta) + 0.4h(t-\delta) \\ \qquad - 0.6T^3, \\ \frac{dh}{dt} = -0.6T(t-\eta) - 0.3h. \end{cases} \quad (44)$$

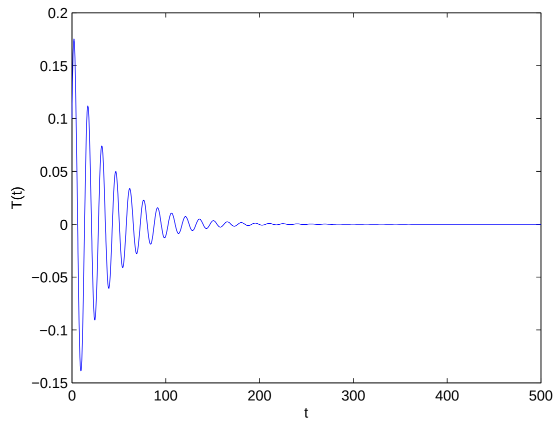
Obviously, system (44) has a unique equilibrium $E(0,0)$. It is east to check that (H1)- (H7) are satisfied. When $\eta = 0$, By Matlab 7.0, we get $\omega_0 \approx 0.4637, \delta_0 \approx 0.37$. When $\delta < \delta_0 \approx 0.37$, then the equilibrium $E(0,0)$ is asymptotically stable. When $\delta > \delta_0 \approx 0.37$, the equilibrium $E(0,0)$ is unstable(see Fig. 1). When $\delta = \delta_0 \approx 0.37$, Hopf bifurcation of system (44) occurs near the equilibrium $E(0,0)$(see Fig. 2).

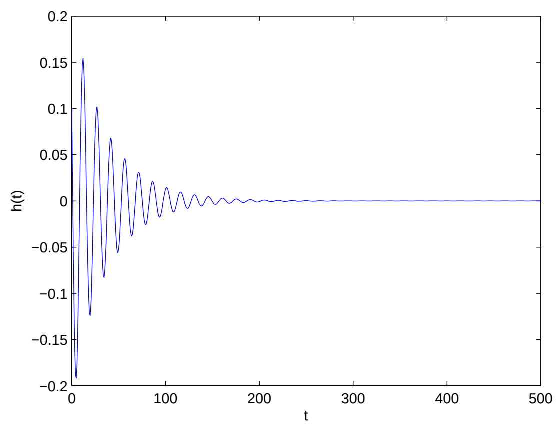
When $\delta = 0$, by Matlab 7.0, we get $\eta_0 \approx 0.25, \eta_0 \approx 0.25$. When $\eta < \eta\eta_0 \approx 0.25$, then the equilibrium $E(0,0)$ of system (44) is asymptotically stable. When $\eta > \eta_0 \approx 0.25$, then the equilibrium $E(0,0)$ of system (44) is unstable(see Fig. 3). When $\eta = \eta_0 \approx 0.25$, Hopf bifurcation occurs near the equilibrium $E(0,0)$. Namely, when $\delta = 0$ and η is close to $\eta_0 = 0.9122$, a small amplitude periodic solution occurs around $E(0,0)$ (see Fig. 4).

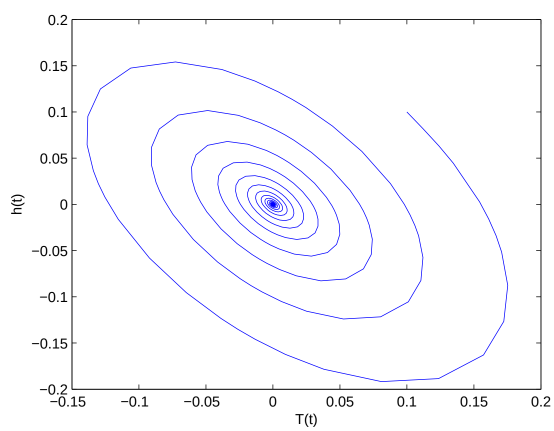
Let $\delta = 0.2 \in (0, 0.37)$ and choose η as parameter. we have $\eta_0 \approx 0.15$. then when $\eta \in [0, 0.15)$, then the equilibrium $E(0,0)$ of system (44) is asymptotically stable. Hopf bifurcation value $\eta_0 \approx 0.15$ (see Figs. 5-6).

Let $\eta = 0.13 \in (0, 0.25)$ and choose δ as bifurcation parameter, we get $\delta_0 \approx 0.35$. When $\delta \in [0, 0.35)$, then the equilibrium $E(0,0)$ of system (44) is asymptotically stable. Hopf bifurcation value of system (44) $\delta_0 \approx 0.35$(see Figs. 7-8).

When $\delta = \eta$, by Matlab 7.0, we get $\theta_0 \approx 0.7428, \eta_0 \approx 0.15$. When $\eta < \eta_0 \approx 0.15$, then the equilibrium $E(0,0)$ of system (44) is asymptotically stable, when $\eta > \eta_0 \approx 0.15$, then the equilibrium $E(0,0)$ of system (44) is unstable(see Fig. 9). When $\eta = \eta_0 \approx 0.15$, a Hopf bifurcation occurs near the equilibrium $E(0,0)$, i.e., when η is close to $\eta_0 = 0.15$, then a small amplitude periodic solution occurs around $E(0,0)$ (see Fig. 10).







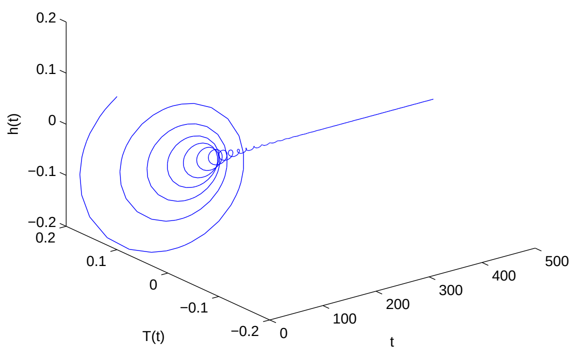
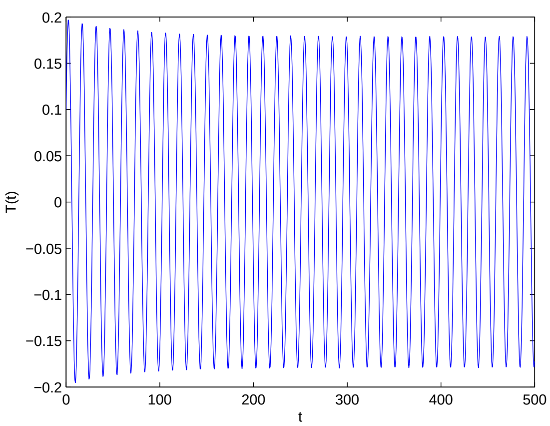
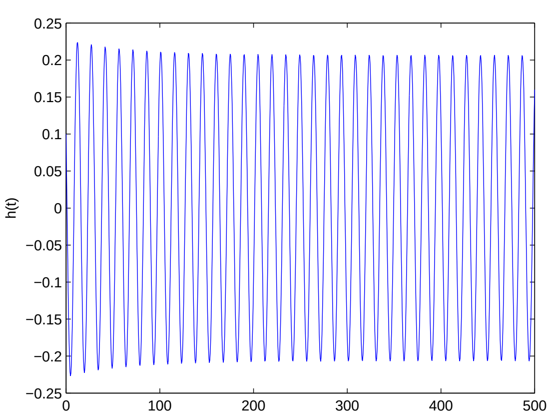
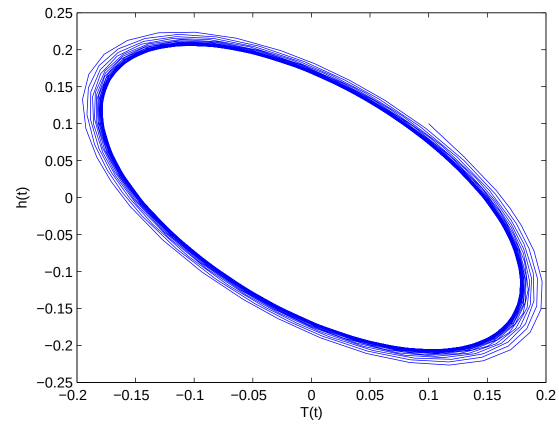


Fig.1. Trajectory portrait and phase portrait of system (44) with $\eta = 0, \delta = 0.2 < \delta_0 \approx 0.37$. The equilibrium $E(0, 0)$ is asymptotically stable. The initial value is (0.1, 0.1).







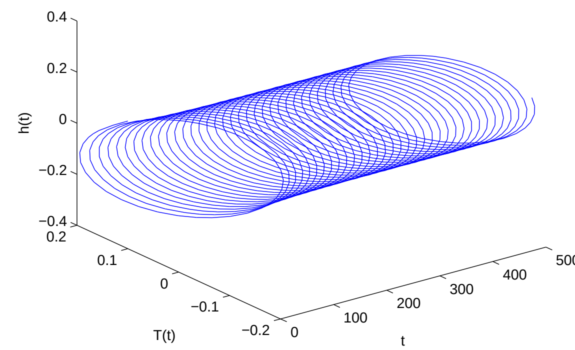
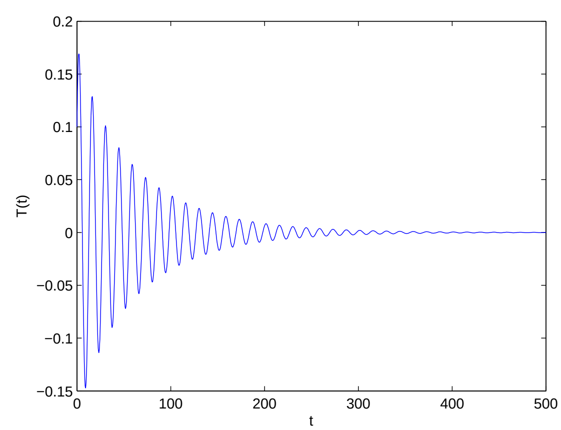


Fig.2. Trajectory portrait and phase portrait of system (44) with $\eta = 0, \delta = 0.48 > \delta_0 \approx 0.37$. Hopf bifurcation occurs from the equilibrium $E(0, 0)$. The initial value is (0.1, 0.1).



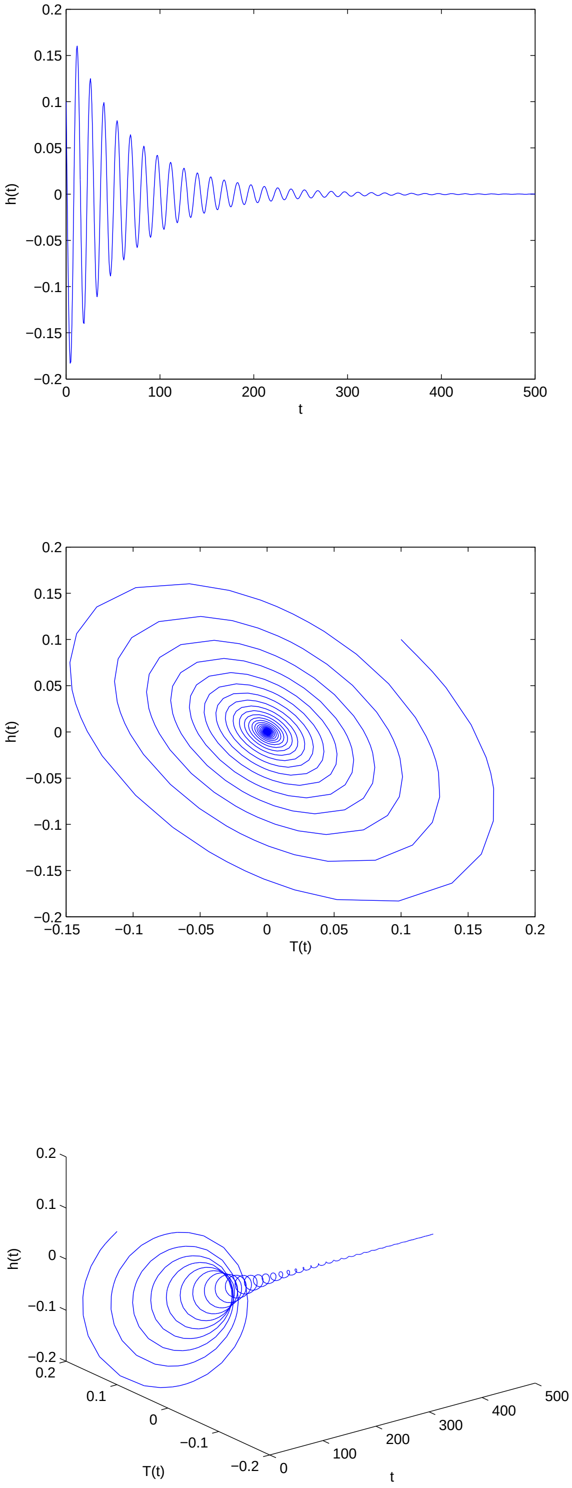
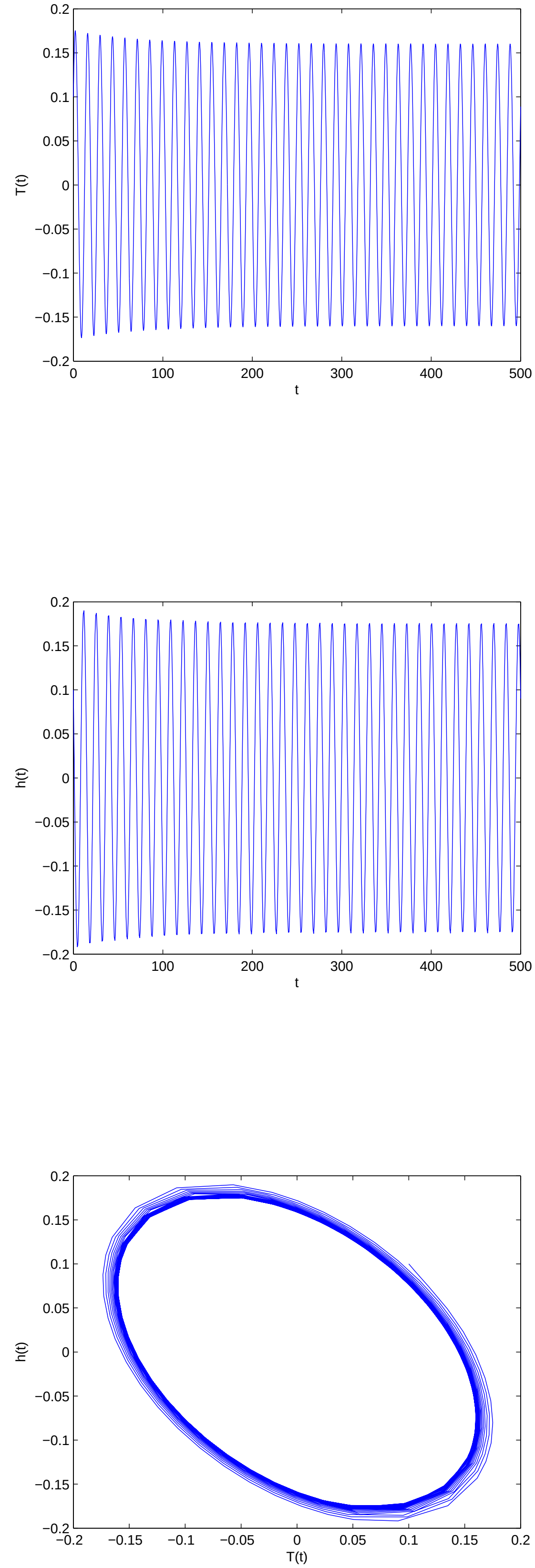


Fig.3. Trajectory portrait and phase portrait of system (44) with $\delta = 0, \eta = 0.2 < \eta_0 \approx 0.25$. The equilibrium $E(0,0)$ is asymptotically stable. The initial value is (0.1, 0.1).



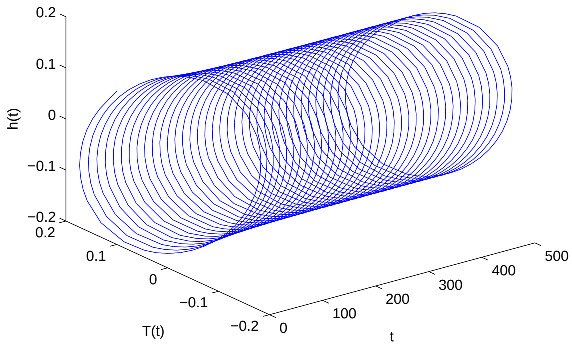
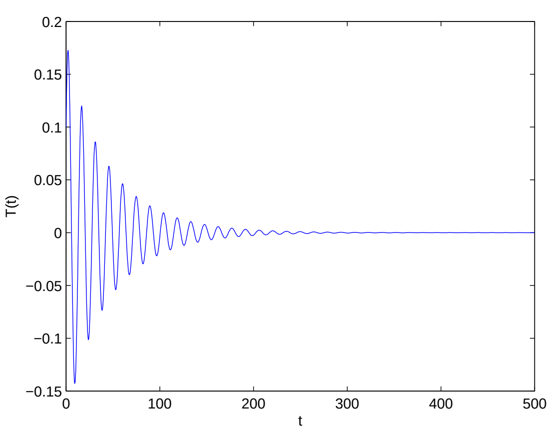
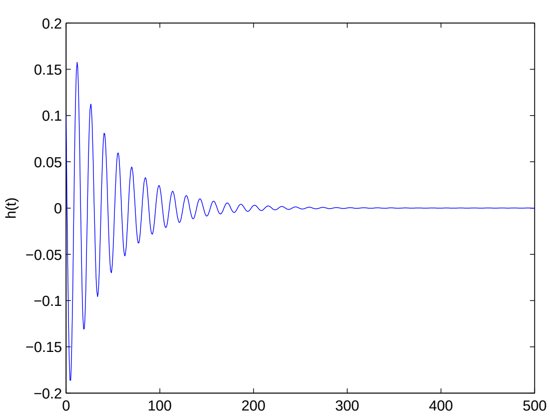
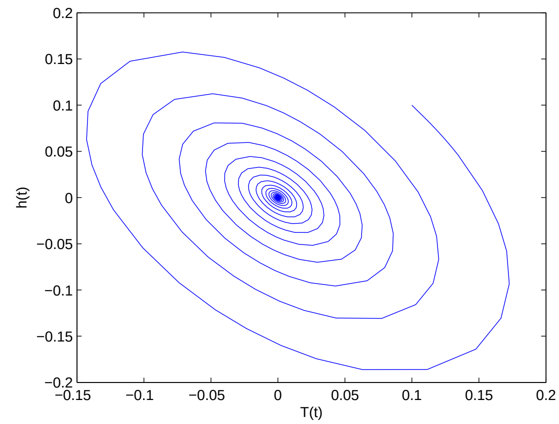


Fig.4. Trajectory portrait and phase portrait of system (44) with $\delta = 0, \eta = 0.3 > \eta_0 \approx 0.25$. Hopf bifurcation occurs from the equilibrium $E(0, 0)$. The initial value is (0.1, 0.1).







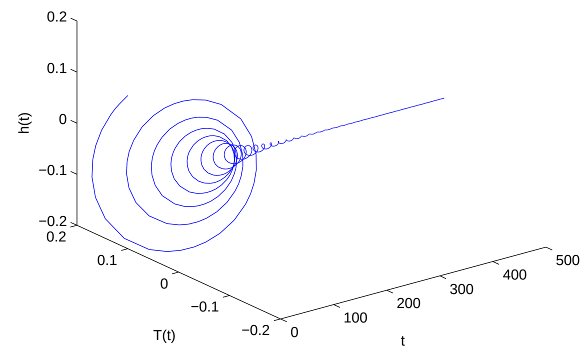
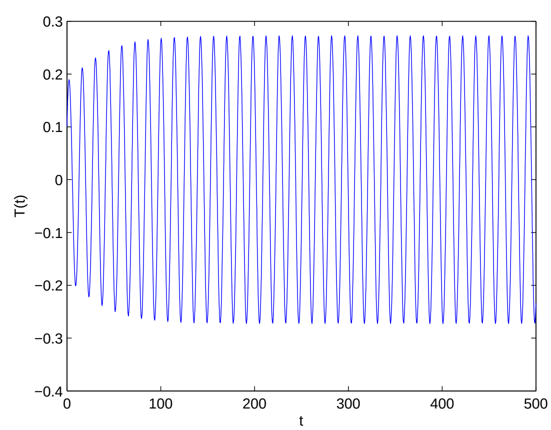
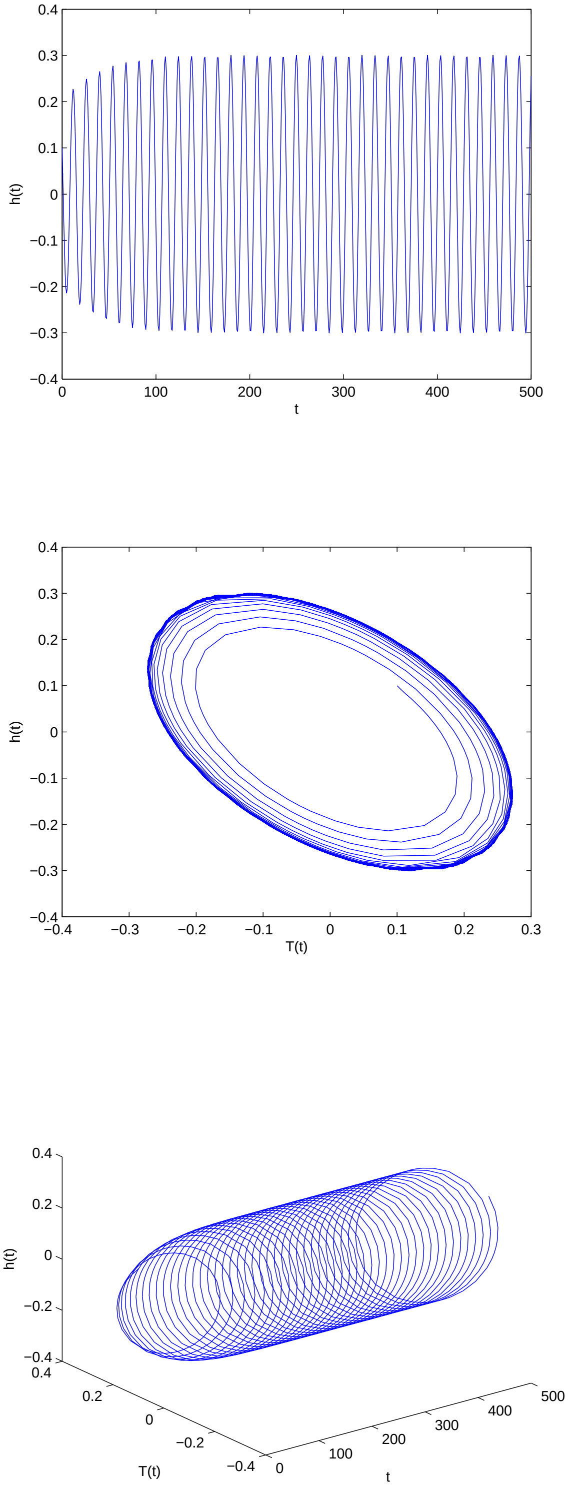


Fig.5. Trajectory portrait and phase portrait of system (44) with $\delta = 0.2, \eta = 0.1 < \eta_0 \approx 0.15$. The equilibrium $E(0, 0)$ is asymptotically stable. The initial value is (0.1, 0.1).





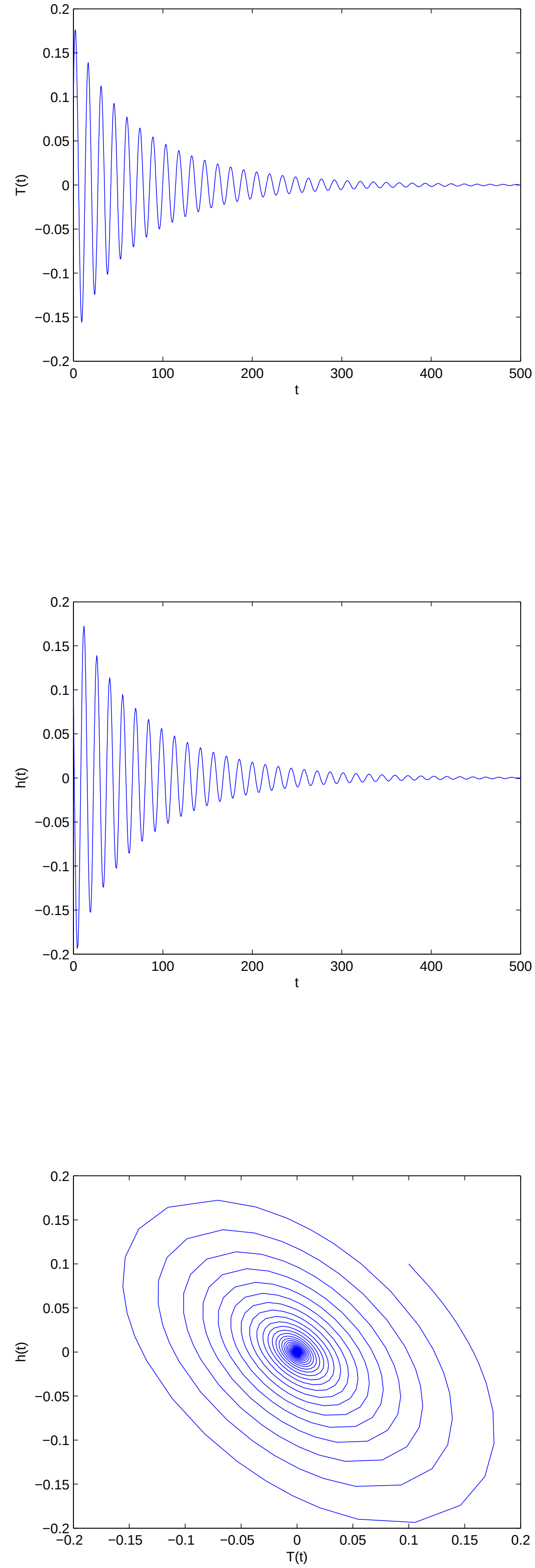


Fig.6. Trajectory portrait and phase portrait of system (44) with $\delta = 0.2, \eta = 0.22 > \eta_0 \approx 0.15$. Hopf bifurcation occurs from the equilibrium $E(0,0)$. The initial value is (0.1, 0.1).

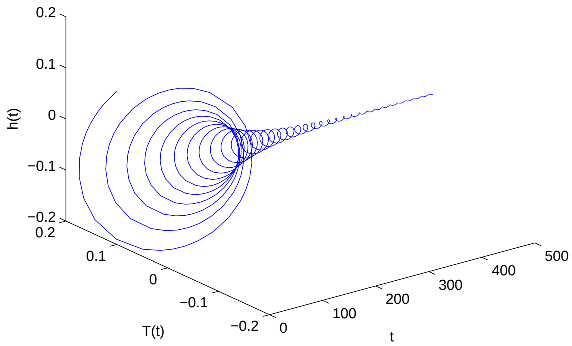
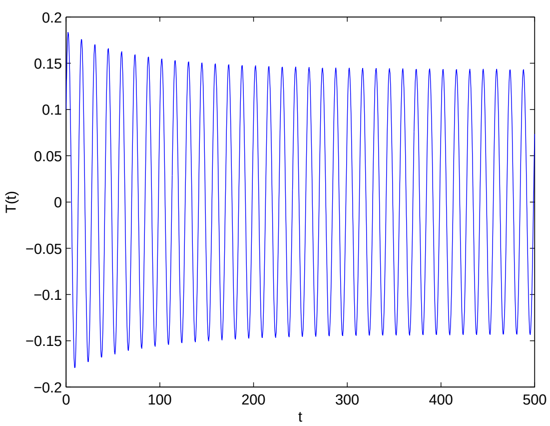
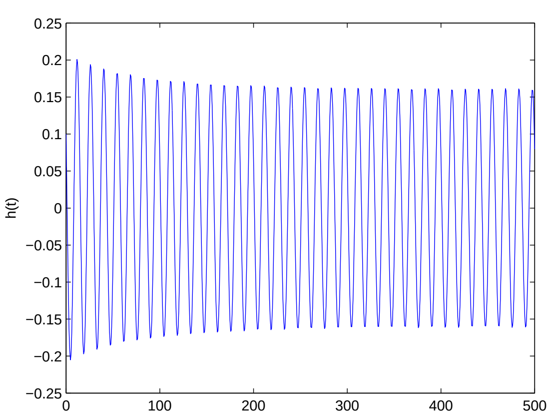
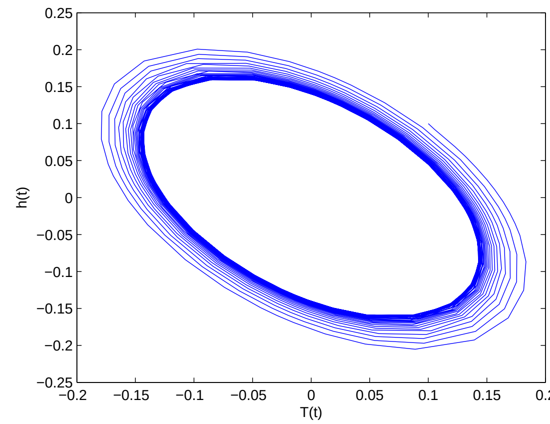


Fig.7. Trajectory portrait and phase portrait of system (44) with $\eta = 0.13, \delta = 0.3 < \delta_0 \approx 0.35$. The equilibrium $E(0,0)$ is asymptotically stable. The initial value is (0.1, 0.1).







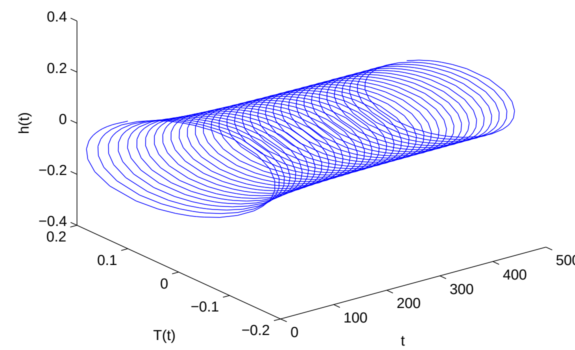
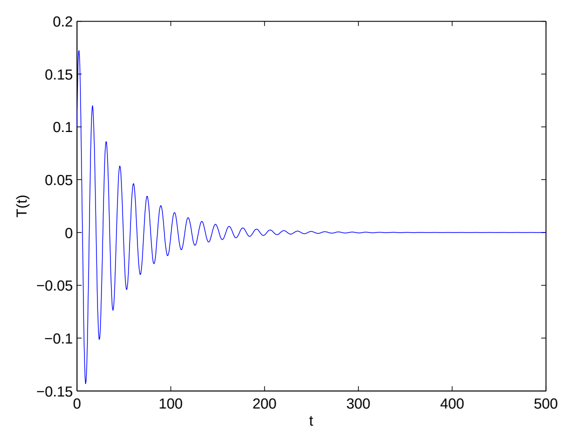
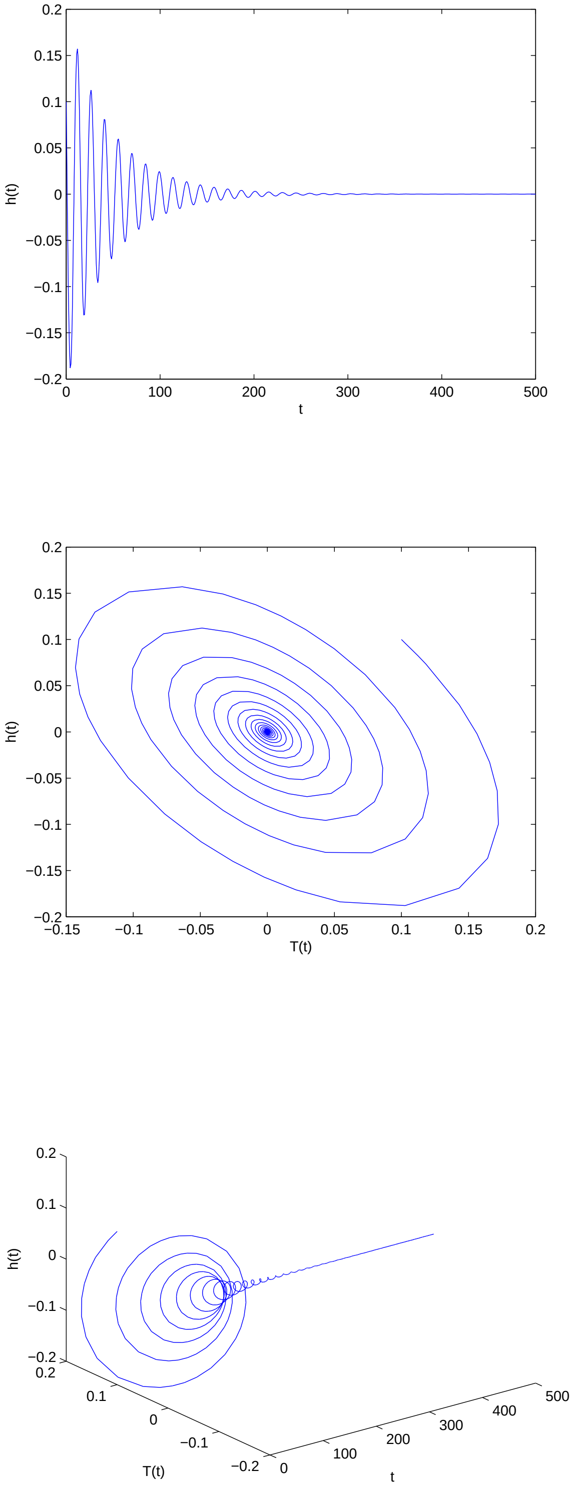


Fig.8. Trajectory portrait and phase portrait of system (44) with $\eta = 0.13, \delta = 0.5 > \delta_0 \approx 0.35$. Hopf bifurcation occurs from the equilibrium $E(0,0)$. The initial value is (0.1, 0.1).





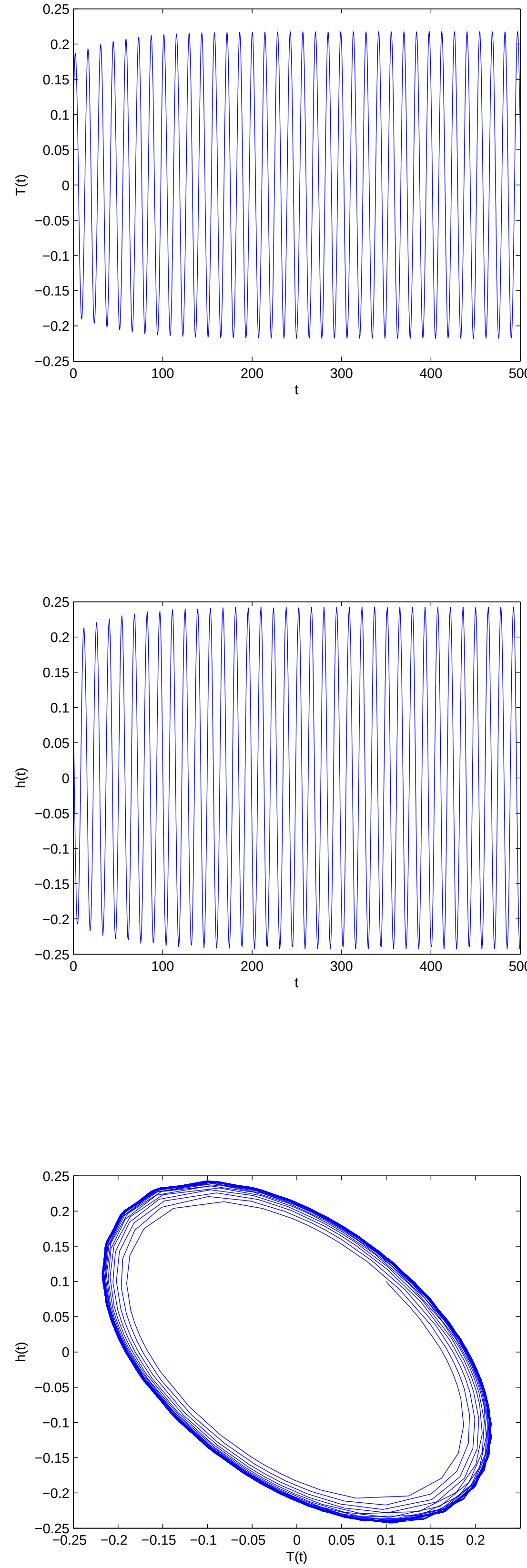


Fig.9. Trajectory portrait and phase portrait of system (44) with $\delta = \eta = 0.1 < \eta_0 \approx 0.15$. The equilibrium $E(0,0)$ is asymptotically stable. The initial value is (0.1, 0.1).

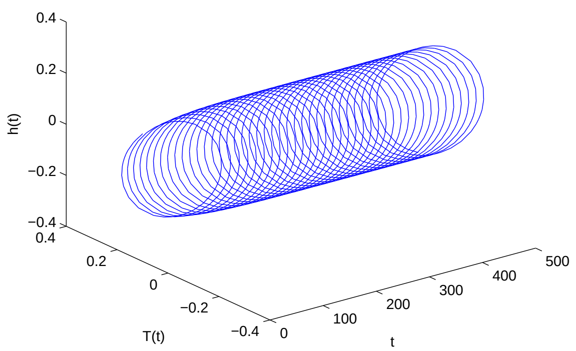


Fig.10. Trajectory portrait and phase portrait of system (44) with $\delta = \eta = 0.2 > \eta_0 \approx 0.15$. Hopf bifurcation occurs from the equilibrium $E(0, 0)$. The initial value is (0.1, 0.1).

4 Conclusions

In this paper, an oscillator model for EI Niño and southern oscillation with two different delays are investigated. By regarding the delays η and δ as bifurcation parameters, we obtain the critical values of two time delays which system undergo Hopf bifurcation. It is shown that when the time delay crosses a certain critical value, the system loses it stability and a family of periodic orbits bifurcate from the equilibrium. Finally, by MATLAB software, some numerical simulations are carried out to illustrate the theoretical results. ENSO is a complex natural phenomenon. Its occurrence will has seriously effect on the climate and ecology all over the world. By investigating what the effect of different time delays have on Hopf bifurcation of this model, we will know the change situation of corresponding physical quantity, then we can makes the forecast on the corresponding situation of the system. Moreover, we can effectively prevent the humanity's disaster from ENSO event.

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