Synchronization Phenomena in Coupled Nonlinear Systems Applied in Economic Cycles

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Abstract: - In this paper, the modelization of coupling between two systems of economic cycles, which adopt the idea that economic fluctuations result from endogenous interactions, is studied. The nonlinear system, which describes the economic system, is a modification of the 2-dimensional Van der Pol oscillator. The coupling strength represents the effect of the capital inflow between the two conjugated economic systems, with identical economic aggregates, such as savings, gross domestic product and foreign capital inflow. Numerical simulations reveal the richness of the coupled system's dynamic behavior, showing interesting nonlinear dynamical and synchronization phenomena. Various tools of nonlinear theory, such as bifurcation diagrams, phase portraits and Lyapunov exponents, for the study of the proposed coupling methods (unidirectional and bidirectional), have been used.

Key-Words: - Nonlinear system**,** economic cycle, Chaos, bifurcation, complete synchronization, inverse lag synchronization, inverse π-lag synchronization.

1 Introduction

In the last decades the research activity in various fields of natural sciences, confirm the fact that nonlinear systems, exhibiting chaotic behavior, have triggered the interest because of the great variety of phenomena that have been observed [1-4]. This is due to the main feature of chaotic systems, which is the great sensitivity on initial conditions and system's parameters.

Also, from the middle of 80's many chaotic phenomena have also been observed in economics. This is the main reason for which a new scientific field namely econophysics, is developed, especially the last few years. So, econophysics provides an alternative approach, aiming to study the particularly complex dynamics of real economic systems, such as structural changes, irregular (erratic) micro- and macro-economic fluctuations.

In many cases the first step for economists, in order to fix an economic model, is to take into consideration, only endogenous variables. But in this way the behavior of the economic model is simplified. After this first approach, the economic model is enriched with exogenous variables, describing forces not directly related to the economic model, such as political events, physical disasters, technological innovations e.t.c. Therefore, complexity of these models makes accurate economic forecasting very difficult.

Also, in many fields of economics such as stocks, funding and social economics, the complexity is unfolded in the internal structure of models that interact with external drives, due to the interaction of nonlinear factors (economic or social) with all kinds of economic problems. Basically, this was the main reason for the use of tools of nonlinear theory to the study of various economic models. Till today many techniques, such as integrating agent based modelling [5], or known nonlinear systems, such as Van der Pol [6, 7] and other [8-12] have been used as economic models for explaining various phenomena.

It is noted in academic literature that economic aggregates such as, consumption, investment, gross domestic product e.t.c., follows an oscillatory motion embedding a vast spectrum of cycles. Some of the most known of these cycles are the seasonal, Kitchin, Juglar, Kuznets and Kondratiev, which are described by a sequence of four phases (boom, recession, depression and recovery). Until 1967, economists believed that the source of these oscillations is various external factors such as natural catastrophes, technological innovations, consumption jumps). That year Goodwin proposed that the fluctuations of the economic aggregates come from endogenous factors [13]. Since then economists have studied this phenomenon by means of mathematical models, including various kinds of linear, nonlinear, and coupled oscillator models.

The aim of this work is to examine the dynamic behavior of two coupled identical systems applied to an economic model of cycles. The system, which is used in this work, it is a modification of the 2-dimensional Van der Poll system [14] with a feedback loop that leads to a full 3-dimensional model with a rich dynamic behavior [15]. Furthermore, from an economic point of view this system simulates the capital flight observed in the less developed countries. So, in this paper the effect of the capital inflow between two economical conjugated systems, with identical economic parameters, such as the marginal propensity to saving, the potential GDP etc, is examined.

The rest of the paper is organized as follows: In Section 2 the basic features of the chaotic systems as well as the synchronization phenomena that are the base of this work, are introduced. Section 3 presents the economic model which is used. In Section 4 the results of the coupling with two different ways (unidirectional and bidirectional) are described. Finally, Section 5 gives the concluding remarks.

2 Chaotic Systems and Synchronization Phenomena

A dynamical system in order to be considered as chaotic must fulfil three basic conditions [16]:

• It must be very sensitive on initial conditions,
• its periodic orbits must be dense and
• it must be topologically mixing.

The first of these conditions is the most important because it means that a small variation in system's initial conditions may cause a totally different dynamic behavior. As we will see in details this feature is the key of the synchronization phenomena which the coupled economic system shows.

Furthermore, the last three decades, the study of the interaction between coupled chaotic systems has been rapidly developed [17-19]. This occurs because of the rich dynamic behavior, which coupled systems show and the multidisciplinary character of this approach. Synchronization phenomena, as an effect of this interaction, has been observed in various fields, such as in complex physical, chemical and biological systems, in broadband communication systems, in secure communication and cryptography [20-25]. So, the study of synchronization phenomena of coupled nonlinear economic models, which describe economic cycles, may have significant future implications.

Until now various types depending on the nature of the interaction systems and of the coupling have been proposed. Phase synchronization, Lag synchronization, Generalized synchronization, Antisynchronization and Anti-phase synchronization, Projective synchronization, Anticipating and Inverse lag synchronization are some of the most interesting types of synchronization [26-33].

Nevertheless the most well-known type of synchronization is the Complete or Full synchronization, in which the interaction between two identical coupled chaotic systems leads to a perfect coincidence of their chaotic trajectories, i.e.

$$x_1(t) = x_2(t), \quad \text{as } t \to \infty \tag{1}$$

where x_1 and x_2 are the variables of the coupled chaotic systems.

However, in 2011 a new synchronization phenomenon, the Inverse π-lag synchronization, between two mutually coupled identical nonlinear systems, has been observed [34]. This new type of synchronization is observed when each one of the coupled systems satisfies a specific type of symmetry, i.e.

$$S: (x, y, z) \to (-x, -y, -z) \tag{2}$$

Also, in this case, the coupled systems are in a phase locked (periodic) state, depending on the coupling factor and it can be characterized by eliminating the sum of two relevant periodic variables (x_1 and x_2) with a time lag τ, which is equal to T/2, where T is the period of x_1 and x_2.

$$x_1(t) = -x_2(t+\tau), \quad \tau = T/2 \qquad (3)$$

Nevertheless, depending on the coupling factor and the chosen set of system's initial conditions, the inverse π-lag synchronization coexists with the complete synchronization.

3 The Economic Model

In this paper a 3-dimensional system of autonomous differential equations, which is a modification of the 2-dimensinal Van der Poll oscillator [14], is used. It can be interpreted as an idealized macroeconomic model with foreign capital investments introduced by Bouali et al. [35]. This system can be described by the following set of three normalized differential equations:

$$\begin{cases} \dot{x} = m \cdot y + p \cdot x \cdot \left(d - y^2\right) \\ \dot{y} = -x + c \cdot z \\ \dot{z} = s \cdot x - r \cdot y \end{cases} \qquad (4)$$

The state variables, x, y, and z of the system represent the savings of households, the Gross Domestic Product (GDP) and the foreign capital inflow, respectively. Also, dot denotes the derivative with respect to time. Positive parameters represent corresponding ratios: m is the marginal propensity to saving, p is the ratio of capitalized profit, d is the value of the potential GDP, c is the output/capital ratio, s is the capital inflow/savings ratio and r is the debt refund/output ratio.

The proposed economic model induces patterns similar to the data series of the real economy. Furthermore, it satisfies the condition of symmetry (2) and it has also three equilibrium points: the origin E_0 and two antisymmetric points E_1 and E_2, where, E_1: [α, (s/r)α, (1/c)α], E_2: [–α, –(s/r)α, –(1/c)α], with $\alpha = [pd + m(s/r)]/[(s/r)^2 p]^{1/2}$. Also, the system (4) shows a complex dynamic behavior with various phenomena, such as periodic behavior of different limit cycles, chaotic behavior and blue sky bifurcations [35,36].

Additional, the set of parameters which is used, are: (m, p, d, c, s, r) = (0.02, 0.4, 1, 50, 10, 0.1), so as each one of the coupled system behaves chaotically. Fig.1 shows the phase portraits of y versus x and z versus y, which confirm the chaotic behavior. Furthermore, the Lyapunov exponents of the system for the above mentioned parameters and for initial conditions: (x_0, y_0, z_0) = (0.05, 0.2, 0.02) were calculated:

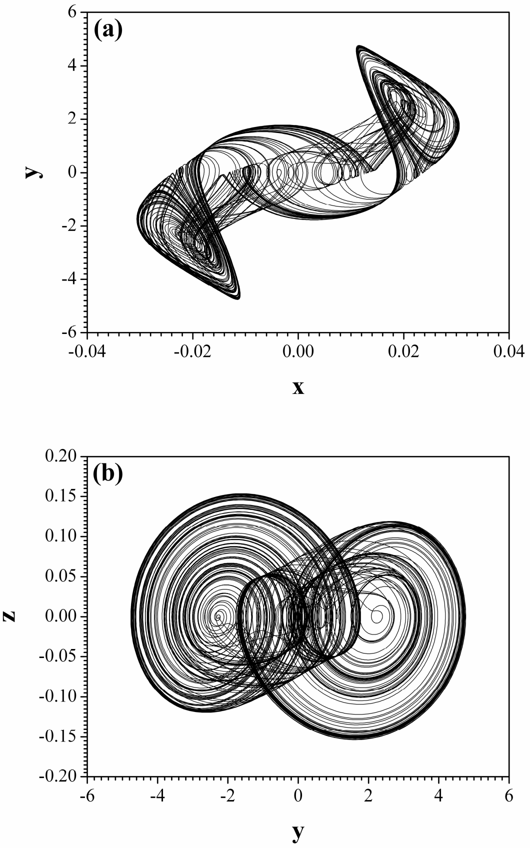

Fig.1 Simulation phase portrait in (a) y versus x and (b) z versus y plane. Double-scroll chaotic attractors are observed.

(LE_1, LE_2, LE_3) = (0.40598, 0, –3.25420), by employing the Wolf et al. algorithm [37]. So, this numerical analysis ensures that the system operates in a desired chaotic mode because of the existence of a positive Lyapunov exponent.

So, the central question of this work is how the coupling of two similar (identical in the ideal scenario) economic systems, which are described by the system (4), changes the qualitative properties of a growth path of the dynamical economic system.

4 The Coupled Systems

Generally, there are various methods of coupling between coupled nonlinear systems available in the literature. However, two are the most interesting. In the first method due to Pecora and Caroll [19], a stable subsystem of a chaotic system could be synchronized with a separate chaotic system under certain suitable conditions. In the second method, chaos synchronization between two nonlinear systems is achieved due to the effect of coupling without requiring to construct any stable subsystem [38-40].

This second method can be divided into two classes: drive-response or unidirectional coupling and bidirectional or mutual coupling. In the first case, one system drives another one called the response or slave system. The system of two unidirectional coupled identical systems is described by the following set of differential equations:

$$\begin{cases} \dot{\mathbf{x}}_1 = F(\mathbf{x}_1) \\ \dot{\mathbf{x}}_2 = F(\mathbf{x}_2) + C \cdot (\mathbf{x}_1 - \mathbf{x}_2) \end{cases} \tag{5}$$

where $F(\mathbf{x})$ is a vector field in a phase space of dimension n and C a matrix of constants, which describes the nature and strength of the coupling between the oscillators. It is obvious from (5) that only the first system influences the dynamic behavior of the other.

In the second case, both the coupled systems are connected and each one influences the dynamics of the other. This is the reason for which this method is called mutual (or bidirectional). The coupled system of two mutually coupled chaotic oscillators is described by the following set of differential equations:

$$\begin{cases} \dot{\mathbf{x}}_1 = F(\mathbf{x}_1) + C \cdot (\mathbf{x}_2 - \mathbf{x}_1) \\ \dot{\mathbf{x}}_2 = F(\mathbf{x}_2) + C \cdot (\mathbf{x}_1 - \mathbf{x}_2) \end{cases} \tag{6}$$

To identify the behavior of the coupled systems in these two cases, the numerical computations are carried out with the fourth order Runge-Kutta integration method. All simulations start with initial conditions: $(x_1, y_1, z_1, x_2, y_2, z_2) = (0.05, 0.2, 0.02, -0.08, -0.1, -0.05)$.

4.1 Unidirectional Coupling

Based on (5) the system of two unidirectional identical economic systems, described by (4), is given by the following set of differential equations:

$$\begin{cases} \dot{x}_1 = m \cdot y_1 + p \cdot x_1 \cdot \left(d - y_1^2\right) \\ \dot{y}_1 = -x_1 + c \cdot z_1 \\ \dot{z}_1 = s \cdot x_1 - r \cdot y_1 \\ \dot{x}_2 = m \cdot y_2 + p \cdot x_2 \cdot \left(d - y_2^2\right) \\ \dot{y}_2 = -x_2 + c \cdot z_2 \\ \dot{z}_2 = s \cdot x_2 - r \cdot y_2 + \xi \cdot (z_1 - z_2) \end{cases} \tag{7}$$

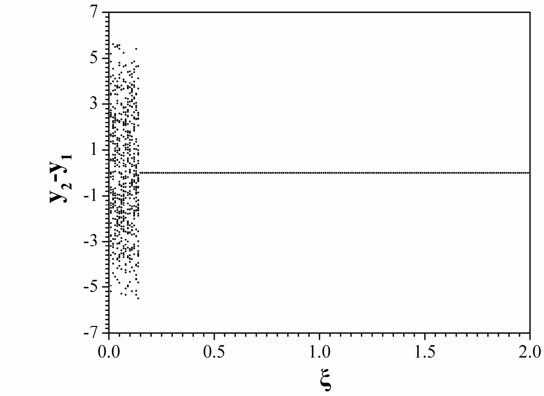

Fig.2 Bifurcation diagram of $(y_2 - y_1)$ versus ξ, in the case of unidirectional coupling.

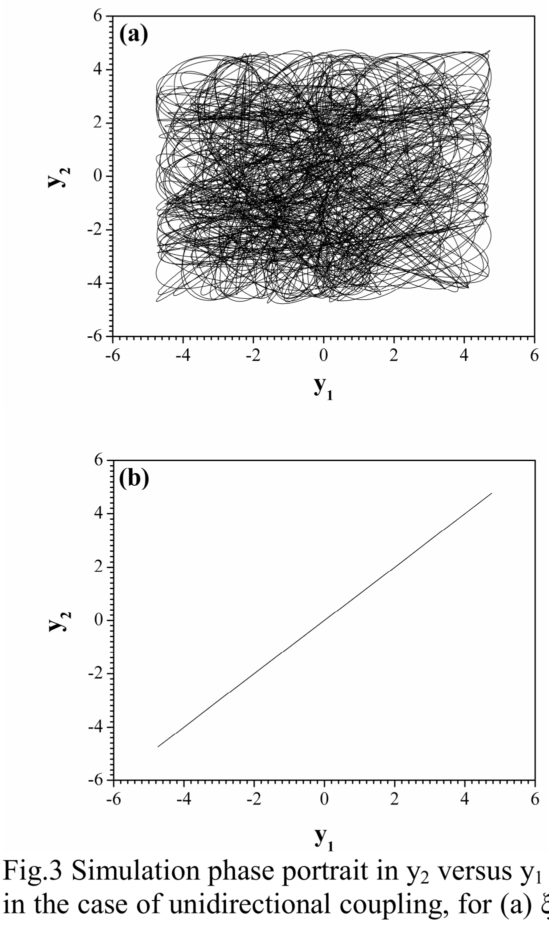

Fig.3 Simulation phase portrait in y_2 versus y_1 plane, in the case of unidirectional coupling, for (a) $\xi = 0.1$ (full desynchronization) and (b) $\xi = 1$ (full synchronization).

where ξ is the coupling coefficient. In system's equations, the first three equations describe the first of the two unidirectional coupled economic systems, while the other three describe the second one.

In Fig.2 the bifurcation diagram of $(y_2 - y_1)$ versus ξ is shown. From this diagram a classical transition from the full desynchronization, for low values of coupling factor ($0 < \xi \leq 0.14$), to full synchronization for higher values of coupling factor ($\xi > 0.14$), is confirmed. This is a very common dynamic behavior, especially in the case of unidirectional coupling, because as the coupling factor is increased from zero, there is a desynchronized state in which each one of the coupled systems is in different chaotic states. So, each one of the economies, which are described by the system (4), follow its growth path independently of the coupling. However, as the coupling factor becomes greater than the critical value of $\xi_{cr} = 0.14$ a full chaotic synchronization is observed. In this case, the strong coupling of the economies results to the convergence of their growth paths. So, the influence of the first system to the other, through the capital inflow, drives the coupled system to identical chaotic behavior. Phase portraits of y_2 versus y_1 of Figs. 3(a) & 3(b) confirm the two distinct dynamic behaviors (desynchronization and synchronization).

4.1 Bidirectional Coupling

The system of two bidirectional or mutual coupled systems of Eq.(4) is described by the following set of differential equations:

$$\begin{cases} \dot{x}_1 = m \cdot y_1 + p \cdot x_1 \cdot \left(d - y_1^2\right) \\ \dot{y}_1 = -x_1 + c \cdot z_1 \\ \dot{z}_1 = s \cdot x_1 - r \cdot y_1 + \xi \cdot (z_2 - z_1) \\ \dot{x}_2 = m \cdot y_2 + p \cdot x_2 \cdot \left(d - y_2^2\right) \\ \dot{y}_2 = -x_2 + c \cdot z_2 \\ \dot{z}_2 = s \cdot x_2 - r \cdot y_2 + \xi \cdot (z_1 - z_2) \end{cases} \tag{8}$$

The coupling coefficient ξ is present in the equations of both systems, since the coupling between them is mutual.

Always the bidirectional coupling between identical nonlinear systems produces more complex dynamic behavior for the system because both the coupled systems influence each other. So, in this case of coupling a variety of dynamical behaviors, including various types of synchronization phenomena and regions of desynchronization depending on the coupling factor and the initial conditions, are observed. Also, in many systems the final state, for a given coupling strength, may be different for the same of system's parameters but for different initial conditions. This phenomenon, that

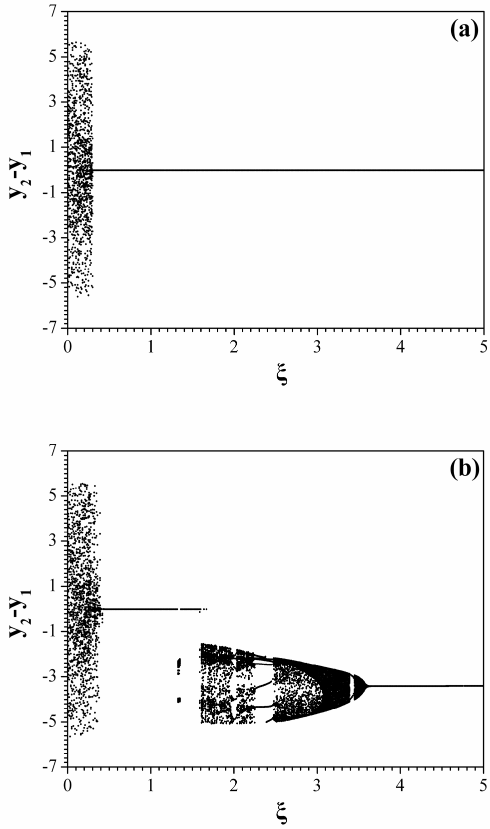

Fig.4 Bifurcation diagrams of ($y_2 - y_1$) versus ξ, in the case of (a) different and (b) same initial conditions, in each iteration.

influences significantly the dynamic behavior of the system, is the multistability [41].

In order to study how the coupled economic systems depends on the coupling factor and the initial conditions, two bifurcation diagrams have been produced numerically with different methods. In the first diagram (Fig.4(a)), which is produced with the same technique as in unidirectional coupling, the coupling factor is increased from $\xi = 0$ (uncoupled systems) to $\xi = 5$ with step $\Delta\xi = 0.01$, while initial conditions in each iteration have different values. This occurs because the last initial conditions in previous iteration become the first in the next iteration. So, in this case a classical transition from full desynchronization to full synchronization, for $\xi \geq 0.3$, is observed. This behavior is exactly the same as in unidirectional coupling but for greater value of the coupling factor.

In the second bifurcation diagram (Fig.4(b)) the initial conditions in each iteration remain the same. This method reveals the richness of the dynamics of the coupled system, which is due to the phenomenon of multistability. The full synchronization, which is observed in the previous bifurcation diagram (Fig.4(a)) coexists with various

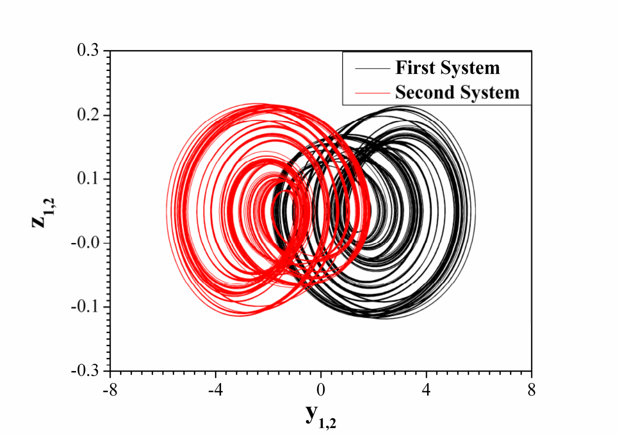

Fig.5 Simulation phase portrait in z versus y plane. Antisymmetric single-scroll chaotic attractors are observed.

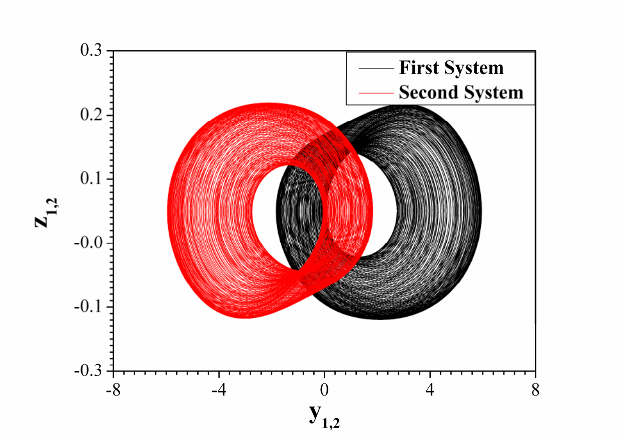

Fig.6 Simulation phase portrait in z versus y plane. Antisymmetric quasiperiodic attractors are observed.

other dynamic behaviors, such as chaotic desynchronization, periodic and quasiperiodic states.

In details, the system is again in a chaotic desynchorinzation state, for $0 < \xi \leq 0.42$. For greater values of coupling factor $(0.42 < \xi \leq 1.58)$ the system remains in a chaotic synchronization state through which it inserts to another desynchronization state $(1.58 < \xi \leq 2.25)$. In this region of values of the coupling factor each one of the coupled circuits produces single-scroll chaotic attractors which are mirrored to each other (Fig.5). The chaotic behavior in this region is confirmed by the calculation of system's Lyapunov exponents: (LE_1, LE_2, LE_3, LE_4, LE_5, LE_6) = (0.04920, 0, –0.75382, –3.0541, –4.74163, –4.99227). So, as it is known from the theory, if the system's Lyapunov exponents are: one positive, one zero and the other negative, then the system is in chaotic state. The specific behavior in this region is due to the increase of the coupling strength which produces the reduction of the size of the double-scroll attractor of each coupled circuit and finally to the observation of antisymmetric single-scroll chaotic attractors. Furthermore, this region is interrupted by smaller windows of periodic behavior, which is a very common feature in similar coupled systems.

As the coupling coefficient increases the system passes through a periodic window to a region of quasiperiodic state $(2.25 < \xi \leq 3.60)$ as the two largest Lyapunov exponents are zero, i.e. for $\xi = 3$, (LE_1, LE_2, LE_3, LE_4, LE_5, LE_6) = (0, 0, –0.33267, –2.9451, –3.81573, –8.77741). In this region each one of the coupled economic systems shows antisymmetric quasiperiodic attractors (Fig.6).

In 2010 an interesting phenomenon between coupled identical double-scroll chaotic circuits is observed. When that coupled system was in quasiperiodic state the signal of the first circuit (x_1) and the opposite signal of the second circuit ($-x_2$) are approximately equal with a time lag τ. This phenomenon is called "Inverse Lag Synchronization" [42]. Exactly the same behavior is observed in this coupled economic system for the region of quasiperiodic behavior.

In order to quantify the possible lag time τ between the similar variables y_1 and y_2 of the coupled economic systems the well known similarity function S is used [27]:

$$S(\tau) = \sqrt{\frac{\left\langle \left[y_2'(t+\tau) - y_1(t) \right]^2 \right\rangle}{\left[\left\langle (y_1(t))^2 \right\rangle \times \left\langle (y_2'(t))^2 \right\rangle \right]^{1/2}}} \qquad (6)$$

where, $y_2' = -y_2$. When the condition: $S_{min} = 0$ for $\tau_{min} \neq 0$, is satisfied, the system is in Inverse Lag Synchronization. So, the normalized time τ_{min} for which $S_{min} = 0$ is calculated, $\tau_{min} = 22.11$, from the similarity function (Fig.7), for $\xi = 3$. As a conclusion for this region, it must be mentioned that variables y, the GDP of each economic system, are approximately equal with a time lag, which is confirmed of the waveform of $y_1 + y_2(t + \tau_{min}) \approx 0$ (Fig.8).

Finally, the coupled system after the region of quasiperiodic behavior inserts to a stable periodic region (period-1). In this case each one of the coupled economic systems shows periodic limit cycles (Fig.9), which coexist with the phenomenon of full chaotic synchronization. Also, the variable y_1 and the opposite of y_2 are synchronized with a time lag, $\tau_{min} = T/2$, where T is a period of y_1 and y_2 (Fig.10). This phenomenon is called Inverse π-Lag Synchronization [33]. Furthermore, for the calculation of the similarity function of y_1 and $-y_2$,

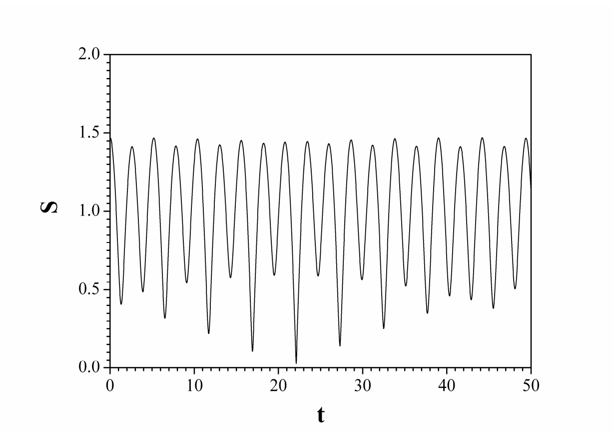

Fig.7 Similarity function (S) versus normalized time (t), for $\xi = 3$. Time lag $\tau_{min} = 22.11$ is calculated.

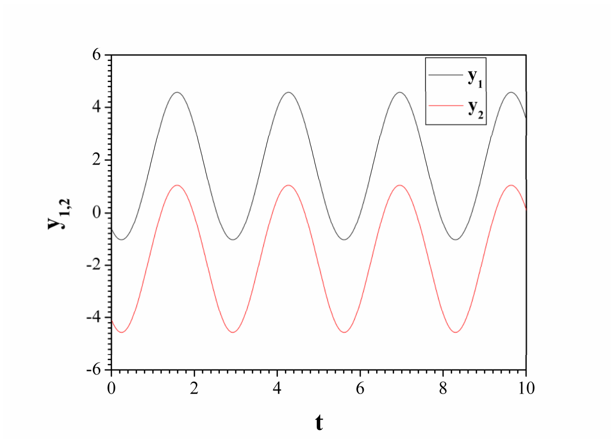

Fig.10 Waveforms of y_1 (black line) and y_2 (red line), for $\xi = 4$. Inverse π-Lag Synchronization is observed.

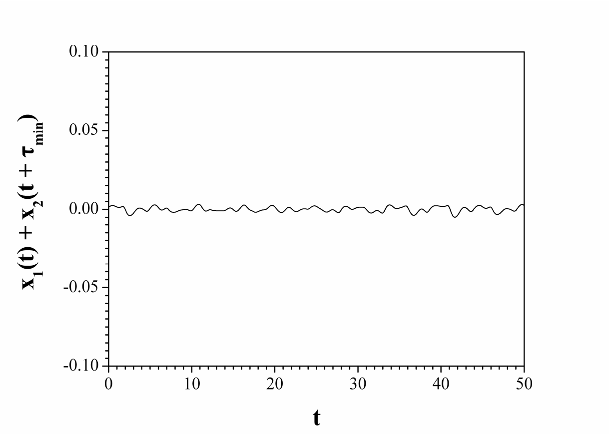

Fig.8 Waveform of $y_1 + y_2(t + \tau_{min})$, for $\xi = 3$. Inverse Lag Synchronization is confirmed.

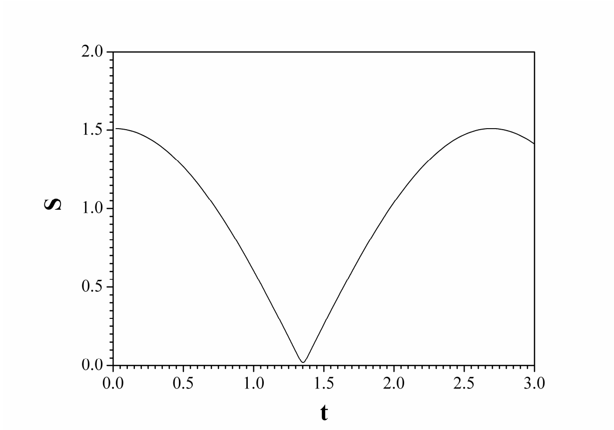

Fig.11 Similarity function (S) versus normalized time (t), for $\xi = 4$. Time lag is: $\tau_{min} = T/2 = 1.34$.

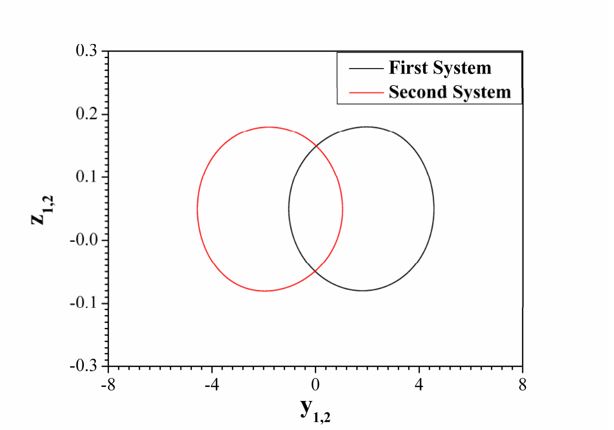

Fig.9 Simulation phase portrait in z versus y plane. Antisymmetric periodic attractors are observed.

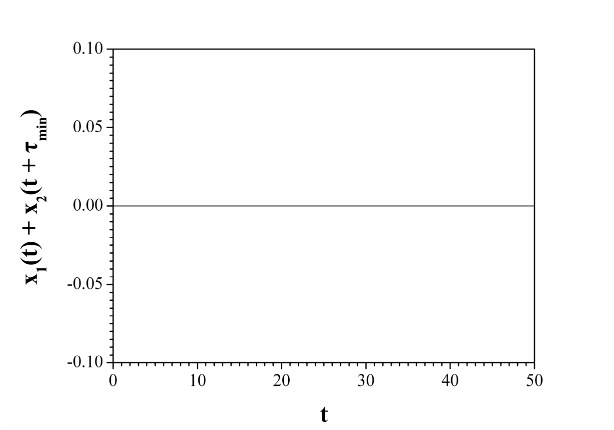

Fig.12 Waveform of $y_1 + y_2(t + \tau_{min})$, for $\xi = 4$. Inverse π-Lag Synchronization is confirmed.

for $\xi = 4$, the above mentioned phenomenon is confirmed (Fig.11), as the value of $y_1 + y_2(t + \tau_{min})$ is equal to zero (Fig.12).

So, the system of coupled economic systems, for values of ξ greater than 3.6, behaves periodically with a specific way. The phenomenon of Inverse π-Lag Synchronization, for the economic point of view, describes the fact that the two economic systems are bounded to a space around the equilibrium points E_1 and E_2. For this reason two set of initial conditions with opposite signs have been chosen. Also, the property, that the proposed economic system (4) is odd-symmetric, plays a crucial role to the observation of this phenomenon. Therefore, the growth's paths of each coupled economic system are periodic limit cycles antisymmetric to each other with a time lag. As a result the normalized time-series of the Gross Domestic Product (Fig.10) shows clearly that each economic system locks in antisymmetric regions showing the same periodic behavior without the systems having the ability to escape from this situation.

So, to get away the economic systems from this situation, two are the choices. The first is the interruption of the coupling between them so that each one of the economic system returns to its previous chaotic behavior. The second is the use of an external shock to one of the two coupled economic systems, so that the set of variables of the system changes and the system goes to the other possible dynamic state, which is the chaotic synchronization. In real world such external shocks may be political events, natural catastrophes, technological innovations, consumption jumps e.t.c.

5 Conclusion

In this work an interesting case of coupling between two nonlinear systems of identical economic cycles, was presented. So, this work is the first step in the study of coupled economic models, which are very prevalent nowadays due to the globalization, by using tools of nonlinear theory. The proposed coupling, which represents the capital inflow between the coupled economic systems, was studied by using two different methods, the unidirectional and bidirectional coupling.

In the first case of coupling (unidirectional), the coupled system passes from full desynchronization, in which each system follows its own growth path, to a full chaotic synchronization. This type of synchronization, is developed when the coupling of the economies is strong, which results to the convergence of their growth paths. So, the influence of the first economic system to the other, through the capital inflow, drives the coupled system to a synchronized behavior.

In the case of bidirectional coupling each one of the coupled economic systems influences the dynamics of the other through the capital inflow. Many other interesting dynamic behaviors, except of the full synchronization, for different set of initial conditions, were observed. Full chaotic desynchronization of single-scroll attractors, periodic states and quasiperiodic states, were the observed phenomena. However, the most interesting case is the recently new proposed Inverse π-Lag Synchronization, in which the system results for strong coupling. From the economic point of view we are led to the conclusion that the growth's paths of each coupled economic system are locked to antisymmetric periodic states with a time lag. This behavior is not desirable especially for the system with negative economic aggregates. So, as it is mentioned, two are the solutions: the interruption of the coupling or the use of an external shock in one of the coupled economic system.

As a future work, the interaction between non-identical economic systems of this type, which is more close to the reality, will be studied. In an economical interconnected world a great need of explaining the results of economic coupling has been risen. So, this coupling between economies with different characteristics, such as in the case of the European Union, would be an interesting case. It is known that the EU includes countries with similar economic aggregates and other countries with large differences. So, the use of different values of parameters in each coupled economic system, which correspond to the real characteristics of each economy (marginal propensity to saving, ratio of capitalized profit, value of GDP, etc) may explain from the physical point of view many of the structural problems of these coupled economic systems, which finally may lead to economic crisis.

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