Wave Dispersion in the Linearised Fractional Korteweg – de Vries equation

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Abstract: In this paper we discuss some properties of linear fractional dispersive waves. In particular, we compare the dispersion relations emerging from the D'Alembert equation and from the linearized Korteweg – deVries equation with the corresponding time-fractionalized versions. For this purpose, we evaluate the expressions for the phase velocity and for the group velocity, highlighting the differences not only analytically, but also by means of illuminating plots.

Key–Words: dispersion, waves, phase velocity, group velocity

1 Introduction

Linear dispersive waves are defined as physical phenomena for which the relation that connects the wave number k with the angular frequency ω is non-trivial. This leads to different dependences in the behaviours of the phase velocity v_p and of the group velocity v_g as we vary the wave number.

In general, the relation between ω and k, known as the dispersion relation, takes the form

$$\mathcal{D}(\omega, k) = 0 \tag{1}$$

where $\mathcal{D}$ is a suitable real function of ω and k. Such a relation is, in general, satisfied by certain $\omega,\ k \in \mathbb{C}$.

Let us assume that (1) can be solved explicitly in terms of a real variable (k or ω) by means of complex valued branches:

$$\overline{\omega}_\ell(k) \ \in \ \mathbb{C}, \quad k \in \mathbb{R}, \tag{2}$$

$$\overline{k}_m(\omega) \ \in \ \mathbb{C}, \quad \omega \in \mathbb{R}, \tag{3}$$

where ℓ, m are two positive integers called mode indices. These branches provide the so-called Normal Mode Solutions for our physical system

$$\varphi_\ell(t, x; k) \ = \ \mathrm{Re}\left\{A_\ell(k)\, \exp\left[i(\overline{\omega}_\ell\, t - k\, x)\right]\right\} \ , \tag{4}$$

$$\varphi_m(t, x; \omega) \ = \ \mathrm{Re}\left\{A_\ell(\omega) \exp\left[i(\omega\, t - \overline{k}_m\, x)\right]\right\} \tag{5}$$

For sake of simplicity, in the following we will denote a normal mode simply by $\varphi_\ell(k)$ and $\varphi_m(\omega)$ so dropping the dependence on the space-time coordinates $x,\ t$, respectively.

The normal mode solutions represent a sort of pseudo-monochromatic modes since generally they are not sinusoidal in both space and time.

Now, for sake of brevity, we will omit the mode labels. Then we define, for the to cases (4) and (5) respectively, the phase velocity as

$$v_p(k) := \frac{\mathrm{Re}\, \overline{\omega}(k)}{k} \, , \tag{6}$$

$$v_p(\omega) := \frac{\omega}{\mathrm{Re}\, \overline{k}(\omega)} \, . \tag{7}$$

In this paper, we will consider the relation (6) for the phase velocity, depending on k.

Furthermore, restricting our discussion on the case of real k, we define the corresponding group velocity as

$$v_g(k) := \frac{\mathrm{Re}\, \overline{\omega}(k)}{\partial k} \, . \tag{8}$$

Despite the fact that the theory of linear dispersive waves is a very well established and developed field of mathematical physics, the effects of fractional extensions of such linear systems on the dispersion of waves can still represent an interesting, and utterly non-trivial, research topic. The aim of this paper is to present some examples of dispersion relations related to fractional properties of mechanical systems.

Particularly, in Section 2 we introduce the problem of dispersion for the simple case of the D'Alembert wave equation. Then, in Section 3 we deal with waves satisfying the linearised Korteweg – de Vries (KdV) equation.

2 The wave equation

We first introduce the problem of dispersion in fractional viscoelasticity showing the case of the wave equation.

The well-known wave equation usually found in literature is

$$\frac{\partial^2 u(x,t)}{\partial t^2} - c_0^2 \frac{\partial^2 u(x,t)}{\partial x^2} = 0\,, \tag{9}$$

where the velocity of the waves c_0 is set to one in the following for sake of simplicity.

This equation leads to a dispersion equation

$$\omega = |k|\,, \tag{10}$$

from which one can easily infer $v_p = v_g$. Therefore, this is a clear example of a non-dispersive scenario.

We can appreciate a different behavior replacing the time derivative with the fractional derivative of order α. Applying this change, our wave equation, will be in the form

$$D_t^\alpha u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} = 0\,, \tag{11}$$

where D_t^α is the well known Caputo derivative (see [4]). Here, the dispersion relation becomes

$$\overline{\omega}(k) = i^{-1+3/\alpha} k^{2/\alpha} \tag{12}$$

Thus, the frequency presents both a real and an imaginary part. Indeed, for $k > 0$,

$$\begin{aligned} \text{Re}\,\overline{\omega}(k) &= \cos\left(\left(-\tfrac{1}{2} + \tfrac{3}{2\alpha}\right)\pi\right) k^{2/\alpha} = \\ &= \cos\left(\left(1 + \tfrac{1}{\alpha}\right)\tfrac{3}{2}\pi\right) k^{2/\alpha}\,, \end{aligned} \tag{13}$$

$$\begin{aligned} \text{Im}\,\overline{\omega}(k) &= \sin\left(\left(-\tfrac{1}{2} + \tfrac{3}{2\alpha}\right)\pi\right) k^{2/\alpha} = \\ &= \sin\left(\left(1 + \tfrac{1}{\alpha}\right)\tfrac{3}{2}\pi\right) k^{2/\alpha}\,. \end{aligned} \tag{14}$$

At this point, we can easily evaluate the velocities, respectively the *complex* phase velocity

$$\overline{v_p}(k) = i^{-1+3/\alpha} k^{(2-\alpha)/\alpha} \tag{15}$$

and the group velocity

$$\overline{v_g}(k) = i^{-1+3/\alpha} \frac{2}{\alpha} k^{(2-\alpha)/\alpha}\,. \tag{16}$$

It is then important to remark that one can immediately infer that a value of $\alpha \neq 2$ introduces dispersion effects.

2.1 Numerical Results

Comparing the plots of certain relevant quantities can therefore be useful to understand the phenomenon. Firstly, it could be helpful to separate the real value and the imaginary value of the expressions (15) and (16). Indeed, one immediately finds that

$$v_p(k) = \text{Re}\,\overline{v_p}(k) = \cos\left(\left(1+\frac{1}{\alpha}\right)\frac{3}{2}\pi\right) k^{(2-\alpha)/\alpha}\,, \tag{17}$$

$$\text{Im}\,\overline{v_p}(k) = \sin\left(\left(1+\frac{1}{\alpha}\right)\frac{3}{2}\pi\right) k^{(2-\alpha)/\alpha}\,, \tag{18}$$

and

$$v_g = \text{Re}\,\overline{v_g}(k) = \frac{2}{\alpha}\cos\left(\left(1+\frac{1}{\alpha}\right)\frac{3}{2}\pi\right) k^{(2-\alpha)/\alpha}\,, \tag{19}$$

$$\text{Im}\,\overline{v_g}(k) = \frac{2}{\alpha}\sin\left(\left(1+\frac{1}{\alpha}\right)\frac{3}{2}\pi\right) k^{(2-\alpha)/\alpha}\,. \tag{20}$$

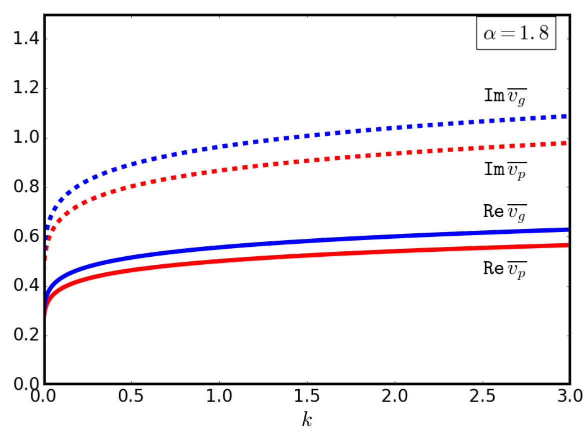

Figure 1: Comparison between phase velocity and group velocity, for the wave equation with fractional derivative of order $\alpha = 1.8$. The straight lines represent real values, the dashed lines represent imaginary values.

From Figure 1 and Figure 2 we can qualitatively estimate the differences respectevely for $\alpha = 1.8$ and $\alpha = 1.5$. Interestingly, one finds that the real part vanishes for certain values of α.

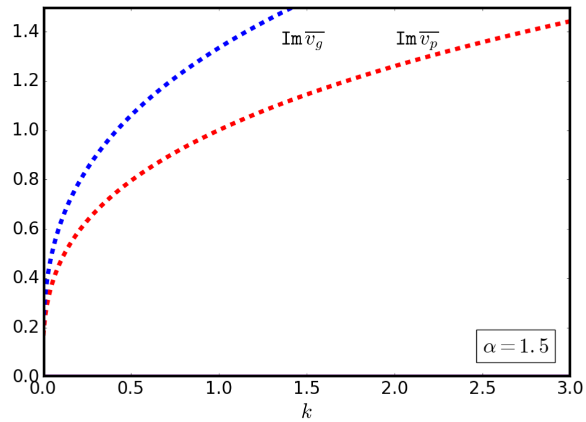

Figure 2: Comparison between phase velocity and group velocity, for the wave equation with fractional derivative of order $\alpha = 1.5$. For $\alpha = 1.5$ the two velocities are purely imaginary functions of the wave number.

3 The Korteweg – de Vries equation

Now, we discuss a similar situation for the KdV equation. It is a non-linear equation with several applications, such as in the study of waves on shallow water surfaces (see [1]) or solitons descriptions (see [7]). The most general form of KdV equation is

$$\frac{\partial u(x,t)}{\partial t} + c_0 \frac{\partial u(x,t)}{\partial x} + \lambda u(x,t) \frac{\partial u(x,t)}{\partial x} + \\ + \mu \frac{\partial^3 u(x,t)}{\partial x^3} = 0 \qquad (21)$$

where λ, μ and c_0 are real numbers, as argued in [2].

However, in this paper, we will deal the linearised KdV equation, that we recover setting $\lambda = 0$, namely

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 u(x,t)}{\partial x^3} = 0 \,, \qquad (22)$$

fixing also $c_0 = \mu = 1$ for sake of simplicity.

We can now focus our attention on the waves described by the related dispersion relation

$$\omega(k) = k - k^3 \,. \qquad (23)$$

Thanks to the latter equation it is not difficult to compute the phase velocity

$$v_p(k) = 1 - k^2 \qquad (24)$$

and the group velocity

$$v_g(k) = 1 - 3k^2 \,. \qquad (25)$$

It is worth remarking that, in this case, we have dispersive effects even for the unmodified wave equation.

Now, following a procedure akin to the one discussed in the previous section, we get:

$$D_t^\alpha u(x,t) + \frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 u(x,t)}{\partial x^3} = 0 \,. \qquad (26)$$

The resulting dispersion relation reads

$$\overline{\omega}(k) = i^{(1-\alpha)/(\alpha)} \left(k - k^3\right)^{1/\alpha} \,, \qquad (27)$$

and, again, the angular frequency can virtually be a complex number with non-vanishing imaginary part.

Once more, from this expression of $\omega(k)$ we get,

$$\overline{v_p}(k) = i^{(1-\alpha)/(\alpha)} \left(k^{1-\alpha} - k^{3-\alpha}\right)^{1/\alpha} \qquad (28)$$

and for the group velocity

$$\overline{v_g}(k) = i^{(1-\alpha)/(\alpha)} \left(k - k^3\right)^{1/\alpha - 1} \left(1 - 3k^2\right) \,, \qquad (29)$$

which can be split as follows

$$v_p(k) = \mathrm{Re}\, \overline{v_p}(k) = \cos\left(\left(\frac{1}{\alpha} - 1\right)\frac{\pi}{2}\right) \times \\ \times \left(k^{1-\alpha} - k^{3-\alpha}\right)^{1/\alpha} \,, \qquad (30)$$

$$\mathrm{Im}\, \overline{v_p}(k) = \sin\left(\left(\frac{1}{\alpha} - 1\right)\frac{\pi}{2}\right) \times \\ \times \left(k^{1-\alpha} - k^{3-\alpha}\right)^{1/\alpha} \,, \qquad (31)$$

and

$$v_g = \mathrm{Re}\, \overline{v_g}(k) = \frac{2}{\alpha} \cos\left(\left(\frac{1}{\alpha} - 1\right)\frac{\pi}{2}\right) \times \\ \times \left(k - k^3\right)^{1/\alpha - 1} \left(1 - 3k^2\right) \,, \qquad (32)$$

$$\mathrm{Im}\, \overline{v_g}(k) = \frac{2}{\alpha} \sin\left(\left(\frac{1}{\alpha} - 1\right)\frac{\pi}{2}\right) \times \\ \times \left(k - k^3\right)^{1/\alpha - 1} \left(1 - 3k^2\right) \,. \qquad (33)$$

3.1 Numerical Results

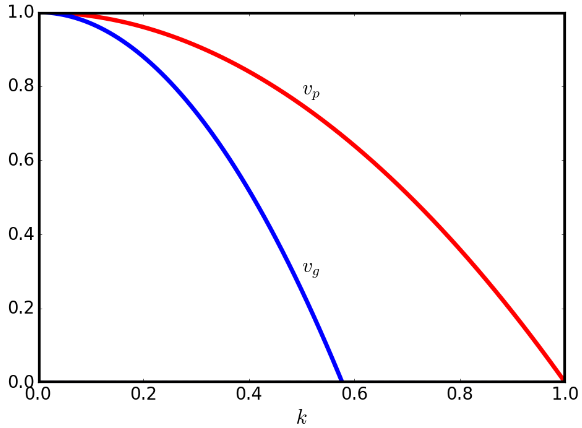

Figure 3: Comparison between phase velocity and group velocity, for the linearised KdV equation with ordinary derivative.

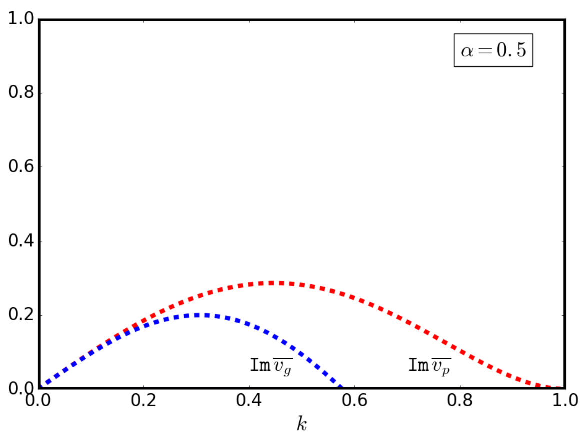

Figure 4: Comparison between phase velocity and group velocity, for the linearised KdV equation with frational derivative of order $1/2$.

We can conclude that for $\alpha = 1$, so in the ordinary case, phase velocity and group velocity are real-valued, as well shown by the Figure 3, and totally imaginary-valued for other values (e.g. $\alpha = 1/4$, $\alpha = 1/2$), as can be stated looking at Figure 4. For real values of α, we find a mixed behavior, as we can see from Figure 5.

Furthermore, from this latter plot, it is remarkable to note that for $k \to 0$, then $\mathrm{Re}\,\overline{v_p}(k) = \mathrm{Re}\,v_g(k)$, so we have not dispersion for small values of the wave number k.

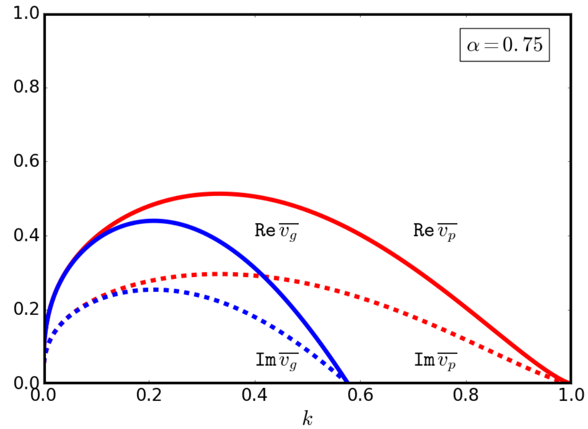

Figure 5: Comparison between phase velocity and group velocity, for the linearised KdV equation with fractional derivative of order $\alpha = 3/4$.

4 Conclusion

In conclusion, it seems that the procedure of fractionalizing a linear wave equation leads to major modifications of the corresponding dispersion relation.

This analysis can surely be extended further by considering fractional derivative with respect to the space spatial coordinate, however this discussion is left for future investigations.

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