A Novel Hybrid Fuzzy Weighted Average for MCDM with Interval Triangular Type-2 Fuzzy Sets

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Abstract: - Weight plays an important role in multi-criteria decision making (MCDM), as it would have a deep effect on the evaluation results. Fuzzy weighted average is one of the popular weight in MCDM method. However, the inherent uncertainty of this method can result in weighting errors. Therefore, this paper presents an interval triangular type-2 fuzzy set (ITT2FS) to capture uncertainty in multi-criteria decision making (MCDM) problems. In this paper, a new weighted average is developed using the concept of interval triangular type-2 fuzzy sets. Based on the concept of the relative closeness coefficients, we construct a simpler interval triangular type-2 fractional programming in weighted average to calculate the closeness coefficients, which can be employed to generate the ranking order of alternatives. The proposed method is illustrated with three numerical examples. As a result, we found that the proposed method is practical for solving the type-2 fuzzy TOPSIS problems. Besides, it seems that the proposed method is flexible, easy to use and low computational volume. Moreover, it has acceptable accurate.

Key-Words: - Weighted average; Multiple attribute decision making; Interval type-2 fuzzy sets; Interval triangular type-2 fuzzy sets

1 Introduction

One of the important parts in multi-criteria decision making (MCDM) method is the weighting part. In MCDM, the weights of the criteria are crucial for measuring the importance of the criteria [1]. The determination of weights is the key point in comprehensive evaluation. The propriety of weights subsets will influence the results of the comprehensive evaluation [2].
Therefore, how to determine the weights in MCDM will be an interesting and important research topic. At present, many methods have been proposed to determine the weights of MCDM, for example, Singh and Lyes Benyoucef presented a methodology for solving the sealed bid, multi-attribute reverse auction problem of e-sourcing using an entropy method [3]. Yue presented a new approach for determining weights of decision makers (DMs) in group decision environment based on an extended TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) method [4]. Besides, Yue developed a method for determining weights of decision makers under group decision environment, in which the each individual decision information is expressed by a matrix in interval numbers [5]. Moreover, Chen et al. proposed a hybrid approach integrating OWA (Ordered Weighted Averaging) aggregation into TOPSIS (technique for order performance by similarity to ideal solution) to tackle multiple criteria decision analysis (MCDA) problems [6]. Then, Chen proposed two objectives of maximal

closeness coefficient and minimal deviation values, an integrated programming model to compute optimal weights for the criteria and corresponding closeness coefficients for alternative rankings [7]. Next, Liu presented an Extended TOPSIS Method deals with multiple attribute group decision making problems in which the attribute values and weights took the form of the generalized interval-valued trapezoidal fuzzy numbers (GIVTFN) [8]. Moreover, Mehdi et al. applied a new fuzzy distance formula to compute distance between each alternative and positive as well as negative ideal solution and used triangular fuzzy numbers (TFN) among decision making process to evaluate the weighted different alternatives versus various criteria and a fuzzy group weight is made by different experts [9]. Furthermore, Wang et al. proposed a new fuzzy TOPSIS for evaluating outsourcing software companies both using subjective and objective weights [10]. Whereas, a fuzzy multiple criteria group decision making (FMCGDM) problem with the Technique for Order Performance by Similarity to Ideal Solution (TOPSIS) based on the concept of positive and negative ideal solution and triangular fuzzy numbers (TFN) among decision making process to evaluate the weighted different alternatives versus various criteria and a fuzzy group weight is made by different experts were presented in [11].

In line with other weighted concepts, fuzzy weighted average (FWA) has also become one of the important empirically based studies on weighted problems worldwide. For example, Park and Kim presented characteristic or attribute of a fuzzy weighted additive rule (FWAR) [12]. Chen et al. proposed a fuzzy weighted average method in the fuzzy expected value operator [13]. On the other hand, Hung et al. presented a fuzzy integrated approach to assess the performance of design concepts. And those criteria rating, relative weights and performance levels are captured by fuzzy numbers, and the overall performance of each alternative is calculated through an enhanced fuzzy weighted average (FWA) approach [2]. Moreover, Mokhtarian proposed a new fuzzy weighted average (FWA) method based on left and right scores for fuzzy MCDM problems [14]. Lin and Roopaei adjusted weights, centers and widths of fuzzy neural network (FNN), based on the adaptive interval type-2 fuzzy logic [15]. Furthermore, Liu and Mendel combined the fuzzy weighted average (FWA) and the generalized centroid of an interval type-2 fuzzy set to get a new α -cut algorithm for solving the FWA problem [16]. Wu and Mendel focused on the linguistic weighted average (LWA), where the weights were always words modeled as interval type-2 fuzzy sets (IT2 FSs), and the attributes may also (but do not have to) be words modeled as IT2 FSs [17].

However, the inherent uncertainty and subjectivity of these methods can result in weighting errors and difficulties in the criteria weight selection process. As a result, the subjectivity (i.e., the fuzzy numbers) of the criteria weight selection process varies among the experts. However, no specific literature was found to develop a new weight with linear programming approach in type-2 fuzzy sets concepts. Besides, there exist little investigation on MCDM problems with both ratings of alternatives on attributes and weights being expressed with interval triangular type-2 fuzzy sets (ITT2FS). Therefore, in this paper, a weighted distance between ITT2FS is defined using weights of ITT2FS. Then, based on the concept of the relative closeness coefficients, a pair of nonlinear fractional programming models is constructed to calculate the relative closeness coefficients of alternatives with respect to the interval triangular type-2 (ITT2) positive ideal solution, which can be used to generate ranking order of the alternatives. The nonlinear fractional programming models can be transformed into two auxiliary linear programming models, respectively.

Thus, the rest of this paper is organized as follows. In Section 2, we briefly introduced the background related to this paper which include the basic concepts on interval triangular type-2 fuzzy sets and weighted average. In Section 3, we propose a modified weighted average for MCDM with interval triangular type-2 fuzzy sets. In Section 4, we apply the proposed method for solving three numerical examples and show the computations and results in

Section 5. Then, the paper is concluded in Section 6.

2 Backgrounds

In order to comprehend the method, some definitions and properties of triangular interval type-2 fuzzy sets (in Section 2.1) and weighted average (in Section 2.2) are presented.

2.1 Interval Triangular Type-2 Fuzzy Sets

We do some modification on the existed type-2 fuzzy sets by building up an interval triangular type-2 fuzzy sets (ITT2FS). This ITT2FS is described from Definition 2 till 6. This ITT2FS is used for handling fuzzy multi-criteria decision making problems, where the reference points and the heights of the upper and lower membership functions of ITT2FS are used to characterize interval type-2 fuzzy sets (as shown in Definition 1) without losing the novelty of it. Figure 1 describes the upper and lower triangular membership function of the ITT2FS, $\tilde{\tilde{a}}$.

Definition 1: (Mendel et al. [18])
Let $\tilde{\tilde{a}}$ be a type-2 fuzzy set in the universe of discourse X represented by the type-2 membership function $\mu_{\tilde{\tilde{a}}}$. If all $\mu_{\tilde{\tilde{a}}} = 1$, then a is called an interval type-2 fuzzy sets. An interval type-2 fuzzy set $\tilde{\tilde{a}}$ can be regarded as a special case of a type-2 fuzzy set, represented as follows:

$$\tilde{\tilde{a}} = \int_{x \in X} \int_{u \in J_x} 1/(x, u), \tag{2.1}$$

where $J_x \subseteq [0,1]$.

Definition 2:
The triangular interval type-2 fuzzy number is defined as in the following:
Let $\tilde{\tilde{a}}_i = (\tilde{a}_i^l, \tilde{a}_i^u) = ([a_{i1}^l, a_{i2}^l], [a_{i1}^u, a_{i2}^u]; \tilde{s}_i^l, \tilde{s}_i^u)$, where $\tilde{a}_i^l$ and $\tilde{a}_i^u$ are type-1 fuzzy sets, $a_{i1}^l, a_{i2}^l, a_{i1}^u, a_{i2}^u$ are the reference points of the interval type-2 fuzzy set $\tilde{\tilde{a}}_i$, $\tilde{s}_i^l$ denotes the upper triangular membership value function and $\tilde{s}_i^u$ denotes the lower triangular membership value function, where $\tilde{s}_i^l \in [0,1]$ and $\tilde{s}_i^u \in [0,1]$, and $1 \le i \le n$.

Definition 3:
Let $\tilde{\tilde{a}}_1 = (\tilde{a}_1^l, \tilde{a}_1^u) = ([a_{11}^l, a_{12}^l], [a_{11}^u, a_{12}^u]; \tilde{s}_1^l, \tilde{s}_1^u)$
and $\tilde{\tilde{b}}_2 = (\tilde{b}_2^l, \tilde{b}_2^u) = ([b_{21}^l, b_{22}^l], [b_{21}^u, b_{22}^u]; \tilde{s}_2^l, \tilde{s}_2^u)$
be two triangular interval type-2 fuzzy numbers. Thus, the addition of $\tilde{\tilde{a}}_1$ and $\tilde{\tilde{b}}_2$ is defined as:

$$\tilde{\tilde{a}}_1 + \tilde{\tilde{b}}_2 = (\tilde{a}_1^l, \tilde{a}_1^u) + (\tilde{b}_2^l, \tilde{b}_2^u) = \begin{bmatrix} \left(a_{11}^l + b_{21}^l, a_{12}^l + b_{22}^l; \min\left(\tilde{s}_1^l, \tilde{s}_2^l\right)\right), \\ \left(a_{11}^u + b_{21}^{ul}, a_{12}^u + b_{22}^u; \min\left(\tilde{s}_1^u, \tilde{s}_2^u\right)\right) \end{bmatrix} \tag{2.2}$$

Definition 4:
Let $\tilde{\tilde{a}}_1 = (\tilde{a}_1^l, \tilde{a}_1^u) = ([a_{11}^l, a_{12}^l], [a_{11}^u, a_{12}^u]; \tilde{s}_1^l, \tilde{s}_1^u)$
and $\tilde{\tilde{b}}_2 = (\tilde{b}_2^l, \tilde{b}_2^u) = ([b_{21}^l, b_{22}^l], [b_{21}^u, b_{22}^u]; \tilde{s}_2^l, \tilde{s}_2^u)$
be two triangular interval type-2 fuzzy numbers. Thus, the subtraction of $\tilde{\tilde{a}}_1$ and $\tilde{\tilde{b}}_2$ is defined as:

$$\tilde{\tilde{a}}_1 - \tilde{\tilde{b}}_2 = (\tilde{a}_1^l, \tilde{a}_1^u) - (\tilde{b}_2^l, \tilde{b}_2^u) = \begin{bmatrix} \left(a_{11}^l - b_{21}^l, a_{12}^l - b_{22}^l; \min\left(\tilde{s}_1^l, \tilde{s}_2^l\right)\right), \\ \left(a_{11}^u - b_{21}^{ul}, a_{12}^u - b_{22}^u; \min\left(\tilde{s}_1^u, \tilde{s}_2^u\right)\right) \end{bmatrix} \tag{2.3}$$

Definition 5:
Let $\tilde{\tilde{a}}_1 = (\tilde{a}_1^l, \tilde{a}_1^u) = ([a_{11}^l, a_{12}^l], [a_{11}^u, a_{12}^u]; \tilde{s}_1^l, \tilde{s}_1^u)$
and $\tilde{\tilde{b}}_2 = (\tilde{b}_2^l, \tilde{b}_2^u) = ([b_{21}^l, b_{22}^l], [b_{21}^u, b_{22}^u]; \tilde{s}_2^l, \tilde{s}_2^u)$
$\tilde{\tilde{a}} = \int_{x \in X} \int_{u \in J_x} 1/(x, u),$
be two triangular interval type-2 fuzzy numbers. Thus, the multiplication of $\tilde{\tilde{a}}_1$ and $\tilde{\tilde{b}}_2$ is defined as:

$$\tilde{\tilde{a}}_1 \times \tilde{\tilde{b}}_2 = (\tilde{a}_1^l, \tilde{a}_1^u) \times (\tilde{b}_2^l, \tilde{b}_2^u) = \begin{bmatrix} \left(a_{11}^l \times b_{21}^l, a_{12}^l \times b_{22}^l; \min\left(\tilde{s}_1^l, \tilde{s}_2^l\right)\right), \\ \left(a_{11}^u \times b_{21}^{ul}, a_{12}^u \times b_{22}^u; \min\left(\tilde{s}_1^u, \tilde{s}_2^u\right)\right) \end{bmatrix} \tag{2.4}$$

Definition 6:
Let $\tilde{\tilde{a}}_1 = (\tilde{a}_1^l, \tilde{a}_1^u) = ([a_{11}^l, a_{12}^l], [a_{11}^u, a_{12}^l]; \tilde{s}_1^l, \tilde{s}_1^u)$
and $\tilde{\tilde{b}}_2 = (\tilde{b}_2^l, \tilde{b}_2^u) = ([b_{21}^l, b_{22}^l], [b_{21}^u, b_{22}^u]; \tilde{s}_2^l, \tilde{s}_2^u)$

be two triangular interval type-2 fuzzy numbers. Thus, the division of $\tilde{\tilde{a}}_1$ and $\tilde{\tilde{b}}_2$ is defined as:

$$\tilde{\tilde{a}}_1 \div \tilde{\tilde{b}}_2 = \left(\tilde{a}_1^{\,l}, \tilde{a}_1^{\,u}\right) \div \left(\tilde{b}_2^{\,l}, \tilde{b}_2^{\,u}\right) = \begin{bmatrix} \left(a_{11}^{l} \div b_{21}^{l}, a_{12}^{l} \div b_{22}^{l}; \min\left(\tilde{s}_1^{\,l}, \tilde{s}_2^{\,l}\right)\right), \\ \left(a_{11}^{u} \div b_{21}^{ul}, a_{12}^{u} \div b_{22}^{u}; \min\left(\tilde{s}_1^{\,u}, \tilde{s}_2^{\,u}\right)\right) \end{bmatrix} \tag{2.5}$$

2.2. Weighted Average

Definition 1: (Guh et al. [19])

The weighted average is frequently expressed in fuzzy numbers as follows:

$$y = f\left(x_1, x_2, \dots, x_n, w_1, w_2, \dots, w_n\right) = \frac{w_1 x_1 + w_2 x_2 + \dots + w_n x_n}{w_1 + w_2 + \dots + w_n} \tag{2.6}$$

Where $x_1, x_2, \dots, x_n$ are fuzzy numbers in fuzzy sets $A_1, A_2, \dots, A_n$; $w_1, w_2, \dots, w_n$ are fuzzy weights in fuzzy sets $W_1, W_2, \dots, W_n$, and y is the dependent or output fuzzy variable in fuzzy set B.

Definition 2: (Charness and Cooper [20]; Charness and Cooper [21])

The minimum and maximum for the fuzzy weighted average for each given α_j can be obtained by solving the following two fractional programming problems:

$$\begin{aligned} &\min \quad f_L = \frac{w_1 a_1 + w_2 a_2 + \dots + w_n a_n}{w_1 + w_2 + \dots + w_n} \\ &\text{s.t. } c_i \le w_i \le d_i, \quad i = 1,2,\dots,n, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\max \quad f_U = \frac{w_1 b_1 + w_2 b_2 + \dots + w_n b_n}{w_1 + w_2 + \dots + w_n} \\ &\text{s.t. } c_i \le w_i \le d_i, \quad i = 1,2,\dots,n, \end{aligned} \tag{2.8}$$

where c_i and d_i are the two end points of the interval w_i for a given α_j level cut.

The Charnes and Cooper's linear transformation is summarized in the following. Consider the following simple fractional programming problem:

$$\begin{aligned} &\min \quad \frac{px}{qx} \\ &\text{s.t. } Ax \le b, \\ &\qquad x \ge 0, \end{aligned} \tag{2.9}$$

Where p and q are two n-dimensional constant vectors, x is the n-dimensional variable, A is an $m \times n$ matrix, and b is an m-dimensional constant vector.

To transform the above fractional programming problem into a linear problem, let

$$z = \frac{1}{qx} \text{ and } zx = y, \tag{2.10}$$

Where we assume that $qx \ne 0$. Multiplying both the objective function and the constraints by z and using the definitions given in Equation 2.10, we obtain

$$\begin{aligned} &\min \quad py \\ &\text{s.t. } Ay \le bz, \\ &\qquad qy = 1, \\ &\qquad y \ge 0, \\ &\qquad z \ge 0, \end{aligned} \tag{2.11}$$

which is a linear programming problem.

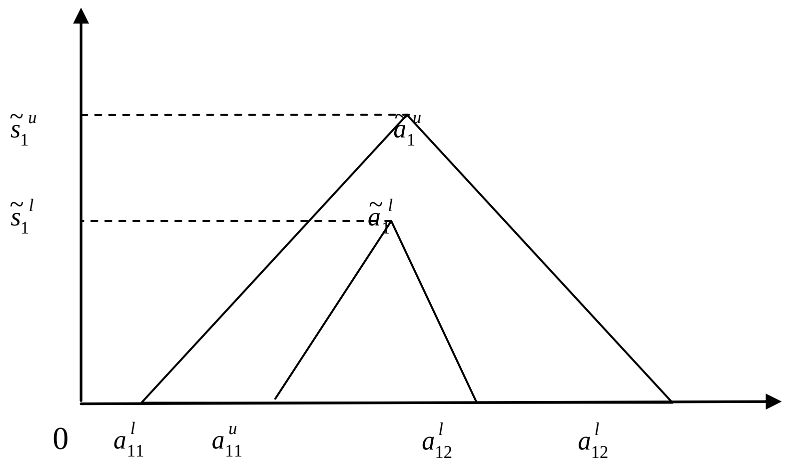

Fig. 1 The upper and lower triangular membership function of the interval triangular type-2 fuzzy sets $\tilde{\tilde{a}}$

3 The Proposed Technique for MCDM

This section divided into two small sub-sections which are sub-section 3.1 and 3.2. Sub-section 3.1 develops a modified method for separation measure of alternatives from positive-ideal solutions and negative-ideal solutions. Modifications are made to accommodate the objective of the research and also to simplify the computational procedures without losing the novelty of the existed method. The comparative of the separation measure of alternatives is important to be considered to enhance the better performance of MCDM evaluation. Thus, this modified separation measure of alternatives is further applied into an interval triangular type-2 fuzzy TOPSIS (ITT2FT). Details are described in sub-section 3.2. Fig. 2 illustrates a conceptual framework of the proposed method.

3.1. An Interval Triangular Type-2 Fuzzy Weighted Average (ITT2FWA)

This section presents a modified version of the separation measure for each alternative from the positive ideal solution I^* and negative ideal solution I^- for relative closeness coefficient by using the concept of fuzzy weighted average. Here, the existed fuzzy weighted average has been transformed into a modified fuzzy weighted average based on fractional programming models under the concept of interval triangular type-2 fuzzy sets (ITT2FS). Thus, motivated from the idea of Li [22], the modified weighted average separation measure between positive and negative ideal solution is defined as follows:

The weighted distance between an alternative of $x_i \in X$ and x^+ is described as follows:

$$d\left(x_i, x^+\right) = \sum_{j=1}^{n}\left[w_j\left(1-\mu_{ij}\right)+\rho_j v_{ij}\right] \qquad (3.1)$$

Similarly, the weighted distance between an alternative $x_i \in X$ and x^- is defined as follows:

$$d\left(x_i, x^-\right) = \sum_{j=1}^{n}\left[w_j\left(\mu_{ij}\right)+\rho_j\left(1-v_{ij}\right)\right] \qquad (3.2)$$

Using Equation 3.1 and 3.2, the relative closeness coefficients of alternatives $x_i \in X$ with respect to the interval triangular type-2 positive ideal solutions (ITT2 PIS), x^+ is defined as follows:

$$C_i\left(\mu_{ij}, v_{ij}; s\right) = \frac{d\left(x_i, x^-\right)}{d\left(x_i, x^-\right)+d\left(x_i, x^+\right)} + s^l + s^u \qquad (3.3)$$

$C_i = 1,2,\ldots,n$ respectively.

Where

$$C_i\left(\mu_{ij}, v_{ij}; s\right) = g + s^l + s^u \tag{3.4}$$

$$= \frac{\Sigma_{j=1}^{n}\left[w_j \mu_{ij} + \rho_j\left(1 - v_{ij}\right)\right]}{\Sigma_{j=1}^{n}\left[w_j + \rho_j\right]} + s^l + s^u \tag{3.5}$$

where g,

$$\frac{\Sigma_{j=1}^{n}\left[w_j \mu_{ij} + \rho_j\left(1 - v_{ij}\right)\right]}{\Sigma_{j=1}^{n}\left[w_j \mu_{ij} + \rho_j\left(1 - v_{ij}\right)\right] + \Sigma_{j=1}^{n}\left[w_j\left(1 - \mu_{ij}\right) + \rho_j\left(v_{ij}\right)\right]}$$

When $C_i\left(\mu_{ij}, v_{ij}; s\right)$ reaches its maximum at the upper bounds $\mu_{ij}^{\ u}$ of the intervals $\left[\mu_{ij}^{\ l}, \mu_{ij}^{\ u}\right]$. So, the nonlinear fractional programming model can be stated as follows:

$$C_i^{\ u}\left(\mu_{ij}, v_{ij}; s\right) =$$

$$\max\left\{\frac{\Sigma_{j=1}^{n}\left[w_j \mu_{ij}^{\ u} + \rho_j\left(1 - v_{ij}^{\ l}\right)\right]}{\Sigma_{j=1}^{n}\left[w_j + \rho_j\right]} + s^l + s^u\right\} \tag{3.6}$$

$$\text{s.t.}\begin{cases} w_j^{\ l} \le w_j \le w_j^{\ u} & \left(j = 1,2,\dots,n\right) \\ \rho_j^{\ l} \le \rho_j \le \rho_j^{\ u} & \left(j = 1,2,\dots,n\right) \\ s^l = n & \left(n = 0,\dots,1\right) \\ s^u = n & \left(n = 0,\dots,1\right) \end{cases}$$

Similarly, when $C_i\left(\mu_{ij}, v_{ij}; m\right)$ reaches its minimum at the upper bounds $\mu_{ij}^{\ l}$ of the intervals $\left[\mu_{ij}^{\ l}, \mu_{ij}^{\ u}\right]$, and the upper bounds $v_{ij}^{\ u}$ of the intervals $\left[v_{ij}^{\ l}, v_{ij}^{\ u}\right]$ .So, the nonlinear fractional programming model can be stated as follows:

$$C_i^{\ l}\left(\mu_{ij}, v_{ij}; s\right)$$

$$= \min\left\{\frac{\Sigma_{j=1}^{n}\left[w_j \mu_{ij}^{\ u} + \rho_j\left(1 - v_{ij}^{\ l}\right)\right]}{\Sigma_{j=1}^{n}\left[w_j + \rho_j\right]} + s^l + s^u\right\} \tag{3.7}$$

$$\text{s.t.}\begin{cases} w_j^{\ l} \le w_j \le w_j^{\ u} & \left(j = 1,2,\dots,n\right) \\ \rho_j^{\ l} \le \rho_j \le \rho_j^{\ u} & \left(j = 1,2,\dots,n\right) \\ s^l = n & \left(n = 0,\dots,1\right) \\ s^u = n & \left(n = 0,\dots,1\right) \end{cases}$$

The nonlinear fractional programming models (Equations 3.6 and 3.7) are simplified and transformed into linear fractional programming models to have an easy computation procedure. Thus, let

$$z = \frac{1}{\sum_{j=1}^{n}\left[w_j + \rho_j\right]} \tag{3.8}$$

and $t_j = zw_j$ (3.9)

$$y_j = z\rho_j \qquad \left(j = 1,2,\dots,n\right) \tag{3.10}$$

Since, $z = \dfrac{1}{\Sigma_{j=1}^{n}\left[w_j + \rho_j\right]}$

$$\therefore \quad \frac{1}{z} = \sum_{j=1}^{n}\left[w_j + \rho_j\right] \tag{3.11}$$

and $t_j = zw_j \quad \therefore \quad w_j = \dfrac{t_j}{z}$ (3.12)

$$y_j = z\rho_j \quad \therefore \quad \rho_j = \frac{y_j}{z} \tag{3.13}$$

Thus, by using the Charnes and Cooper's transformations [19], Equations 3.6 can be transformed into the equivalent linear programming models as follows:

$$C_i^{\ u} = \max\left\{\frac{\sum_{j=1}^{n}\frac{t_i}{z}\mu_{ij}^{\ u} + \frac{y_j}{z}\left(1 - v_{ij}^{\ l}\right)}{z}\right\} \tag{3.14}$$

$$\text{s.t.}\begin{cases} zw_j^{\ l} \le t_j \le zw_j^{\ u} & \left(j = 1,2,\dots,n\right) \\ z\rho_j^{\ l} \le y_j \le z\rho_j^{\ u} & \left(j = 1,2,\dots,n\right) \\ \sum_{j=1}^{n}\left(t_j + y_j\right) = 1 & \\ z \ge 0 & \end{cases}$$

where it can be simplified as follows:

$$C_i^{\ u} = \max\left\{\sum_{j=1}^{n} t_i \mu_{ij}^{\ u} + y_j\left(1 - v_{ij}^{\ l}\right)\right\} \tag{3.15}$$

$$\text{s.t.} \begin{cases} zw_j^{\,l} \le t_j \le zw_j^{\,u} & (j = 1,2,\dots,n) \\ z\rho_j^{\,l} \le y_j \le z\rho_j^{\,u} & (j = 1,2,\dots,n) \\ \sum_{j=1}^{n}(t_j + y_j) = 1 & \\ z \ge 0 & \end{cases}$$

and can be stated in interval triangular type-2 fuzzy concepts as follows:

$$C_i^{\,u}(\mu_{ij}, v_{ij}; s) = \max\left\{\sum_{j=1}^{n} t_i \mu_{ij}^{\,u} + y_j\left(1 - v_{ij}^{\,l}\right) + s^l + s^u\right\} \tag{3.16}$$

$$\text{s.t.} \begin{cases} zw_j^{\,l} \le t_j \le zw_j^{\,u} & (j = 1,2,\dots,n) \\ z\rho_j^{\,l} \le y_j \le z\rho_j^{\,u} & (j = 1,2,\dots,n) \\ \sum_{j=1}^{n}(t_j + y_j) = 1 & \\ z \ge 0 & \\ s^l = n & (n = 0,\dots,1) \\ s^u = n & (n = 0,\dots,1) \end{cases}$$

Whereas, also by using the Charnes and Cooper's transformations [19], Equations 3.7 can be transformed into the equivalent linear programming models as follows:

$$C_i^{\,l} = \max\left\{\frac{\sum_{j=1}^{n} \frac{t_i}{z}\mu_{ij}^{\,l} + \frac{y_j}{z}\left(1 - v_{ij}^{\,u}\right)}{z}\right\} \tag{3.17}$$

$$\text{s.t.} \begin{cases} zw_j^{\,l} \le t_j \le zw_j^{\,u} & (j = 1,2,\dots,n) \\ z\rho_j^{\,l} \le y_j \le z\rho_j^{\,u} & (j = 1,2,\dots,n) \\ \sum_{j=1}^{n}(t_j + y_j) = 1 & \\ z \ge 0 & \end{cases}$$

where it can be simplified as follows:

$$C_i^{\,l} = \max\left\{\sum_{j=1}^{n} t_i \mu_{ij}^{\,l} + y_j\left(1 - v_{ij}^{\,ul}\right)\right\} \tag{3.18}$$

$$\text{s.t.} \begin{cases} zw_j^{\,l} \le t_j \le zw_j^{\,u} & (j = 1,2,\dots,n) \\ z\rho_j^{\,l} \le y_j \le z\rho_j^{\,u} & (j = 1,2,\dots,n) \\ \sum_{j=1}^{n}(t_j + y_j) = 1 & \\ z \ge 0 & \end{cases}$$

and can be stated in interval triangular type-2 fuzzy concepts as follows:

$$C_i^{\,l}(\mu_{ij}, v_{ij}; s) = m\;ax\left\{\sum_{j=1}^{n} t_i \mu_{ij}^{\,l} + y_j\left(1 - v_{ij}^{\,u}\right) + s^l + s^u\right\} \tag{3.19}$$

$$\text{s.t.} \begin{cases} zw_j^{\,l} \le t_j \le zw_j^{\,u} & (j = 1,2,\dots,n) \\ z\rho_j^{\,l} \le y_j \le z\rho_j^{\,u} & (j = 1,2,\dots,n) \\ \sum_{j=1}^{n}(t_j + y_j) = 1 & \\ z \ge 0 & \\ s^l = n & (n = 0,\dots,1) \\ s^u = n & (n = 0,\dots,1) \end{cases}$$

Therefore, Equation 3.16 and 3.19 are the modified version of the distance for each alternative from the positive ideal solution I^* and negative ideal solution I^-. Both equations can be solved using the LINGO software package without striking a blowing because their constraint are all linear. Thus, the values for interval triangular type-2 relative closeness coefficient intervals $C_i(\mu_{ij}, v_{ij}; s) = \left[C_i^{\,l}, C_i^{\,u}\right]$ $(i = 1,2,\dots,m)$, are obtained.

3.2. An Interval Triangular Type-2 Fuzzy Weighted Average for MCDM

Suppose a type-2 fuzzy MCDM has n alternatives $(A_1,\dots,A_n)$ and m decision criteria/ attributes $(C_1,\dots,C_m)$. Each alternative is evaluated with respect to the m criteria/ attributes. All the values/ ratings assigned to the alternatives with respect to each criterion from a decision matrix, denoted by $S = (y_{ij})_{n\times m}$, and the relative weight vector about the criteria, denoted by $W = (w_1,\dots,w_m)$, that satisfying $\sum_{j=1}^{m} w_j = 1$. Due to the fact that, in late cases, the exact values for the elements do not represent the uncertainty and the fuzziness of the real world, so, their values are considered as type-2 fuzzy numbers. In other words, in type-2 fuzzy MCDM problems, the values of alternatives with respect to each criterion/ attribute and the values of relative weights with respect to each criterion/ attribute are usually characterized by fuzzy numbers. By considering the fact that, the type-2 fuzzy TOPSIS method can also be used to deal with type-2 fuzzy MCDM problems as a popular, accurate, and easy to use method, in this section, we extend the original TOPSIS for type-2 fuzzy MCDM problems based on type-2 fuzzy weighted average as follows:

Step 1: Establish decision matrix
Construct the type-2 fuzzy decision matrix and type-2 fuzzy weights matrix as follows:

$$\tilde{\tilde{y}} = \begin{pmatrix} \left(\tilde{\tilde{y}}_{11}\right)_N & \cdots & \left(\tilde{\tilde{y}}_{1j}\right)_N & \cdots & \left(\tilde{\tilde{y}}_{1m}\right)_N \\ \vdots & & \vdots & & \vdots \\ \left(\tilde{\tilde{y}}_{i1}\right)_N & \cdots & \left(\tilde{\tilde{y}}_{ij}\right)_N & \cdots & \left(\tilde{\tilde{y}}_{im}\right)_N \\ \vdots & & \vdots & & \vdots \\ \left(\tilde{\tilde{y}}_{n1}\right)_N & \cdots & \left(\tilde{\tilde{y}}_{nj}\right)_N & \cdots & \left(\tilde{\tilde{y}}_{nm}\right)_N \end{pmatrix},$$

$$\tilde{\tilde{W}} = \left(\tilde{\tilde{w}}_1, \ \cdots, \ \tilde{\tilde{w}}_j, \ \cdots, \ \tilde{\tilde{w}}_m\right) \qquad (3.20)$$

where $\tilde{\tilde{y}}$ and $\tilde{\tilde{W}}$ are in triangular interval type-2 fuzzy sets, $1 \le i \le m$ and $1 \le j \le n$.

Step 2: Normalize the decision matrix
Normalize type-2 fuzzy decision matrix and type-2 fuzzy weights matrix as follows:

$$\left(\tilde{\tilde{Y}}\right)_N = \begin{pmatrix} \left(\tilde{\tilde{y}}_{11}\right)_N & \cdots & \left(\tilde{\tilde{y}}_{1j}\right)_N & \cdots & \left(\tilde{\tilde{y}}_{1m}\right)_N \\ \vdots & & \vdots & & \vdots \\ \left(\tilde{\tilde{y}}_{i1}\right)_N & \cdots & \left(\tilde{\tilde{y}}_{ij}\right)_N & \cdots & \left(\tilde{\tilde{y}}_{im}\right)_N \\ \vdots & & \vdots & & \vdots \\ \left(\tilde{\tilde{y}}_{n1}\right)_N & \cdots & \left(\tilde{\tilde{y}}_{nj}\right)_N & \cdots & \left(\tilde{\tilde{y}}_{nm}\right)_N \end{pmatrix} ,$$

$$\left(\tilde{\tilde{W}}\right)_N = \left(\tilde{\tilde{w}}_1, \ \cdots, \ \tilde{\tilde{w}}_j, \ \cdots, \ \tilde{\tilde{w}}_m\right) \qquad (3.21)$$

In above equations, we define $\left(\tilde{\tilde{y}}_{ij}\right)_N$ and $\left(\tilde{\tilde{w}}_j\right)_N$ as normalized fuzzy value/ ratings related to $\tilde{\tilde{y}}_{ij}$ and $\tilde{\tilde{w}}_j$ respectively.

Step 3: Determine the weighted of decision matrix
Construct the weighted decision matrix $\tilde{\tilde{v}}_{ij}$ as follows:

$$\left(\tilde{\tilde{v}}_{ij}\right)_{nm} = \begin{bmatrix} \tilde{\tilde{v}}_{11} & \cdots & \tilde{\tilde{v}}_{1j} & \cdots & \tilde{\tilde{v}}_{1m} \\ \vdots & & \vdots & & \vdots \\ \tilde{\tilde{v}}_{i1} & \cdots & \tilde{\tilde{v}}_{ij} & \cdots & \tilde{\tilde{v}}_{im} \\ \vdots & & \vdots & & \vdots \\ \tilde{\tilde{v}}_{n1} & \cdots & \tilde{\tilde{v}}_{nj} & \cdots & \tilde{\tilde{v}}_{nm} \end{bmatrix} \qquad (3.22)$$

Where $\tilde{\tilde{v}}_{ij} = \left(\tilde{\tilde{Y}}\right)_N \times \left(\tilde{\tilde{W}}\right)_N$ (3.23)

Step 4: Positive ideal solution and negative ideal solution
Determine the matrices that include positive and negative ideal solution as

$$I^* = \left\{x_1^*, \dots, x_m^*\right\} = \left\{\left(\max_j d_{ij} \,\middle|\, j \in \Omega_b\right), \left\{\min_j a_{ij} \,\middle|\, j \in \Omega_c\right\}\right\} \qquad (3.24)$$

$$I^- = \left\{x_1^-, \dots, x_m^-\right\} = \left\{\left(\min_j a_{ij} \,\middle|\, j \in \Omega_b\right), \left\{\max_j d_{ij} \,\middle|\, j \in \Omega_c\right\}\right\} \qquad (3.25)$$

Step 5: Construct the separation of each alternative
Calculate the separation of each alternative from the positive ideal solution I^* and negative ideal solution I^- using the Equation 3.16 and 3.19 (in Section 3.1)·

$$C_i^u\left(\mu_{ij}, v_{ij}; s\right) = \max \left\{\Sigma_{j=1}^n t_i \mu_{ij}^u + y_j\left(1 - v_{ij}^l\right) + s^l + s^u\right\} \qquad (3.26)$$

$$\text{s.t.} \begin{cases} zw_j^{\ l} \le t_j \le zw_j^{\ u} & (j = 1,2,\dots,n) \\ z\rho_j^{\ l} \le y_j \le z\rho_j^{\ u} & (j = 1,2,\dots,n) \\ \sum_{j=1}^{n}\left(t_j + y_j\right) = 1 \\ z \ge 0 \\ s^l = n & (n = 0,\dots,1) \\ s^u = n & (n = 0,\dots,1) \end{cases}$$

and

$$C_i^{\ l}\left(\mu_{ij}, v_{ij}; s\right) = \max \left\{\sum_{j=1}^{n} t_i \mu_{ij}^{\ l} + y_j\left(1 - v_{ij}^{\ u}\right) + s^l + s^u\right\} \tag{3.27}$$

$$\text{s.t.} \begin{cases} zw_j^{\ l} \le t_j \le zw_j^{\ u} & (j = 1,2,\dots,n) \\ z\rho_j^{\ l} \le y_j \le z\rho_j^{\ u} & (j = 1,2,\dots,n) \\ \sum_{j=1}^{n}\left(t_j + y_j\right) = 1 \\ z \ge 0 \\ s^l = n & (n = 0,\dots,1) \\ s^u = n & (n = 0,\dots,1) \end{cases}$$

Step 6: Define the closeness coefficient
Calculate the relative degree of closeness to the ideal solution for each alternative

$$(CC)_i = \frac{C_i^{\ l}}{C_i^{\ u} + C_i^{\ l}} \quad (i = 1,\dots,n) \tag{3.28}$$

Step 7: Rank all alternatives
Sort the values of $(CC)_i$ in a decending sequence, where $1 \le j \le n$. The larger the value of $(CC)_i$, the higher the preference of the alternative for $(CC)_i$.

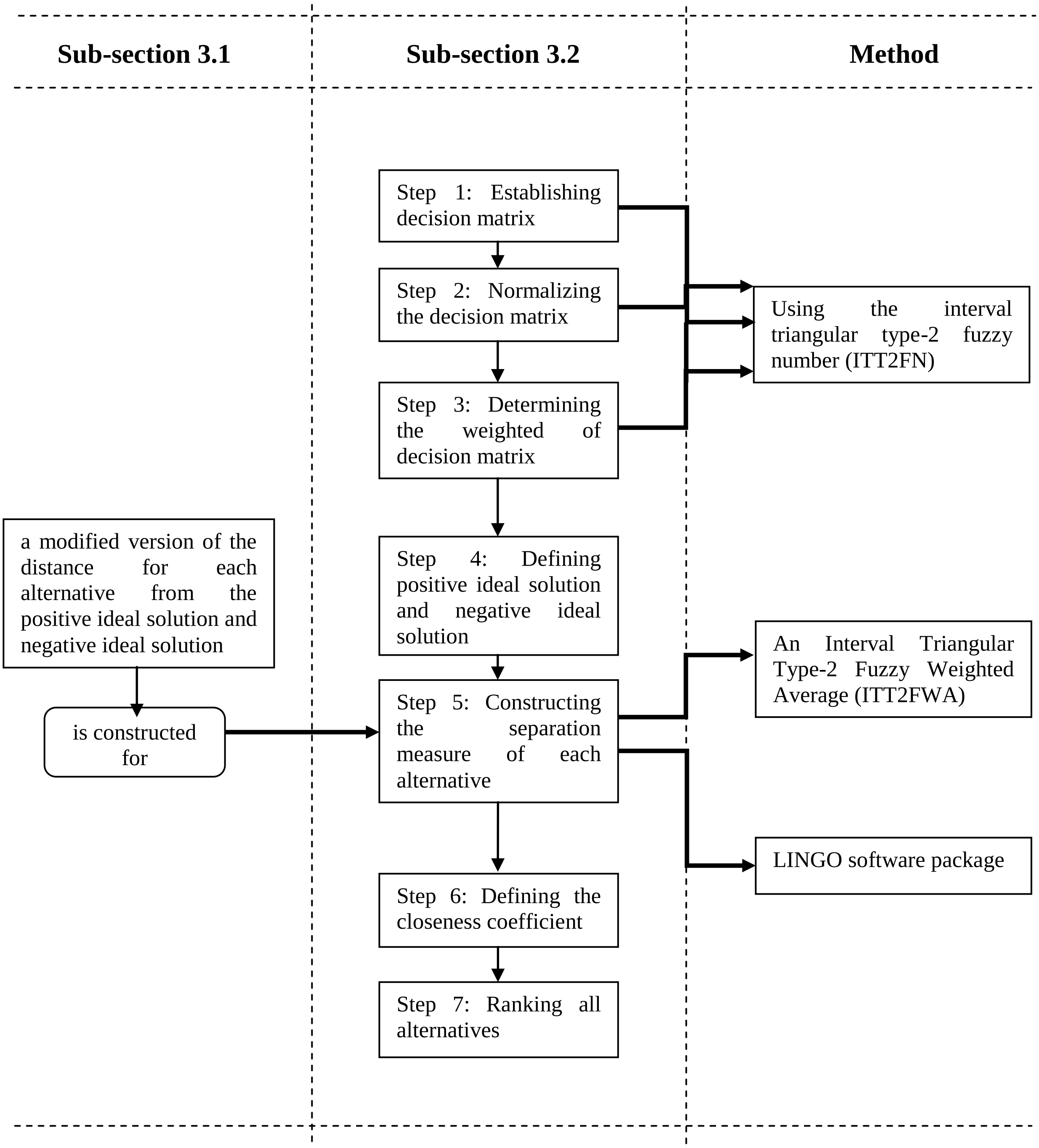

Fig. 2 Conceptual Framework

4 Numerical Examples

Three numerical examples were conducted to elicit linguistic judgment for illustrating the fuzzy multiple attribute group decision-making process of the proposed method. These numerical examples are to test the ability of the proposed method to handle the multiple attribute decision making problems in many areas.

All the relative importance weights and ratings (i.e. the criteria values) in three numerical examples are described using linguistic variables which are defined in (Table 1 and Figure 3) and (Table 2 and Figure 4). These linguistic variables are using the converting

ITT2FNs' linguistic. Table 1 shows the linguistic terms "Very Low" (VL), "Low" (L), "Medium Low" (ML), "Medium" (M), "Medium High" (MH), "High" (H),"Very High" (VH) and their corresponding interval triangular type-2 fuzzy numbers (ITT2FN). Besides, Table 2 shows the linguistic terms for the ratings of criteria "Very Poor" (VP), "Poor" (P), "Medium Poor" (MP), "Medium" (M)/ "Fair" (F), "Medium Good" (MG), "Good" (G),"Very Good" (VG) and their corresponding interval triangular type-2 fuzzy numbers (ITT2FN). The flexibility of linguistic judgment and their ITT2FNs for both Table 1 and 2 can be observed in Figure 3 and 4.

Table 1 Linguistic variables for the relative importance weights of criteria

Linguistic Variable	**Type-1 Fuzzy Sets**
Very Low (VL)	((0,0.1;1),(0,0.5;1))
Low (L)	((0,0.3;1),(0.05,0.2;1))
Medium Low (ML)	((0.1,0.5;1),(0.2,0.4;1))
Medium (M)	((0.3,0.7;1),(0.4,0.6;1))
Medium High (MH)	((0.5,0.9;1),(0.6,0.8;1))
High (H)	((0.7,1.0;1),(0.8,0.95;1))
Very High (VH)	((0.9,1.0;1),(0.95,1.0,1))

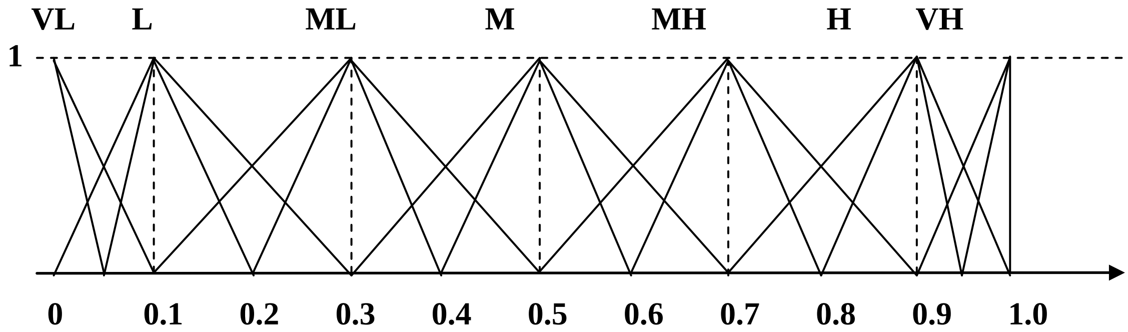

Fig. 3 The relative importance weights of criteria

Table 2 Linguistic variables for the ratings

Linguistic Variable	**Type-1 Fuzzy Sets**
Very Poor (VP)	((0,1;1),(0,0.5;1))
Poor (P)	((0,3;1),(0.5,2;1))
Medium Poor (MP)	((1,5;1),(2,4;1))
Medium (M)/Fair (F)	((3,7;1),(4,6;1))
Medium Good (MG)	((5,9;1),(6,8;1))
Good (G)	((7,10;1),(8,9.5;1))
Very Good (VG)	((9,10;1),(9.5,10,1))

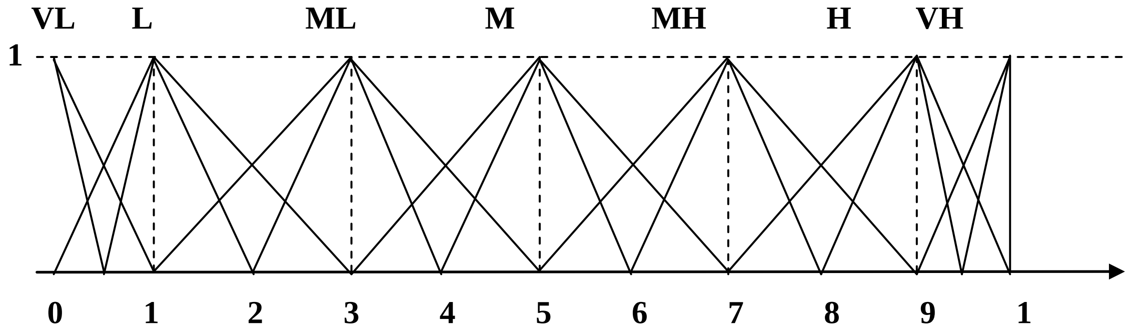

Fig. 4 Linguistic variables for the ratings

Example 1: (Chen [23])
Assume that there are three decision-makers, D_1, D_2, and D_3 of a software company to hire a system analysis engineer and assume that there are three alternatives x_1, x_2, x_3 and five attributes "Emotional Steadiness" (denoted by C_1), "Oral Communication Skill" (denoted by C_2), "Personality" (denoted by C_3), "Past Experience" (denoted by C_4), "Self-Confidence" (denoted by C_5). Let X be the set of alternatives, where $X = \{x_1, x_2, x_3\}$, and let F be the set of attributes, where F={Emotional Steadiness, Oral Communication Skill, Personality, Past Experience, Self-Confidence}. Assume that there are three decision-makers D_1, D_2, and D_3 use the linguistic terms shown in Table 1 to represent the weights of the four attributes, respectively, as shown in Table 3. In Table 3, five benefit attributes are considered, including "Emotional Steadiness", "Oral Communication Skill", "Personality", "Past Experience" and "Self-Confidence". Assume that the three decision-makers D_1, D_2, and D_3 use the linguistic terms shown in Table 2 to represent the evaluating values of the alternatives with respect to different attributes, respectively, as shown in Table 4.

Table 3 Weights of the attributes evaluated by decision-makers

Attributes	**Decision-makers**		
	D_1	**D_2**	**D_3**
C_1	H	VH	MH
C_2	VH	VH	VH
C_3	VH	H	H
C_4	VH	VH	VH
C_5	M	MH	MH

Table 4 Linguistic of Decision Matrix

Attributes	**Alternatives**	**Decision-Makers**		
		D_1	**D_2**	**D_3**
C_1	x_1	MG	G	MG
	x_2	G	G	MG
	x_3	VG	G	F
C_2	x_1	G	MG	F
	x_2	VG	VG	VG
	x_3	MG	G	VG
C_3	x_1	F	G	G
	x_2	VG	VG	G
	x_3	G	MG	VG
C_4	x_1	VG	G	VG
	x_2	VG	VG	VG
	x_3	G	VG	MG
C_5	x_1	F	F	F
	x_2	VG	MG	G
	x_3	G	G	MG

Example 2: (Chen and Lee [24])
Assume that there are three decision-makers D_1, D_2, and D_3 to evaluate cars and assume that there are three alternatives x_1, x_2, x_3 and four attributes, where F={Safety (C_1), Price (C_2), Appearance (C_3), Performance (C_4)}. Assume that the three decision-makers D_1, D_2, and D_3 use the linguistic terms shown in Table 1 to represent the weights of the four attributes, respectively, as shown in Table 5. Where as, assume that the three decision-makers D_1, D_2, and D_3 use the linguistic terms shown in Table 2 to represent the evaluating values of the alternatives with respect to different attributes, respectively, as shown in Table 6.

Table 5 Weights of the attributes evaluated by decision-makers

Attributes	Decision-makers		
	D_1	D_2	D_3
C_1	VH	H	VH
C_2	H	VH	VH
C_3	M	MH	MH
C_4	VH	H	H

Table 6 Linguistic of Decision Matrix

Attributes	Alternatives	Decision-Makers		
		D_1	D_2	D_3
C_1	x_1	MH	H	MH
	x_2	H	MH	H
	x_3	VH	H	MH
C_2	x_1	H	VH	H
	x_2	MH	H	VH
	x_3	VH	VH	H
C_3	x_1	VH	H	H
	x_2	H	VH	VH
	x_3	M	MH	MH
C_4	x_1	VH	H	H
	x_2	H	VH	H
	x_3	H	VH	VH

Table 7 Weights of the attributes evaluated by decision-makers

Attributes	Decision-makers		
	D_1	D_2	D_3
C_1	VH	VH	VH
C_2	H	H	H
C_3	M	M	M
C_4	L	L	L

Table 8 Linguistic of Decision Matrix

Attributes	Alternatives	Decision-Makers		
		D_1	D_2	D_3
C_1	x_1	VG	G	G
	x_2	VG	G	P
	x_5	MG	MG	VG
C_2	x_1	VG	G	G
	x_2	VG	G	VG
	x_5	VG	G	MP
C_3	x_1	VG	G	G
	x_2	VG	G	VG
	x_5	G	MG	MP
C_4	x_1	VG	G	G
	x_2	VG	G	VG
	x_5	G	MG	F

Example 3: (Nurnadiah and Lazim [25])

Assume that, there are four vehicles that most involved in road accidents represent as attributes, there are motorcycle (C_1), car (C_2), bus (C_3), lorry (C_4) and five subjective alternatives has been highlighted as speeding behaviour (x_1), reckless driving (x_2), driver's health (x_3), road condition (x_4) and road environment (x_5). Furthermore, a committee of three decision-makers or experts, D_1, D_2, and D_3 has been identified to seek reliable data over the accidents. Data in form of linguistics variables were collected through interviewing of three authorised personnel from three Malaysian Government agencies. The interview was conducted in three separated sessions to elicit the information about causes that regularly lead to accident as stated in Table 7 (for weight of the attributes) and Table 8 (for linguistic of decision matrix).

The vital part of this conceptual model is how the computation is carried out. Details of computational steps and results are explained in the following section.

5. Computations and Results

The computations steps for Example 1, 2 and 3 are executed in the following manner.

Step 1: Establish decision matrix

Type-2 fuzzy decision matrix and type-2 fuzzy weights matrix are defined using the formulae 3.20.

Step 2: Normalize the decision matrix

Normalization for type-2 fuzzy decision matrix and type-2 fuzzy weights matrix are obtained as Table 9.

Table 9 Decision Matrix

Example 1	C_1	C_2	C_3	C_4	C_5
x_1	((0.5667,0.9333;1), (0.6667,0.85;1))	((0.4,0.7333;1), (0.4833,0.65;1))	((0.4667,0.7667;1), (0.55,0.7;1))	((0.8333,1;1), (0.9,0.9833;1))	((0,0.3;1), (0.05,0.2;1))
x_2	((0.6333,0.9667;1), (0.7333,0.9;1))	((0.9,1;1); (0.95,1;1))	((0.8333,1;1), (0.9,0.9833;1))	((0.9,1;1), (0.95,1;1))	((0.7,0.9667;1), (0.7667,0.9167;1))
x_3	((0.5333,0.7667;1), (0.6,0.7167;1))	((0.7,0.9667;1), (0.7833,0.9167;1))	((0.7,0.9667;1), (0.7833,0.9167;1))	((0.7,0.9667;1), (0.7833,0.9167;1))	((0.6333,0.9667;1), (0.7333,0.9;1))
W	((0.7,0.9667;1), (0.7833,0.9167;1))	((0.9,1;1), (0.95,1;1))	((0.7667,1;1), (0.85,0.9667;1))	((0.9,1;1), (0.95,1;1))	((0.4333,0.8333;1), (0.5333,0.7333;1))

Example 2	C_1	C_2	C_3	C_4
x_1	((0.6333,0.9333;1), (0.7167,0.8667;1))	((0.7667,1;1), (0.85,0.9667;1))	((0.7667,1;1), (0.85,0.9667;1))	((0.7667,1;1), (0.85,0.9667;1))
x_2	((0.7667,0.9667;1), (0.8333,0.9333;1))	((0.7,0.9667;1), (0.7833,0.9167;1))	((0.8333,1;1), (0.9,0.9833;1))	((0.7667,1;1), (0.85,0.9667;1))
x_3	((0.7,0.9667;1), (0.7833,0.9167;1))	((0.8333,1;1), (0.9,0.9833;1))	((0.4333,0.8333;1), (0.5333,0.7333;1))	((0.8333,1;1), (0.9,0.9833;1))
W	((0.8333,1;1), (0.9,0.9833;1))	((0.8333,1;1), (0.9,0.9833;1))	((0.4333,0.8333;1), (0.5333,0.7333;1))	((0.7667,1;1), (0.85,0.9667;1))

Example 3	C_1	C_2	C_3	C_4
x_1	((0.7667,1;1), (0.8,0.9667;1))	((0.7667,1;1), (0.8,0.9667;1))	((0.7667,1;1), (0.8,0.9667;1))	((0.7667,1;1), (0.8,0.9667;1))
x_2	((0.5333,0.7667;1), (0.6,0.7167;1))	((0.8333,1;1), (0.9,0.9833;1))	((0.8333,1;1), (0.9,0.9833;1))	((0.8333,1;1), (0.9,0.9833;1))
x_3	((0.6333,0.9333;1), (0.7167,0.8667;1))	((0.5667,0.8333;1), (0.65,0.7833;1))	((0.4333,0.8;1), (0.5333,0.7167;1))	((0.5,0.8667;1), (0.6,0.7833;1))
W	((0.9,1;1), (0.95,1;1))	((0.7,1;1), (0.8,0.95;1))	((0.3,0.7;1), (0.4,0.6;1))	((0,0.3;1), (0.05,0.2;1))

Step 3: Determine the weighted of decision matrix

Weighted decision matrix $\tilde{\tilde{v}}_{ij}$ is constructed as follows:

Table 10 Weighted Decision Matrix

	C_1	C_2	C_3	C_4	C_5
x_1	((0.3970,0.9022;1), (0.5222,0.7792;1))	((0.36,0.7333;1), (0.4591,0.65;1))	((0.3578,0.7667;1), (0.4675,0.6767;1))	((0.75,1;1), (0.855,0.9833;1))	((0,0.25;1), (0.0267,0.1467;1))
x_2	((0.4433,0.9345;1), (0.5744,0.8250;1))	((0.81,1;1), (0.9025,1;1))	((0.6389,1;1), (0.765,0.9506;1))	((0.81,1;1), (0.9025,1;1))	((0.3033,0.8056;1), (0.4089,0.6722;1))
x_3	((0.3733,0.7412;1), (0.47,0.657;1))	((0.63,0.9667;1), (0.7441,0.9167;1))	((0.5367,0.9667;1), (0.6658,0.8862;1))	((0.63,0.9667;1), (0.7441,0.9167;1))	((0.2744,0.8056;1), (0.3911,0.66;1))

	C_1	C_2	C_3	C_4
x_1	((0.6450,0.8522;1), (0.5277,0.9333;1))	((0.765,0.9506;1), (0.6389,1;1))	((0.4533,0.7089;1), (0.3322,0.8333;1))	((0.7225,0.9345;1), (0.5878,1;1))
x_2	((0.75,0.9177;1), (0.6389,0.9667;1))	((0.705,0.9014;1), (0.5833,0.9667;1))	((0.48,0.7211;1), (0.3611,0.8333;1))	((0.7225,0.9345;1), (0.5878,1;1))
x_3	((0.705,0.9014;1), (0.5833,0.9667;1))	((0.81,0.9669;1), (0.6944,1;1))	((0.2844,0.5377;1), (0.1877,0.6944;1))	((0.765,0.9506;1), (0.6389,1.0;1))

	C_1	C_2	C_3	C_4
x_1	((0.69,0.76;1), (0.5367,0.64;1))	((0.5367,0.64;1), (0.9184,1;1))	((0.23,0.04;1), (0.58,0.7;1))	((0,0.04;1), (0.1933,0.3;1))
x_2	((0.48,0.57;1),	((0.5833,0.72;1),	((0.25,0.36;1),	((0,0.045;1),

	(0.7167,0.7667;1))	(0.9341,1;1))	(0.59,0.7;1))	(0.1967,0.3;1))
x_3	((0.57,0.6809;1), (0.8667,0.9333;1))	((0.3967,0.52;1), (0.7441,0.8333;1))	((0.13,0.2133;1), (0.43,0.56;1))	((0,0.03;1), (0.1567,0.26;1))

Step 4: Positive ideal solution and negative ideal solution
The matrices that include positive and negative ideal solution are determined using Equation 3.24 and 3.25.

Step 5: Construct the separation measure of each alternative
The separation of each alternative from the positive ideal solution I^* and negative ideal solution are given in Table 11.

Step 6: Define the closeness coefficient
Relative degree of closeness to the ideal solution for each alternative are calculated using Equation 3.28.

Step 7: Rank all alternatives
The values of $(CC)_i$ are sorted in a descending sequence, where $1 \le j \le n$. The larger the value of $(CC)_i$, the higher the preference of the alternative for $(CC)_i$.

Table 11 Final ranking order

Example 1	min	max	Closeness Coefficient, $(CC)_i$
x_1	2.3606	2.3760	0.4985
x_2	2.3548	2.3507	0.5004
x_3	2.3365	2.3355	0.5001

Example 2	min	max	Closeness Coefficient, $(CC)_i$
x_1	2.3181	2.3208	0.499708982
x_2	2.3254	2.3267	0.499860278
x_3	2.3367	2.3412	0.499519014

Example 3	min	max	Closeness Coefficient, $(CC)_i$
x_1	2.2905	2.3954	0.488806846
x_2	2.3078	2.4204	0.48809272
x_3	2.3068	2.4310	0.48689265

Lastly, we compare all three existed numerical examples with the numerical examples that have been tested in our proposed method. Thus, the results for all the numerical examples are stated in Table 12 as follows:

Table 12 Results Comparison

	Chen (2000)			
	Chen (2000)'s results	**Chen (2000)'s Ranking**	**Proposed results**	**Proposed Ranking**
x_1	0.19	3	0.4985	3
x_2	1	1	0.5004	1
x_3	0.56	2	0.5001	2

	Chen and Lee (2010)			
	Chen and Lee (2010)'s results	**Chen and Lee (2010)'s Ranking**	**Proposed results**	**Proposed Ranking**
x_1	0.61	2	0.499708982	2
x_2	0.87	1	0.499860278	1

x_3	0.31	3	0.499519014	3

	Nurnadiah and Lazim (2012)			
	Nurnadiah and Lazim (2012)'s results	**Nurnadiah and Lazim (2012)'s Ranking**	**Proposed results**	**Proposed Ranking**
x_1	0.2291	1	0.488806846	1
x_2	0.2259	2	0.48809272	2
x_3	0.1566	3	0.48689265	3

As shown in Table 12, the relative closeness for Chen's method [23] for three alternatives are 0.4985 for x_1, 0.5004 for x_2 and 0.5001 for x_3. Which lead to the ranking of $x_2>x_3>x_1$. Whereas, the relative closeness coefficients for Chen and Lee's method [24] are 0.499708982 for x_1, 0.499860278 for x_2 and 0.499519014 for x_3. Which lead to the ranking of $x_2>x_1>x_3$. Moreover, the relative closeness coefficients for Nurnadiah and Lazim's method [25] are 0.488806846 for x_1, 0.48809272 for x_2 and 0.48689265 for x_3. Which lead to the ranking of $x_1>x_2>x_3$. All these results coincide with the existed numerical examples.

6. Conclusions

In this paper, we presented a modified interval triangular type-2 fuzzy weighted average for MCDM problems. The existed fuzzy weighted average is modified to become a modified fuzzy weighted average in interval triangular type-2 fuzzy sets (ITT2FS) approach. This proposed method is constructed for solving the distance for each alternative from the positive ideal solution and negative ideal solution in type-2 MCDM method. Three existed numerical examples [23; 24; 25] are used to illustrate the proposed method. Results from three proposed methods are compared with the results from the three existed method. All the proposed results coincide with the existed results. As a result, we found that the proposed method is practical for solving the decision making problems. Moreover, it seems that the proposed method is flexible and easy to use. Besides, it represents with the low computational volume. Therefore, this proposed method is more fuzziness and uncertainties due to the fact that it uses interval type-2 fuzzy sets rather than existed fuzzy sets to represent the evaluating values and the weights of attributes.

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