Robust Recursive Least-Squares Wiener Fixed-Interval Smoother Based on Innovation Approach in Linear Continuous-Time Stochastic Systems with Uncertainties

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Abstract: - This study develops a robust recursive least-squares (RLS) Wiener fixed-interval (FI) smoother by exploiting covariance information for linear continuous-time systems that face uncertainties in both their system and observation matrices. Uncertainties in the state-space model cause degradations in the signal and observed values. The robust FI smoothing and filtering methods introduced do not assume that the system and the observation matrix have norm-bounded uncertainties. An observable companion form represents the state space model of the degraded signal. Robust RLS FI smoothing is to minimize the mean-square value of the smoothing errors of the system state over a fixed interval. Section 3 introduces an integral equation satisfied by the impulse response function that is optimal for robust FI smoothing estimation of the system state. An integral equation for the impulse response function, which provides a filtering estimate of the state of the degraded system, is also shown. Theorem 1 presents the robust RLS FI smoothing and filtering algorithm for the signal and the system state using covariance information. Theorem 2 presents the robust RLS Wiener (RLSW) FI smoothing and filtering algorithm for the signal and the system state. Robust RLS FI smoother outperforms robust RLS filter in estimation accuracy, as shown by the FI smoothing error covariance function in Section 5. Numerical simulation examples demonstrate that the robust RLSW FI smoother achieves superior signal estimation accuracy compared to the robust RLSW filter.

Key-Words: - Robust RLS Wiener fixed-interval smoother, innovation process, minimum-variance unbiased smoother, degraded system, observable companion form, covariance information, Wiener-Hopf integral equation, semi-degenerate kernel, orthogonal projection lemma.

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1 Introduction

The problem of estimation has received a lot of attention, e.g. in [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]. The classification of estimators includes filter, smoothers, and predictor within both linear and nonlinear stochastic frameworks. One can divide the smoothers into the fixed-lag (FL), fixed-point (FP), and fixed-interval (FI) smoothers. Conventional Kalman estimation requires information from a state-space model. Researchers in [3], [12] developed robust estimators under the assumption that the uncertain components of the system and observation matrices are norm-bounded.

This study addresses the robust recursive least squares Wiener (RLSW) FI smoothing problem. The FI smoother has been studied in continuous-time stochastic systems [13] and discrete-time stochastic systems [14], [15]. The FI smoothing algorithms in [1], [2], [3], [13], [14], [15] use the state-space model. Robust estimators in [3], [12] estimate the signal or state of a system with uncertain components in the system and observation matrices. The RLSW filter [16] has an advantage over the Kalman filter because it uses neither the input matrix nor the covariance of the input noise in the state-space model. In [17], [18], [19], [20], [21], [22], robust sequential fusion Kalman estimation techniques are studied for networked uncertain sensor systems.

In [23], the robust RLSW fixed-interval smoother is proposed in linear discrete-time stochastic systems with uncertainties. [24] proposes a robust RLSW filter for linear continuous-time stochastic systems with uncertain parameters. To improve the estimation accuracy of the robust RLSW filter, this paper designs a robust FI smoother for linear continuous-time stochastic systems with uncertainties. Uncertain parameters are assumed to exist in the system and observation matrices. A linear integral transformation of the innovation process for the degraded system with uncertain parameters expresses the FI smoothing estimate for the system state. From the orthogonal projection lemma, the optimal impulse response function satisfies the Wiener-Hopf integral equation. With stochastic properties, the Wiener-Hopf integral equation results in the second kind Volterra type integral equation for the optimal impulse response function. The integral equation for the impulse response function, which provides a filtering estimate of the state of the degraded system, is also shown. A robust RLSW FI smoothing and filtering algorithm is derived to estimate the signal and the system state from the two integral equations. The observable companion form represents the state-space model of the degraded signal. Here, estimates of the system and observation matrices of the degraded system are calculated based on the method in [24]. Theorem 1 represents the robust RLS FI smoothing and filtering algorithm for the signal $z(t)$ and the system state $x(t)$. In addition to the degraded observation matrix $\breve{H}$, the algorithm uses information from the cross-covariance function $K_{x\breve{x}}(t,s)$ of the system state $x(t)$ with the state $\breve{x}(s)$ of the degraded system and the covariance function $K_{\breve{x}}(t,s)$ of the state $\breve{x}(t)$ of the degraded system. In this context, $K_{x\breve{x}}(t,s)$ and $K_{\breve{x}}(t,s)$ are represented in the semi-degenerate kernel form. Based on Theorem 1, Theorem 2 presents the robust RLSW FI smoothing and filtering algorithm for the signal $z(t)$ and the system state $x(t)$.The algorithm uses covariance information $K_{x\breve{x}}(t,t)$ and $\breve{K}(t,t)$, the degraded observation matrix $\breve{H}$, the system matrix Φ, and the degraded system matrix $\breve{\Phi}$. Numerical simulation examples show that the robust RLSW FI smoother achieves better signal estimation accuracy than the robust RLSW filter.

The remainder of this paper is organized as follows. Section 2 presents the nominal and degraded state-space models and the realization of the degraded system. Section 3 introduces the robust FI smoothing problem. Section 4 presents the robust RLS FI smoothing and filtering algorithm using the covariance information in Theorem 1 and the robust RLSW FI smoothing and filtering algorithm in Theorem 2. Section 5 presents the estimation error covariance functions for the robust FI smoother and filter in Theorem 1. Section 6 provides numerical simulation examples.

2 Nominal and Degraded State-Space Models and Realization of Degraded System

Let the state-space model for the system state $x(t)$ and the observed value $y(t)$ be given by (1).

$$
\begin{aligned}
&y(t)=z(t)+v(t), z(t)=Hx(t),\\
&\frac{dx(t)}{dt}=\Phi x(t)+\Gamma w(t), x(0)=c,\\
&E[v(k)v^T(s)]=R\delta(t-s), R>0,\\
&E[w(t)w^T(s)]=Q\delta(t-s), Q>0,\\
&E[v(t)w^T(s)]=0, E[x(0)w^T(t)]=0
\end{aligned}
\tag{1}
$$

Here, $x(t)\in R^n$ is the system state, and $z(t)\in R^m$ is the signal. The input noise $w(t)\in R^l$ and the observation noise $v(t)$ are independent white Gaussian noises with zero mean. Γ is the $n\times l$ input matrix, and H is the $m\times n$ observation matrix. The covariance functions for the input noise $w(t)$ and the observation noise $v(t)$ are described in (1). The state and observation equations, which include uncertain parameters, are provided in (2).

$$
\begin{aligned}
&\breve{y}(t)=\breve{z}(t)+v(t),\\
&\breve{z}(t)=\overleftrightarrow{H}(t)\overleftrightarrow{x}(t), \overleftrightarrow{H}(t)=H+\Delta H(t),\\
&\frac{d\overleftrightarrow{x}(t)}{dt}=\overleftrightarrow{\Phi}(t)\overleftrightarrow{x}(t)+\Gamma w(t),\\
&\overleftrightarrow{\Phi}(t)=\Phi+\Delta\Phi(t), \overleftrightarrow{x}(0)=\overleftrightarrow{c},\\
&\mathrm{E}[\mathrm{v}(\mathrm{t})w^T(\mathrm{s})]=0, \mathrm{E}[\Delta\Phi(\mathrm{t})w^T(\mathrm{s})]=0,\\
&E[\Delta \mathrm{C}(t)v^T(s)]=0, E[\overleftrightarrow{x}(0)w^T(t)]=0,\\
&E[\overleftrightarrow{x}(0)v^T(t)]=0
\end{aligned}
\tag{2}
$$

In (2), $\overleftrightarrow{\Phi}(t)$ and $\overleftrightarrow{H}(t)$ represent the degraded system matrix and the degraded observation matrix, respectively. $\Delta \mathrm{A}(t)$ and $\Delta H(t)$ are the uncertain matrices. The initial state of the degraded system, $\overleftrightarrow{x}(0)$, is a random vector that is uncorrelated with both the system input noise $w(t)$ and the measurement noise $v(t)$.

Assume that the degraded signal is represented as $\breve{z}(t)=\breve{H}\breve{x}(t)$ using the degraded state vector $\breve{x}(t)$, where $\breve{x}(t)$ has n components.

$$
\begin{aligned}
&\breve{z}(t)=\breve{H}\breve{x}(t), \breve{z}(t)=\breve{x}_1(t),\\
&\breve{H}=\begin{bmatrix} I_{m\times m} & 0 & 0 & \cdots & 0\end{bmatrix},\\
&\breve{x}(t)=\begin{bmatrix}\breve{x}_1(t)\\ \breve{x}_2(t)\\ \vdots\\ \breve{x}_{n-1}(t)\\ \breve{x}_n(t)\end{bmatrix}
\end{aligned}
\tag{3}
$$

Let $\breve{x}_1(t)$ satisfy a differential equation

$$\frac{d\breve{x}_1^n(t)}{dt^n} = -\breve{\alpha}_1 \frac{d\breve{x}_1^{n-1}(t)}{dt^n} - \breve{\alpha}_2 \frac{d\breve{x}_1^{n-2}(t)}{dt^{n-2}} \cdots - \breve{\alpha}_{n-1} \frac{d\breve{x}_1(t)}{dt} - \breve{\alpha}_n \breve{x}_1(t) + \xi(t). \quad (4)$$

(4) is transformed into the state differential equations in the observable companion form:

$$\frac{d\breve{x}(t)}{dt} = \breve{\Phi}\breve{x}(t) + \breve{\Gamma}\xi(t),$$
$$E[\xi(t)\xi^T(s)] = \breve{Q}\delta(t-s),$$
$$\breve{\Phi} = \begin{bmatrix} 0 & I_{m\times m} & 0 & \cdots & 0 \\ 0 & 0 & I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m\times m} \\ -\breve{\alpha}_n & -\breve{\alpha}_{n-1} & -\breve{\alpha}_{n-2} & \cdots & -\breve{\alpha}_1 \end{bmatrix}, \quad (5)$$
$$\breve{\Gamma} = [0 \quad 0 \quad \cdots \quad 0 \quad I_{I_{m\times m}}]^T,$$

where $\xi(t)$ represents the residual to approximate the degraded signal $\breve{z}(t)$. The degraded system matrix $\breve{\Phi}$ is estimated using (6) [24].

$$\breve{\Phi} = E\left[\frac{d\breve{x}(t)}{dt}\breve{x}^T(s)\right]E[\breve{x}(s)\breve{x}^T(s)]^{-1},$$
$$0 \le s < t,$$
$$\breve{x}^T(t) = [\breve{x}_1(t) \quad \breve{x}_2(t) \quad \cdots \quad \breve{x}_{n-1}(t) \quad \breve{x}_n(t)], \quad (6)$$
$$\breve{x}_1(t) = \breve{z}(t), \breve{x}_2(t) = \frac{d\breve{z}(t)}{dt},$$
$$\cdots, \breve{x}_{n-1}(t) = \frac{d^{n-2}\breve{z}(t)}{dt^{n-2}}, \breve{x}_n(t) = \frac{d^{n-1}\breve{z}(t)}{dt^{n-1}}$$

Also, the degraded observation matrix $\breve{H}$ is estimated by

$$\breve{H} = E[\breve{y}(t)\breve{x}^T(t)]E[\breve{x}(t)\breve{x}^T(t)]^{-1} \quad (7)$$

[24].

3 Robust RLS FI Smoothing Problem

Assume that (8) provides a FI smoothing estimate, denoted by $\hat{x}(t,T)$, of the system state $x(t)$, as a linear transformation of the innovation process $\breve{\upsilon}(\tau) = \breve{y}(\tau) - \breve{H}\hat{\breve{x}}(\tau,\tau), \quad 0 \le \tau \le T$. $\hat{\breve{x}}(\tau,\tau)$ represents the filtering estimate of the state $\breve{x}(\tau)$ of the degraded system.

$$\hat{x}(t,T) = \int_0^T g(t,\tau)\breve{\upsilon}(\tau)d\tau \quad (8)$$

Here, $g(t,\tau)$ is the impulse response function. Let us consider minimizing the mean-square value

$$J = E[(x(t) - \hat{x}(t,T))^T(x(t) - \hat{x}(t,T))] \quad (9)$$

of the FI smoothing error $x(t) - \hat{x}(t,T)$. The FI smoothing estimate $\hat{x}(t,T)$ that minimizes the cost function J satisfies the relationship

$$x(t) - \hat{x}(t,T) \perp \breve{\upsilon}(s), 0 \le t,s \le T \quad (10)$$

from the orthogonal projection lemma [24]. Here, " ⊥ " denotes the notation of orthogonality. Hence, the optimal impulse response function satisfies the Wiener-Hopf integral equation.

$$E[x(t)\breve{\upsilon}^T(s)] = \int_0^T g(t,\tau)E[\breve{\upsilon}(\tau)\breve{\upsilon}^T(s)]d\tau \quad (11)$$

Since $E[\breve{\upsilon}(\tau)\breve{\upsilon}^T(s)] = R\delta(\tau - s)$, (11) is rewritten as:

$$g(t,s)R = E[x(t)\breve{\upsilon}^T(s)]. \quad (12)$$

Let the filtering estimate $\hat{\breve{x}}(t,t)$ of the degraded system state $\breve{x}(t)$ be given by:

$$\hat{\breve{x}}(t,t) = \int_0^t g_0(t,\tau)\breve{\upsilon}(\tau)d\tau \quad (13)$$

as a linear transformation of the innovation process $\breve{\upsilon}(\tau), 0 \le \tau \le t$. $E[x(t)\breve{\upsilon}^T(s)]$ in (12) is developed as follows:

$$E[x(t)\breve{\upsilon}^T(s)] = E\left[x(t)(\breve{y}(s) - \breve{H}\hat{\breve{x}}(s,s))^T\right]$$
$$= K_{x\breve{y}}(t,s)$$
$$- \int_0^s E[x(t)\breve{\upsilon}^T(\tau)]\, {g_0}^T(s,\tau)\breve{H}^T d\tau \quad (14)$$
$$= K_{x\breve{y}}(t,s) - \int_0^s g(t,\tau)R\, {g_0}^T(s,\tau)\breve{H}^T d\tau.$$

Hence, the optimal impulse response function $g(t,s)$ satisfies:

$$g(t,s)R = K_{x\breve{y}}(t,s) - \int_0^s g(t,\tau)R\, {g_0}^T(s,\tau)\breve{H}^T d\tau. \quad (15)$$

Let the cross-covariance function $K_{x\breve{x}}(t,s)$ of the system state $x(t)$ with the degraded system state $\breve{x}(s)$ be given by (16) in the semi-degenerate kernel form [24].

$$K_{x\breve{x}}(t,s) = \begin{cases} \alpha(t)\beta^T(s), 0 \le s \le t, \\ \gamma(t)\delta^T(s), 0 \le t \le s, \end{cases}$$
$$\alpha(t) = e^{\Phi t}, \beta^T(s) = e^{-\Phi s}K_{x\breve{x}}(s,s), \quad (16)$$
$$\gamma(t) = K_{x\breve{x}}(t,t)e^{-\Phi^T t}, \delta^T(s) = e^{\Phi^T s}$$

Let the covariance function $K_{\breve{x}}(t,s)$ of the degraded system state $\breve{x}(t)$ be given by (17) in the semi-degenerate kernel form [24].

$$\breve{K}(t,s) = \begin{cases} \breve{A}(t)\breve{B}^T(s), 0 \le s \le t, \\ \breve{B}(t)\breve{A}^T(s), 0 \le t \le s, \end{cases} \quad (17)$$
$$\breve{A}(t) = e^{\breve{\Phi} t}, \breve{B}^T(s) = e^{-\breve{\Phi} s}K_{\breve{x}}(s,s)$$

Similarly to the derivation of (15) for the optimal impulse response function $g(t,s)$, it is observed that the optimal impulse response function $g_0(t,s)$ in (13) satisfies:

$$g_0(t,s)R = \breve{K}(t,s)\breve{H}^T - \int_0^t g_0(t,\tau)R\, g_0{}^T(s,\tau)\breve{H}^T d\tau. \quad (18)$$

Starting from (15) and (18), Theorem 1 proposes the robust FI smoothing and filtering algorithm for the signal $\boldsymbol{z(t)}$ and the system state $\boldsymbol{x(t)}$. Besides the degraded observation matrix $\breve{\boldsymbol{H}}$, the algorithm in Theorem 1 uses the covariance information $\boldsymbol{K_{x\breve{x}}(t,s)}$ from (16) and $\breve{\boldsymbol{K}}\boldsymbol{(t,s)}$ from (17). Based on Theorem 1, Theorem 2 presents the robust RLSW FI smoothing and filtering algorithm for the signal $\boldsymbol{z(t)}$ and the system state $\boldsymbol{x(t)}$. Besides $\boldsymbol{\Phi}$, $\breve{\boldsymbol{H}}$, and $\breve{\boldsymbol{\Phi}}$, the use of covariance information $\boldsymbol{K_{x\breve{x}}(t,t)}$ and $\breve{\boldsymbol{K}}\boldsymbol{(t,t)}$ characterizes the robust RLSW FI smoothing and filtering algorithm in Theorem 2.

4 Robust RLSW FI Smoothing and Filtering Algorithm

Theorem 1 presents the robust RLS FI smoothing and filtering algorithm for the signal $z(t)$ and the system state $x(t)$. In addition to the degraded observation matrix $\breve{H}$, the algorithm uses the covariance information $K_{x\breve{x}}(t,s)$ from (16) and $\breve{K}(t,s)$ from (17).

Theorem 1 *Let the state-space model for the system state $x(t)$ and the observed value $y(t)$ be given by (1). Let the state-space model for the state $\breve{x}(t)$ of the degraded system and the degraded observed value $\breve{y}(t)$ be given by (2). Let the degraded signal be given by $\breve{z}(t)=\breve{H}\breve{x}(t)$ in (3) and the state differential equations for the state $\breve{x}(t)$ of the degraded system be given by (5). Let the cross-covariance function $K_{x\breve{x}}(t,s)$ of the system state $x(t)$ with the degraded observed value $\breve{x}(s)$ be given by (16). Let the covariance function $\breve{K}(t,s)$ of the state $\breve{x}(t)$ of the degraded system be given by (17). Then, the robust RLS FI smoothing and filtering algorithm for the signal $z(t)$ and the system state $x(t)$ consists of (19)- (31).*

Filtering estimate of the signal $z(t)$: $\hat{z}(t,t)$

$$\hat{z}(t,t)=H\hat{x}(t,t) \quad (19)$$

FI smoothing estimate of the signal $z(t)$: $\hat{z}(t,T)$

$$\hat{z}(t,T)=H\hat{x}(t,T) \quad (20)$$

FI smoothing estimate of the system state $x(t)$: $\hat{x}(t,T)$

$$\hat{x}(t,T)=\hat{x}(t,t)+\gamma(t)q_1(t,T) -\alpha(t)r_{13}(t)q_2(t,T) \quad (21)$$

Partial differential equation for $q_1(t,T)$ in the backward direction of t:

$$\frac{\partial q_1(t,T)}{\partial t} = -\delta^T(t)\breve{H}^T R^{-1}(\breve{y}(t)-\breve{H}\hat{\breve{x}}(t,t)) +\delta^T(t)\breve{H}^T J_3^T(t)q_2(t,T), \quad q_1(T,T)=0 \quad (22)$$

Partial differential equation for $q_2(t,T)$ in the backward direction of t:

$$\frac{\partial q_2(t,T)}{\partial t} = -\breve{A}^T(t)\breve{H}^T R^{-1}(\breve{y}(t)-\breve{H}\hat{\breve{x}}(t,t)) +\breve{A}^T(t)\breve{H}^T J_3^T(t)q_2(t,T), \quad q_2(T,T)=0 \quad (23)$$

Filtering estimate of the state $\breve{x}(t)$ for the degraded system: $\hat{\breve{x}}(t,t)$

$$\hat{\breve{x}}(t,t)=\breve{A}(t)e_3(t) \quad (24)$$

Differential equation for $e_3(t)$:

$$\frac{de_3(t)}{dt}=J_3(t)\left(\breve{y}(t)-\breve{H}\hat{\breve{x}}(t,t)\right), \quad e_3(0)=0 \quad (25)$$

Equation for $J_3(t)$:

$$J_3(t)=(\breve{B}^T(t)\breve{H}^T-r_{33}(t)\breve{A}^T(t)\breve{H}^T)R^{-1} \quad (26)$$

Differential equation for $r_{33}(t)$:

$$\frac{dr_{33}(t)}{dt}=J_3(t)RJ_3{}^T(t), \quad r_{33}(0)=0 \quad (27)$$

Filtering estimate of the system state $x(t)$: $\hat{x}(t,t)$

$$\hat{x}(t,t)=\alpha(t)e_1(t) \quad (28)$$

Differential equation for $e_1(t)$:

$$\frac{de_1(t)}{dt}=J_1(t)(\breve{y}(t)-\breve{H}\hat{\breve{x}}(t,t)), \quad e_1(0)=0 \quad (29)$$

Equation for $J_1(t)$:

$$J_1(t)=(\beta^T(t)\breve{H}^T-r_{13}(t)\breve{A}^T(t)\breve{H}^T)R^{-1} \quad (30)$$

Differential equation for $r_{13}(t)$:

$$\frac{dr_{13}(t)}{dt}=J_1(t)RJ_3{}^T(t), \quad r_{13}(0)=0 \quad (31)$$

Appendix A presents the proof of Theorem 1.

The partial differential equation for $q_2(t,T)$ in (23) is asymptotically stable if and only if the real part of each eigenvalue of the matrix $\breve{A}^T(t)\breve{H}^T J_3^T(t)$ is negative.

Based on Theorem 1, Theorem 2 presents the robust RLSW FI smoothing and filtering algorithm for the signal $z(t)$ and the system state $x(t)$. In addition to the observation matrix $\breve{H}$, the system matrix Φ, and the degraded system matrix $\breve{\Phi}$, the algorithm uses the covariance information $K_{x\breve{x}}(t,t)$ and $\breve{K}(t,t)$.

Theorem 2 *Let the state-space model for the system state $x(t)$ and the observed value $y(t)$ be given by (1). Let the state-space model for the state $\breve{x}(t)$ of the degraded system and the degraded observed value $\breve{y}(t)$ be given by (2). Let the degraded signal be given by $\breve{z}(t)=\breve{H}\breve{x}(t)$ in (3) and the state differential equations for $\breve{x}(t)$ be given by (5). Let the cross-covariance function $K_{x\breve{x}}(t,t)$ of the system state $x(t)$ with the degraded system state $\breve{x}(t)$ and the covariance function $\breve{K}(t,t)$ of the degraded system state $\breve{x}(t)$ be given, in addition to the observation matrix $\breve{H}$, the system matrix Φ, and the degraded system matrix $\breve{\Phi}$. Then, the robust RLSW FI smoothing and filtering algorithm for the signal $z(t)$ and the system state $x(t)$ consists of (32)-(40).*

Filtering estimate of the signal $z(t)$: $\hat{z}(t,t)$

$$\hat{z}(t,t)=H\hat{x}(t,t) \tag{32}$$

FI smoothing estimate of the signal $z(t)$: $\hat{z}(t,T)$

$$\hat{z}(t,T)=H\hat{x}(t,T) \tag{33}$$

FI smoothing estimate of the system state $x(t)$: $\hat{x}(t,T)$

$$\hat{x}(t,T)=\hat{x}(t,t)+K_{x\breve{x}}(t,t)\breve{q}_1(t,T) -S_{13}(t)\breve{q}_2(t,T) \tag{34}$$

Filtering estimate of the system state $x(t)$ $\hat{x}(t,t)$

$$\frac{d\hat{x}(t,t)}{dt}=\Phi\hat{x}(t,t) +(K_{x\breve{x}}(t,t)\breve{H}^T-S_{13}(t)\breve{H}^T) \times R^{-1}(\breve{y}(t)-\breve{H}\hat{\breve{x}}(t,t)),\quad \hat{x}(0,0)=0 \tag{35}$$

Filtering estimate of the state $\breve{x}(t)$ for the degraded system: $\hat{\breve{x}}(t,t)$

$$\frac{d\hat{\breve{x}}(t,t)}{dt}=\breve{\Phi}\hat{\breve{x}}(t,t) +(\breve{K}(t,t)\breve{H}^T-S_{33}(t)\breve{H}^T)^T \times R^{-1}(\breve{y}(t)-\breve{H}\hat{\breve{x}}(t,t)),\quad \hat{\breve{x}}(0,0)=0 \tag{36}$$

Partial differential equation for $\breve{q}_1(t,T)$:

$$\frac{\partial\breve{q}_1(t,T)}{\partial t}=-\breve{\Phi}^T\breve{q}_1(t,T) -\breve{H}^T R^{-1}\left(\breve{y}(t)-\breve{H}\hat{\breve{x}}(t,t)\right) +\breve{H}^T R^{-1}(\breve{K}(t,t)\breve{H}^T-S_{33}(t)\breve{H}^T)^T\breve{q}_2(t,T),\quad \breve{q}_1(t,T)=0 \tag{37}$$

$$\breve{q}_1(T,T)=0$$

Partial differential equation for $\breve{q}_2(t,T)$:

$$\frac{\partial\breve{q}_2(t,T)}{\partial t}=-\breve{\Phi}^T\breve{q}_2(t,T) -\breve{H}^T R^{-1}(\breve{y}(t)-\breve{H}\hat{\breve{x}}(t,t)) +\breve{H}^T R^{-1}(\breve{K}(t,t)\breve{H}^T-S_{33}(t)\breve{H}^T)^T\breve{q}_2(t,T),\quad \breve{q}_2(T,T)=0 \tag{38}$$

Differential equation for $S_{13}(t)$:

$$\frac{dS_{13}(t)}{dt}=\Phi S_{13}(t)+S_{13}(t)\breve{\Phi}^T +(K_{x\breve{x}}(t,t)\breve{H}^T-S_{13}(t)\breve{H}^T)R^{-1} \times(\breve{K}(t,t)\breve{H}^T-S_{33}(t)\breve{H}^T)^T,\quad S_{13}(0)=0 \tag{39}$$

Differential equation for $S_{33}(t)$:

$$\frac{dS_{33}(t)}{dt}=\breve{\Phi}S_{33}(t)+S_{33}(t)\breve{\Phi}^T +(\breve{K}(t,t)\breve{H}^T-S_{33}(t)\breve{H}^T)R^{-1} \times(\breve{K}(t,t)\breve{H}^T-S_{33}(t)\breve{H}^T)^T,\quad S_{33}(0)=0 \tag{40}$$

For the stability of the RLSW FI smoothing and filtering algorithm, the following conditions are required.

(1) The differential equation (35) for $\hat{x}(t,t)$ is asymptotically stable if and only if the real parts of the eigenvalues of the system matrix Φ are all negative.
(2) The differential equation (36) for $\hat{\breve{x}}(t,t)$ is asymptotically stable if and only if the real parts of the eigenvalues of the matrix $\breve{\Phi}-(\breve{K}(t,t)\breve{H}^T-S_{33}(t)\breve{H}^T)^T R^{-1}\breve{H}$ are all negative.
(3) The partial differential equation (37) for $\breve{q}_1(t,T)$ is asymptotically stable if and only if the real parts of the eigenvalues of the matrix $-\breve{\Phi}^T$ are all negative.
(4) The partial differential equation (38) for $\breve{q}_2(t,T)$ is asymptotically stable if and only if

the real parts of the eigenvalues of the matrix $-\breve{\Phi}^T + +\breve{H}^T R^{-1}(\breve{K}(t,t)\breve{H}^T - S_{33}(t)\breve{H}^T)^T$ are all negative.

Appendix B shows the proof of Theorem 2.

5 FI Smoothing Error Covariance Function

Let $K_x(t,t)$ be the covariance of $x(t)$, $\tilde{P}(t,T)$ be the FI smoothing error covariance function of $x(t)$, and $\tilde{P}(t,t)$ be the filtering error covariance function of $x(t)$. From (A-28), $\tilde{P}(t,T)$ is given by (41).

$$\begin{aligned}\tilde{P}(t,T) &= E[(x(t) - \hat{x}(\mathrm{t,T})) \\ &\times (x(t) - \hat{x}(\mathrm{t,T}))^T] \\ &= K_x(t,t) - E[\hat{x}(\mathrm{t,T})\hat{x}(\mathrm{t,T})^T] \\ &= K_x(t,t) - E[(\hat{x}(t,t) \\ &+ \int_t^T g(t,\tau)\breve{\nu}(\tau)d\tau) \\ &(\hat{x}(t,t) + \int_t^T g(t,\tau)\breve{\nu}(\tau)d\tau)^T] \\ &= \tilde{P}(t,t) - \int_t^T g(t,\tau)Rg^T(t,\tau)d\tau\end{aligned} \quad (41)$$

Since $\tilde{P}(t,T)$, $\tilde{P}(t,t)$, $K_x(t,t)$, and $\int_t^T g(t,\tau)Rg^T(t,\tau)d\tau$ are positive semidefinite matrices, it is seen that (42) is valid.

$$\tilde{P}(t,T) \le \tilde{P}(t,t) \quad (42)$$

(42) shows that the estimation accuracy of the robust FI smoother is superior to that of the robust filter. Substituting $g(t,s) = \alpha(t)J_1(s)$ in (A-13) into (41) yields (43).

$$\begin{aligned}\tilde{P}(t,T) &= \tilde{P}(t,t) \\ &-\alpha(t)\int_t^T J_1(\tau)RJ_1^T(\tau)d\tau\alpha^T(t)\end{aligned} \quad (43)$$

Introducing

$$S(t,T) = \int_t^T J_1(\tau)RJ_1^T(\tau)d\tau, \quad (44)$$

the FI smoothing error covariance function $\tilde{P}(t,T)$ becomes (45).

$$\tilde{P}(t,T) = \tilde{P}(t,t) - \alpha(t)S(t,T)\alpha^T(t) \quad (45)$$

Differentiating (44) regarding t gives a partial differential equation (46) for $S(t,T)$.

$$\frac{\partial S(t,T)}{\partial T} = J_1(T)RJ_1^T(T),\ S(t,t) = 0 \quad (46)$$

Here, (30) and (31) compute $J_1(t)$.

The filtering error covariance function $\tilde{P}(t,t)$ is given by (47).

$$\tilde{P}(t,t) = K_x(t,t) - E[\hat{x}(\mathrm{t,t})\hat{x}(\mathrm{t,t})^T] \quad (47)$$

Substitution of (A-14) into (47) yields (48).

$$\begin{aligned}\tilde{P}(t,t) &= K_x(t,t) - \\ &- \int_0^t \int_0^t g(t,\tau)E[\breve{\nu}(\tau)\breve{\nu}^T(\tau')] \\ &\times g^T(t,\tau')d\tau'd\tau \\ &= K_x(t,t) - \int_0^t g(t,\tau)Rg^T(t,\tau)d\tau\end{aligned} \quad (48)$$

Substitution of (A-13) into (48) and use of (44) gives (49).

$$\begin{aligned}\tilde{P}(t,t) &= K_x(t,t) \\ &-\alpha(t)\int_0^t J_1(\tau)RJ_1^T(\tau)\alpha^T(t)d\tau \\ &= K_x(t,t) - \alpha(t)S(0,t)\alpha^T(t)\end{aligned} \quad (49)$$

Here, $S(0,t)$ is given by (50).

$$S(0,t) = \int_0^t J_1(\tau)RJ_1^T(\tau)d\tau \quad (50)$$

Differentiating (50) with respect to t yields (51).

$$\frac{dS(0,t)}{dt} = J_1(t)RJ_1^T(t),\ \ S(0,0) = 0 \quad (51)$$

$J_1(t)$ is computed by (30) and (31).

6 Examples of Numerical Simulation

Example 1

Suppose that (52) contains the observation equation for the signal $z(t)$ and the second-order state differential equation for the system state $x(t)$ [25].

$$\begin{aligned}&y(t) = z(t) + v(t), z(t) = Hx(t), \\ &H = [1 \quad 0], \\ &\frac{dx(t)}{dt} = \Phi x(t) + \Gamma w(t), x(0) = c, \\ &x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ &\Phi = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \omega_n = \sqrt{3}, \\ &\zeta = \frac{2}{\omega_n}, \Gamma = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix}, \\ &E[v(k)v(s)] = R\delta(t-s), R > 0, \\ &E[w(t)w(s)] = Q\delta(t-s), Q = 1, \\ &E[v(t)w(s)] = 0, E[x(0)w(t)] = 0, \\ &E[x(0)v(t)] = 0\end{aligned} \quad (52)$$

Since the damping ratio ζ is 1.15470, the signal $z(t)$ decays over time. Suppose that (53) contains the observation equation for the signal $\breve{z}(t)$ and the state differential equation for the state $\breve{x}(t)$ in the degraded system.

$$\breve{y}(t) = \breve{z}(t) + v(t),$$
$$\breve{z}(t) = \breve{H}(t)\breve{x}(t), \breve{H}(t) = H + \Delta H(t),$$
$$\Delta H(t) = [0.1 \quad 0],$$
$$\frac{d\breve{x}(t)}{dt} = \breve{\Phi}(t)\breve{x}(t) + \Gamma w(t),$$
$$\breve{\Phi}(t) = \Phi + \Delta\Phi(t) \quad (53)$$
$$\Delta\Phi(t) = \begin{bmatrix} 0 & 0 \\ -0.1 * rand & -0.1 * rand \end{bmatrix},$$
$$\mathrm{E}[v(t)w(s)] = 0, \mathrm{E}[\Delta\Phi(t)w(s)] = 0,$$
$$E[\Delta C(t)v(s)] = 0, E[\breve{x}(0)w(t)] = 0,$$
$$E[\breve{x}(0)v(t)] = 0$$

In (53), $\Delta\Phi(t)$ denotes an uncertain matrix that is additional to the system matrix Φ. "$rand$" represents a scalar random number from a uniform distribution in the interval $(0, 1)$.

By substituting the cross-covariance function $K_{x\breve{x}}(t,t)$ of the system state $x(t)$ with the state $\breve{x}(t)$ of the degraded system, the covariance function $\breve{K}(t,t)$ of $\breve{x}(t)$, the observation matrix $\breve{H}$, the system matrix Φ, and the degraded system matrix $\breve{\Phi}$ into the robust RLSW FI smoothing and filtering algorithm of Theorem 2, the FI smoothing and filtering estimates are recursively computed. Figure 1 illustrates the signal $z(t)$ and its filtering estimate $\hat{z}(t,t)$ vs. t for the white Gaussian observation noise (WGON) $N(0,0.3^2)$. Figure 2 illustrates the signal $z(t)$ and its FI smoothing estimate $\hat{z}(t,T)$, $T = 2$, vs. t for the WGON $N(0,0.3^2)$.

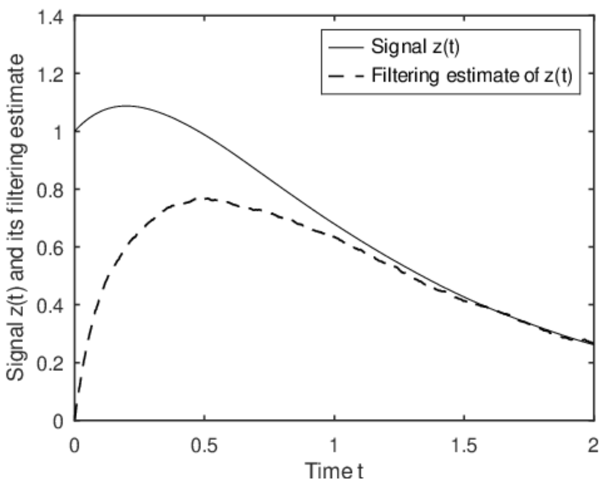

Fig. 1: Signal $z(t)$ and its filtering estimate $\hat{z}(t,t)$ vs. t for the WGON $N(0,0.3^2)$

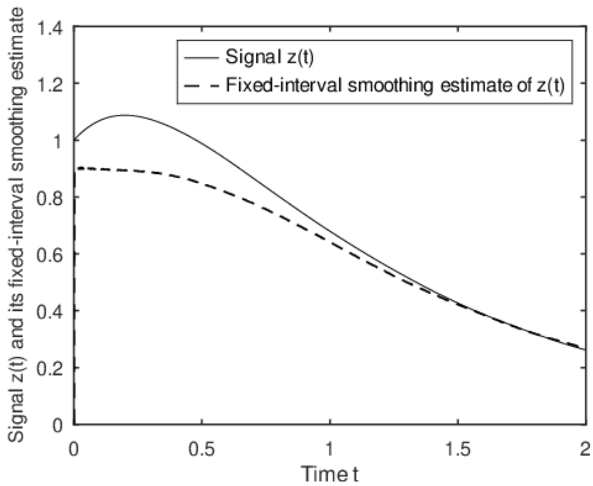

Fig. 2: Signal $z(t)$ and its FI smoothing estimate $\hat{z}(t,T)$, $T = 2$, vs. t for the WGON $N(0,0.3^2)$

Table 1 shows the mean-square values (MSVs) of filtering errors $z(t) - \hat{z}(t,t)$ and FI smoothing errors $z(t) - \hat{z}(t,T)$, $T = 2$, by the robust RLSW filtering and FI smoothing algorithm in Theorem 2 for the WGONs $N(0, 0.1^2)$, $N(0, 0.3^2)$, and $N(0, 0.5^2)$. Here, the MSVs of the filtering and FI smoothing errors are evaluated using $\frac{1}{2000}\sum_{i=1}^{2000}(z(i\Delta) - \hat{z}(i\Delta, i\Delta))^2$, and $\frac{1}{2000}\sum_{i=1}^{2000}(z(i\Delta) - \hat{z}(i\Delta, T))^2$, $T = 2$, respectively.

Table 1. MSVs of filtering errors $z(t) - \hat{z}(t,t)$ and FI smoothing errors $z(t) - \hat{z}(t,T)$, $T = 2$, by the robust RLSW filtering and FI smoothing algorithm in Theorem 2 for the WGONs $N(0, 0.1^2)$, $N(0, 0.3^2)$, and $N(0, 0.5^2)$

White Gaussian observation noise	**MSV of filtering errors**	**MSV of FI smoothing errors**
$N(0, 0.1^2)$	8.66602×10^{-3}	2.65194×10^{-4}
$N(0, 0.3^2)$	6.96936×10^{-2}	9.29714×10^{-3}
$N(0, 0.5^2)$	0.154461	4.04420×10^{-2}

Table 1 shows that the robust RLSW FI smoother for $z(t)$ outperforms the robust RLSW filter for each observation noise regarding estimation accuracy. The MSV increases as the variance of the white Gaussian observation noise becomes larger. The results demonstrate the effectiveness of the robust RLSW FI smoothing and filtering algorithm proposed in Theorem 2 during estimating signal damping.

Example 2

Assume that equation (54) includes the observation equation for the signal $z(t)$ and the second-order state differential equation for the system state $x(t)$ [25].

$$
\begin{aligned}
&y(t) = z(t) + v(t), z(t) = Hx(t),\\
&H = [1 \quad 0],\\
&\frac{dx(t)}{dt} = \Phi x(t) + \Gamma w(t), x(0) = c,\\
&x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\\
&\Phi = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \omega_n = \sqrt{20}, \qquad (54)\\
&\zeta = \frac{2}{\omega_n}, \Gamma = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix},\\
&E[v(k)v(s)] = R\delta(t-s), R > 0,\\
&E[w(t)w(s)] = Q\delta(t-s), Q = 1,\\
&E[v(t)w(s)] = 0, E[x(0)w(t)] = 0,\\
&E[x(0)v(t)] = 0
\end{aligned}
$$

The damping ratio ζ in this example is 0.447215. Natural angular frequency is $\omega_n = 4.47214$ [rad/s]. Since the damping ratio is less than 1, the signal $z(t)$ exhibits a damped oscillatory waveform as time progresses. Suppose, as in Example 1, that (53) contains the observation equation for the signal $\breve{z}(t)$ and the state differential equation for the state $\breve{x}(t)$ of the degraded system.

By substituting the cross-covariance function $K_{x\breve{x}}(t,t)$, the covariance function $\breve{K}(t,t)$, the observation matrix $\breve{H}$, the system matrix Φ, and the degraded system matrix $\breve{\Phi}$ into the robust RLSW FI smoothing and filtering algorithm of Theorem 2, the FI smoothing and filtering estimates are recursively computed. Figure 3 illustrates the signal $z(t)$ and its filtering estimate $\hat{z}(t,t)$ vs. t for the WGON $N(0,0.3^2)$. Figure 4 illustrates the signal $z(t)$ and its FI smoothing estimate $\hat{z}(t,T)$, $T = 2$, vs. t for the WGON $N(0,0.3^2)$.

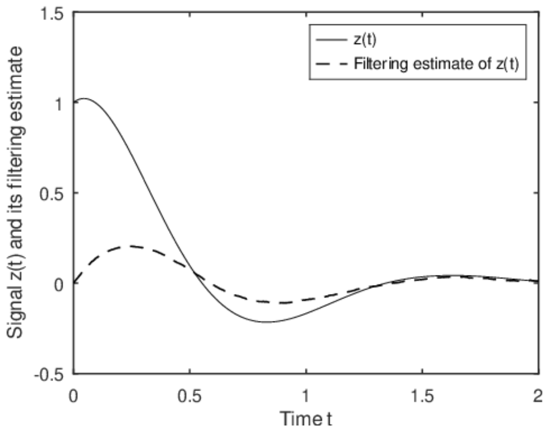

Fig. 3: Signal $z(t)$ and its filtering estimate $\hat{z}(t,t)$ vs. t for the WGON $N(0,0.3^2)$.

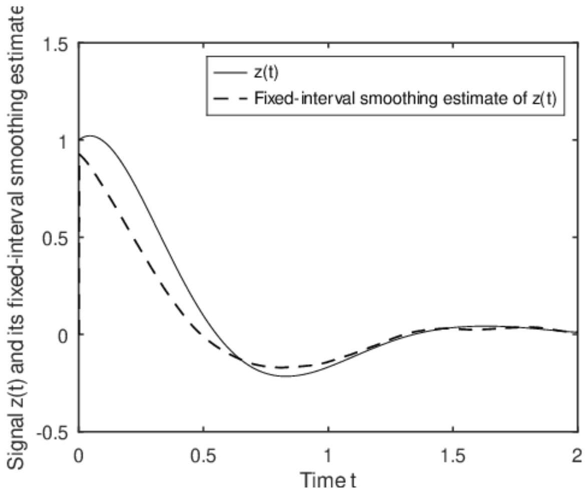

Fig. 4: Signal $z(t)$ and its FI smoothing estimate $\hat{z}(t,T)$, $T = 2$, vs. t for the WGON $N(0,0.3^2)$.

Table 2 shows the MSVs of filtering errors $z(t) - \hat{z}(t,t)$ and FI smoothing errors $z(t) - \hat{z}(t,T)$, $T = 2$, by the robust RLSW filtering and FI smoothing algorithm in Theorem 2 for the WGONs $N(0,0.1^2)$, $N(0,0.3^2)$, and $N(0,0.5^2)$. Here, the MSVs of the filtering and FI smoothing errors are evaluated using $\frac{1}{2000}\sum_{i=1}^{2000}(z(i\Delta) - \hat{z}(i\Delta, i\Delta))^2$, and $\frac{1}{2000}\sum_{i=1}^{2000}(z(i\Delta) - \hat{z}(i\Delta, T))^2$, $T = 2$, respectively.

Table 2. MSVs of filtering errors $z(t) - \hat{z}(t,t)$ and FI smoothing errors $z(t) - \hat{z}(t,T)$, $T = 2$, by the robust RLSW filtering and FI smoothing algorithm in Theorem 2 for the WGONs $N(0,0.1^2)$, $N(0,0.3^2)$, and $N(0,0.5^2)$.

White Gaussian observation noise	**MSV of filtering errors**	**MSV of FI smoothing errors**
$N(0,0.1^2)$	2.54783×10^{-2}	1.60368×10^{-3}
$N(0,0.3^2)$	9.14111×10^{-2}	1.30889×10^{-2}
$N(0,0.5^2)$	0.117375	4.76564×10^{-2}

Table 2 shows that the robust RLSW FI smoother for $z(t)$ outperforms the robust RLSW filter regarding estimation accuracy for each observation noise. The MSV increases as the variance of the WGON becomes larger. The results show that the robust RLSW FI smoothing and filtering algorithm in Theorem 2 effectively estimates the signal for the damping and vibration system.

7 Conclusion

Using covariance information, Theorem 1 presented the robust RLS FI smoothing and filtering algorithm for the signal $z(t)$ and the system state $x(t)$. Following Theorem 1, Theorem 2 presented the robust RLSW FI smoothing and filtering algorithm for the signal $z(t)$ and the system state $x(t)$. The algorithm works with the cross-covariance function $K_{x\breve{x}}(t,t)$ of the system state $x(t)$ with the state $\breve{x}(t)$ of the degraded system, the covariance function $\breve{K}(t,t)$ of the degraded system state, $\breve{x}(t)$, the observation matrix $\breve{H}$, the system matrix Φ, and the degraded system matrix $\breve{\Phi}$. Numerical simulation results indicate that the robust RLSW FI smoother for the signal $z(t)$ achieves higher estimation accuracy than the robust RLSW filter for every observation noise. The RLSW FI smoothing and filtering algorithm presented in Theorem 2 effectively estimates the signal in uncertain stochastic systems.

Research on robust estimation techniques will likely apply to image restoration and acoustic signal estimation in linear discrete-time stochastic systems.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

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Conflict of Interest

The authors have no conflicts of interest to declare.

APPENDIX A

Proof of Theorem 1

Introducing an integral equation

$$J_3(s)R = \breve{B}^T(s)\breve{H}^T - \int_0^t J_3(\tau)R\, {g_0}^T(s,\tau)\breve{H}^T d\tau, \quad \text{(A-1)}$$

from (17) and (18), $g_0(t,s)$ is given by

$$g_0(t,s) = \breve{A}(t)J_3(s), 0 \le s \le t. \quad \text{(A-2)}$$

Substituting (A-2) into (13) yields (A-3).

$$\hat{\breve{x}}(t,t) = \breve{A}(t)\int_0^t J_3(s)\breve{\nu}(s)ds \quad \text{(A-3)}$$

Putting

$$e_3(t) = \int_0^t J_3(s)\breve{\nu}(s)ds, \quad \text{(A-4)}$$

the filtering estimate $\hat{\breve{x}}(t,t)$ of $\breve{x}(t)$ is given by

$$\hat{\breve{x}}(t,t) = \breve{A}(t)e_3(t). \quad \text{(A-5)}$$

Differentiating (A-4) with respect to t, we obtain (A-6).

$$\frac{de_3(t)}{dt} = J_3(t)(\breve{y}(t) - \breve{H}\hat{\breve{x}}(t,t)), \quad e_3(0) = 0 \quad \text{(A-6)}$$

From (A-1) and (A-2), $J_3(t)$ satisfies (A-7):

$$\begin{aligned} J_3(t)R =& \breve{B}^T(t)\breve{H}^T - \int_0^t J_3(\tau)R\, {g_0}^T(t,\tau)\breve{H}^T d\tau \\ =& \breve{B}^T(t)\breve{H}^T - \int_0^t J_3(\tau)R\, {J_3}^T(\tau)d\tau \breve{A}^T(t)\breve{H}^T. \end{aligned} \quad \text{(A-7)}$$

Introducing a function

$$r_{33}(t) = \int_0^t J_3(\tau)R\, {J_3}^T(\tau)d\tau, \quad \text{(A-8)}$$

$J_3(t)$ is given by

$$J_3(t) = (\breve{B}^T(t)\breve{H}^T - r_{33}(t)\breve{A}^T(t)\breve{H}^T)R^{-1}. \quad \text{(A-9)}$$

Differentiating (A-8) with respect to t, we obtain a differential equation for $r_{33}(t)$.

$$\frac{dr_{33}(t)}{dt} = J_3(t)R{J_3}^T(t), \quad r_{33}(0) = 0 \quad \text{(A-10)}$$

The impulse response function for the filtering estimate $\hat{x}(t,t)$ of the system state $x(t)$ satisfies integral equation (A-11), which is the same as (15).

$$g(t,s)R = K_{x\breve{y}}(t,s) - \int_0^t g(t,\tau)R\, {g_0}^T(t,\tau)\breve{H}^T d\tau \quad \text{(A-11)}$$

Introducing an integral equation

$$J_1(s)R = \beta^T(s)\breve{H}^T - \int_0^t J_1(\tau)R\, {g_0}^T(s,\tau)\breve{H}^T d\tau, \quad \text{(A-12)}$$

$g(t,s)$ is given by

$$g(t,s) = \alpha(t)J_1(s). \quad \text{(A-13)}$$

From (8), the filtering estimate $\hat{x}(t,t)$ of the system state $x(t)$ is given by (A-14).

$$\hat{x}(t,t) = \int_0^t g(t,\tau)\breve{\nu}(\tau)d\tau \quad \text{(A-14)}$$

Substituting (A-13) into (A-14), we have (A-15).

$$\hat{x}(t,t) = \alpha(t)\int_0^t J_1(\tau)\breve{\nu}(\tau)d\tau \quad \text{(A-15)}$$

Putting

$$e_1(t) = \int_0^t J_1(\tau)\breve{\nu}(\tau)d\tau, \quad \text{(A-16)}$$

the filtering estimate $\hat{x}(t,t)$ of $x(t)$ is given by

$$\hat{x}(t,t) = \alpha(t)e_1(t). \quad \text{(A-17)}$$

Differentiating (A-16) with respect to t, we obtain

$$\frac{de_1(t)}{dt} = J_1(t)(\breve{y}(t) - \breve{H}\hat{\breve{x}}(t,t)), \quad e_1(0) = 0. \quad \text{(A-18)}$$

From (A-2) and (A-12), $J_1(t)$ satisfies (A-19):

$$\begin{aligned} J_1(t)R =& \beta^T(t)\breve{H}^T d\tau - \int_0^t J_1(\tau)R\, {g_0}^T(t,\tau)\breve{H}^T \\ =& \beta^T(t)\breve{H}^T - \int_0^t J_1(\tau)R\, {J_3}^T(\tau)d\tau\breve{A}^T(t)\breve{H}^T. \end{aligned} \quad \text{(A-19)}$$

Introducing a function

$$r_{13}(t) = \int_0^t J_1(\tau)R\, {J_3}^T(\tau)d\tau, \quad \text{(A-20)}$$

$J_1(t)$ is given by

$$J_1(t) = (\beta^T(t)\breve{H}^T - r_{13}(t)\breve{A}^T(t)\breve{H}^T)R^{-1}. \quad \text{(A-21)}$$

Differentiating (A-20) with respect to t, we obtain a differential equation for $r_{13}(t)$.

$$\frac{dr_{13}(t)}{dt} = J_1(t)R{J_3}^T(t), \quad r_{13}(0) = 0 \quad \text{(A-22)}$$

In the FI smoothing problem, the impulse response function $g(t,s), 0 \le t \le s$, satisfies (15). From $K_{x\breve{y}}(t,s) = K_{x\breve{x}}(t,s)\breve{H}^T$, (15) is rewritten as (A-23):

$$\begin{aligned} g(t,s)R =& K_{x\breve{x}}(t,s)\breve{H}^T - \int_0^s g(t,\tau)R\, {g_0}^T\breve{H}^T d\tau \\ =& K_{x\breve{x}}(t,s)\breve{H}^T - \int_0^t g(t,\tau)R\, {g_0}^T(s,\tau)\breve{H}^T \\ & - \int_t^s g(t,\tau)R\, {g_0}^T(s,\tau)\breve{H}^T d\tau, \\ & 0 \le t \le s. \end{aligned} \quad \text{(A-23)}$$

From $g(t,s) = \alpha(t)J_1(s)$ in (A-13), $g_0(t,s) = \breve{A}(t)J_3(s)$ in (A-2) for $0 \le s \le t$, (A-20), and (16), (A-23) becomes (A-24).

$$\begin{aligned} g(t,s)R &= K_{x\breve{x}}(t,s)\breve{H}^T - \int_0^s g(t,\tau)R\, g_0^T(s,\tau)\breve{H}^T d\tau \\ &= \gamma(t)\delta^T(s)\breve{H}^T(s)\breve{H}^T d\tau - \alpha(t)\int_0^t J_1(\tau)R J_3^T(\tau)\breve{A}^T - \int_t^s g(t,\tau)R\, g_0^T(s,\tau)\breve{H}^T d\tau \\ &= \gamma(t)\delta^T(s)\breve{H}^T - \alpha(t)r_{13}(t)\breve{A}^T(s)\breve{H}^T - \int_t^s g(t,\tau)R\, g_0^T(s,\tau)\breve{H}^T d\tau \end{aligned} \quad \text{(A-24)}$$

Introducing functions $\Delta_1(t,s)$ and $\Delta_2(t,s)$ given by

$$\Delta_1(t,s)R = \delta^T(s)\breve{H}^T - \int_t^s \Delta_1(t,\tau)R\, g_0^T(s,\tau)\breve{H}^T d\tau \quad \text{(A-25)}$$

and

$$\Delta_2(t,s)\, R = \breve{A}^T(s)\breve{H}^T - \int_t^s \Delta_2(t,\tau)R\, g_0^T(s,\tau)\breve{H}^T d\tau, \quad \text{(A-26)}$$

$g(t,s)$ is shown in (A-27).

$$g(t,s) = \gamma(t)\Delta_1(t,s) - \alpha(t)r_{13}(t)\Delta_2(t,s) \quad \text{(A-27)}$$

Hence, the FI smoothing estimate $\hat{x}(t,T)$ is given by (A-28).

$$\begin{aligned} \hat{x}(t,T) &= \hat{x}(t,t) + \int_t^T g(t,\tau)\breve{v}(\tau)d\tau \\ &= \hat{x}(t,t) + \int_t^T \gamma(t)\Delta_1(t,\tau)\,\breve{v}(\tau)d\tau - \int_t^T \alpha(t)r_{13}(t)\Delta_2(t,\tau)\,\breve{v}(\tau)d\tau \end{aligned} \quad \text{(A-28)}$$

Introducing functions $q_1(t,s)$ and $q_2(t,s)$ given by

$$q_1(t,T) = \int_t^T \Delta_1(t,\tau)\,\breve{v}(\tau)d\tau \quad \text{(A-29)}$$

and

$$q_2(t,T) = \int_t^T \Delta_2(t,\tau)\,\breve{v}(\tau)d\tau, \quad \text{(A-30)}$$

the FI smoothing estimate $\hat{x}(t,T)$ of the system state $x(t)$ is given by (A-31).

$$\hat{x}(t,T) = \hat{x}(t,t) + \gamma(t)q_1(t,T) - \alpha(t)r_{13}(t)q_2(t,T) \quad \text{(A-31)}$$

Differentiating (A-29) with respect to t, we have (A-32).

$$\frac{\partial q_1(t,T)}{\partial t} = -\Delta_1(t,t)\breve{v}(t) + \int_t^T \frac{\partial \Delta_1(t,\tau)}{\partial t}\breve{v}(\tau)d\tau \quad \text{(A-32)}$$

Differentiating (A-25) with respect to t, from (A-2), we have (A-33).

$$\begin{aligned} \frac{\partial \Delta_1(t,s)}{\partial t}R &= \Delta_1(t,t)Rg_0^T(s,t)\breve{H}^T - \int_t^s \frac{\partial \Delta_1(t,\tau)}{\partial t}R\, g_0^T(s,\tau)\breve{H}^T d\tau \\ &= \Delta_1(t,t)RJ_3^T(t)\breve{A}^T(s)\breve{H}^T - \int_t^s \frac{\partial \Delta_1(t,\tau)}{\partial t}R\, g_0^T(s,\tau)\breve{H}^T d\tau \end{aligned} \quad \text{(A-33)}$$

From (A-25), we observe that $\Delta_1(t,t)R = \delta^T(t)\breve{H}^T$. From (A-26), we obtain (A-34).

$$\frac{\partial \Delta_1(t,s)}{\partial t} = \delta^T(t)\breve{H}^T J_3^T(t)\Delta_2(t,s) \quad \text{(A-34)}$$

Differentiating (A-26) with respect to t, from (A-2), we have (A-35).

$$\begin{aligned} \frac{\partial \Delta_2(t,s)}{\partial t}R &= \Delta_2(t,t)Rg_0^T(s,t)\breve{H}^T - \int_t^s \frac{\partial \Delta_2(t,\tau)}{\partial t}R\, g_0^T(s,\tau)\breve{H}^T d\tau \\ &= \Delta_2(t,t)RJ_3^T(t)\breve{A}^T(s)\breve{H}^T - \int_t^s \frac{\partial \Delta_2(t,\tau)}{\partial t}R\, g_0^T(s,\tau)\breve{H}^T d\tau \end{aligned} \quad \text{(A-35)}$$

From (A-26), we observe that $\Delta_2(t,t)\, R = \breve{A}^T(t)\breve{H}^T$. From (A-26), we obtain (A-36).

$$\frac{\partial \Delta_2(t,s)}{\partial t} = \breve{A}^T(t)\breve{H}^T J_3^T(t)\Delta_2(t,s) \quad \text{(A-36)}$$

Substituting (A-34) into (A-32) from (A-30) and the relationship $\Delta_1(t,t) = \delta^T(t)\breve{H}^T R^{-1}$, we obtain (A-37).

$$\begin{aligned} \frac{\partial q_1(t,T)}{\partial t} &= -\Delta_1(t,t)\breve{v}(t) + \int_t^T \frac{\partial \Delta_1(t,\tau)}{\partial t}\breve{v}(\tau)d\tau \\ &= -\Delta_1(t,t)\breve{v}(t) + \int_t^T \delta^T(t)\breve{H}^T J_3^T(t)\Delta_2(t,\tau)\,\breve{v}(\tau)d\tau \\ &= -\delta^T(t)\breve{H}^T R^{-1}(\breve{y}(t) - \breve{H}\hat{\breve{x}}(t,t)) + \delta^T(t)\breve{H}^T J_3^T(t)q_2(t,T), \\ q_1(T,T) &= 0 \end{aligned} \quad \text{(A-37)}$$

Differentiating (A-30) with respect to t, from (A-30), (A-36), and the relationship $\Delta_2(t,t) = \breve{A}^T(t)\breve{H}^T R^{-1}$, we obtain (A-38).

$$\frac{\partial q_2(t,T)}{\partial t} = -\Delta_2(t,t)\breve{v}(t) + \int_t^T \frac{\partial \Delta_2(t,\tau)}{\partial t}\breve{v}(\tau)d\tau$$
$$= -\Delta_2(t,t)\breve{v}(t) + \int_t^T \breve{A}^T(t)\breve{H}^T J_3^T(t)\Delta_2(t,\tau)\,\breve{v}(\tau)d\tau$$
$$= -\breve{A}^T(t)\breve{H}^T R^{-1}(\breve{y}(t) - \breve{H}\hat{\breve{x}}(t,t)) + \breve{A}^T(t)\breve{H}^T J_3^T(t) q_2(t,T),$$
$$q_2(T,T) = 0 \quad \text{(A-38)}$$

(Q.E.D.)

APPENDIX B

Proof of Theorem 2

(24) gives the filtering estimate $\hat{\breve{x}}(t,t)$ of the state $\breve{x}(t)$ of the degraded system. Differentiating (24) with respect to t and introducing a function

$$S_{33}(t) = \breve{A}(t) r_{33}(t) \breve{A}^T(t), \quad \text{(B-1)}$$

from (17), (24)-(26), we obtain (B-2).

$$\frac{d\hat{\breve{x}}(t,t)}{dt} = \frac{d\breve{A}(t)}{dt} e_3(t) + \breve{A}(t)\frac{de_3(t)}{dt}$$
$$= \breve{\Phi}\breve{A}(t)e_3(t) + \breve{A}(t)J_3(t) \times (\breve{y}(t) - \breve{H}\hat{\breve{x}}(t,t))$$
$$= \breve{\Phi}\hat{\breve{x}}(t,t) + \breve{A}(t)(\breve{B}^T(t)\breve{H}^T - r_{33}(t)\breve{A}^T(t)\breve{H}^T)R^{-1} \times (\breve{y}(t) - \breve{H}\hat{\breve{x}}(t,t))$$
$$= \breve{\Phi}\hat{\breve{x}}(t,t) + (\breve{K}(t,t)\breve{H}^T - S_{33}(t)\breve{H}^T)R^{-1} \times (\breve{y}(t) - \breve{H}\hat{\breve{x}}(t,t)),$$
$$\hat{\breve{x}}(0,0) = 0 \quad \text{(B-2)}$$

Differentiating (B-1) with respect to t, from (17), (26), and (27), we obtain (B-3).

$$\frac{dS_{33}(t)}{dt} = \breve{\Phi}S_{33}(t) + S_{33}(t)\breve{\Phi}^T + \breve{A}(t)J_3(t)R{J_3}^T(t)\breve{A}^T(t)$$
$$= \breve{\Phi}S_{33}(t) + S_{33}(t)\breve{\Phi}^T + \breve{A}(t)(\breve{B}^T(t)\breve{H}^T - r_{33}(t)\breve{A}^T(t)\breve{H}^T)R^{-1} \times (\breve{B}^T(t)\breve{H}^T - r_{33}(t)\breve{A}^T(t)\breve{H}^T)^T\,\breve{A}^T(t)$$
$$= \breve{\Phi}S_{33}(t) + S_{33}(t)\breve{\Phi}^T + (\breve{K}(t,t)\breve{H}^T - S_{33}(t)\breve{H}^T)R^{-1} (\breve{K}(t,t)\breve{H}^T - S_{33}(t)\breve{H}^T)^T,$$
$$S_{33}(0) = 0 \quad \text{(B-3)}$$

(28) gives the filtering estimate $\hat{x}(t,t)$ of the system state $x(t)$. Differentiating (28) with respect to t and introducing a function

$$S_{13}(t) = \alpha(t) r_{13}(t)\breve{A}^T(t), \quad \text{(B-4)}$$

from (16), (29), and (30), we obtain (B-5).

$$\frac{d\hat{x}(t,t)}{dt} = \frac{d\alpha(t)}{dt} e_1(t) + \alpha(t)\frac{de_1(t)}{dt}$$
$$= \Phi\alpha(t)e_1(t) + \alpha(t)J_1(t)(\breve{y}(t) - \breve{H}\hat{\breve{x}}(t,t))$$
$$= \Phi\hat{x}(t,t) + \alpha(t)(\beta^T(t)\breve{H}^T - r_{13}(t)\breve{A}^T(t)\breve{H}^T) \times R^{-1}(\breve{y}(t) - \breve{H}\hat{\breve{x}}(t,t))$$
$$= \Phi\hat{x}(t,t) + (K_{x\breve{x}}(t,t)\breve{H}^T - S_{13}(t)\breve{H}^T) R^{-1}(\breve{y}(t) - \breve{H}\hat{\breve{x}}(t,t)),$$
$$\hat{x}(0,0) = 0 \quad \text{(B-5)}$$

Differentiating (B-4) with respect to t, from (16), (17), (26), (30), (31), and (B-1), we obtain (B-6).

$$\frac{dS_{13}(t)}{dt} = \Phi S_{13}(t) + S_{13}(t)\breve{\Phi}^T + \alpha(t)J_1(t)R{J_3}^T(t)\breve{A}^T(t)$$
$$= \Phi S_{13}(t) + S_{13}(t)\breve{\Phi}^T + \alpha(t)(\beta^T(t)\breve{H}^T - r_{13}(t)\breve{A}^T(t)\breve{H}^T)R^{-1} \times (\breve{B}^T(t)\breve{H}^T - r_{33}(t)\breve{A}^T(t)\breve{H}^T)^T\,\breve{A}^T(t)$$
$$= \Phi S_{13}(t) + S_{13}(t)\breve{\Phi}^T + (K_{x\breve{x}}(t,t)\breve{H}^T - S_{13}(t)\breve{H}^T)R^{-1} \times (\breve{K}(t,t)\breve{H}^T - S_{33}(t)\breve{H}^T)^T,$$
$$S_{13}(0) = 0 \quad \text{(B-6)}$$

Let us introduce functions $\breve{q}_1(t,T)$ and $\breve{q}_2(t,T)$.

$$\breve{q}_1(t,T) = \left(\breve{A}^T(t)\right)^{-1} q_1(t,T), \quad \breve{q}_2(t,T) = \left(\breve{A}^T(t)\right)^{-1} q_2(t,T) \quad \text{(B-7)}$$

From (17), $\breve{q}_1(t,T)$, and $\breve{q}_2(t,T)$ are rewritten as (B-8).

$$\breve{q}_1(t,T) = e^{-\breve{\Phi}^T t} q_1(t,T), \quad \breve{q}_2(t,T) = e^{-\breve{\Phi}^T t} q_2(t,T), \quad \text{(B-8)}$$

From $\gamma(t) = K_{x\breve{x}}(t,t)e^{-\breve{\Phi}^T t}$ in (16), the term $\gamma(t)q_1(t,T)$ in (A-31) becomes (B-9).

$$\gamma(t)q_1(t,T) = K_{x\breve{x}}(t,t)e^{-\breve{\Phi}^T t}q_1(t,T) = K_{x\breve{x}}(t,t)\breve{q}_1(t,T) \quad \text{(B-9)}$$

From (B-4) and (B-7), $\alpha(t)r_{13}(t)q_2(t,T)$ in (A-31) becomes (B-10).

$$\alpha(t)r_{13}(t)q_2(t,T) = \alpha(t)r_{13}(t)\breve{A}^T(t)\breve{q}_2(t,T) = S_{13}(t)\breve{q}_2(t,T) \quad \text{(B-10)}$$

Hence, (A-31) for the FI smoothing estimate $\hat{x}(t,T)$ is rewritten as (B-11).

$$\hat{x}(t,T) = \hat{x}(t,t) + K_{x\breve{x}}(t,t)\breve{q}_1(t,T) - S_{13}(t)\breve{q}_2(t,T) \quad \text{(B-11)}$$

Differentiating $\breve{q}_1(t,T)$ with respect to t, from (A-37) and $\delta^T(s) = e^{\breve{\Phi}^T s}$ in (16), we have (B-12).

$$\frac{\partial\breve{q}_1(t,T)}{\partial t}=-\breve{\Phi}^T e^{-\breve{\Phi}^T t}q_1(t,T)$$
$$-e^{-\breve{\Phi}^T t}\delta^T(t)\breve{H}^T R^{-1}$$
$$\times(\breve{y}(t)-\breve{H}\hat{\breve{x}}(t,t))$$
$$+e^{-\breve{\Phi}^T t}\delta^T(t)\breve{H}^T J_3^T(t)q_2(t,T) \quad \text{(B-12)}$$
$$=-\breve{\Phi}^T\breve{q}_1(t,T)$$
$$-\breve{H}^T R^{-1}\left(\breve{y}(t)-\breve{H}\hat{\breve{x}}(t,t)\right)$$
$$+\breve{H}^T J_3^T(t)e^{\breve{\Phi}^T t}\breve{q}_2(t,T)$$

From (26), $\breve{A}(t)=e^{\breve{\Phi}t}$ in (17), $\breve{B}^T(s)=e^{-\breve{\Phi}s}K_{\breve{x}}(s,s)$ in (17), $\breve{K}(t,t)=\breve{A}(t)\breve{B}^T(t)$, and (B-1), $J_3^T(t)e^{\breve{\Phi}^T t}$ becomes (B-13).

$$J_3^T(t)e^{\breve{\Phi}^T t}=R^{-1}(\breve{B}^T(t)\breve{H}^T$$
$$-r_{33}(t)\breve{A}^T(t)\breve{H}^T)^T e^{\breve{\Phi}^T t} \quad \text{(B-13)}$$
$$=R^{-1}(\breve{K}(t,t)\breve{H}^T-S_{33}(t)\breve{H}^T)^T$$

Substituting (B-13) into (B-12), we obtain (B-14).

$$\frac{\partial\breve{q}_1(t,T)}{\partial t}=-\breve{\Phi}^T\breve{q}_1(t,T)$$
$$-\breve{H}^T R^{-1}(\breve{y}(t)-\breve{H}\hat{\breve{x}}(t,t))$$
$$+\breve{H}^T R^{-1}(\breve{K}(t,t)\breve{H}^T-S_{33}(t)\breve{H}^T)^T \quad \text{(B-14)}$$
$$\times\breve{q}_2(t,T),$$
$$\breve{q}_1(T,T)=0$$

Differentiating $\breve{q}_2(t,T)$ with respect to t, from (A-38) and $\breve{A}(t)=e^{\breve{\Phi}t}$ in (17), we have (B-15).

$$\frac{\partial\breve{q}_2(t,T)}{\partial t}=-\breve{\Phi}^T e^{-\breve{\Phi}^T t}q_2(t,T)$$
$$-e^{-\breve{\Phi}^T t}\breve{A}^T(t)\breve{H}^T R^{-1}(\breve{y}(t)-\breve{H}\hat{\breve{x}}(t,t))$$
$$+e^{-\breve{\Phi}^T t}\breve{A}^T(t)\breve{H}^T J_3^T(t)q_2(t,T) \quad \text{(B-15)}$$
$$=-\breve{\Phi}^T\breve{q}_2(t,T)$$
$$-\breve{H}^T R^{-1}(\breve{y}(t)-\breve{H}\hat{\breve{x}}(t,t))$$
$$+\breve{H}^T J_3^T(t)e^{\breve{\Phi}^T t}\breve{q}_2(t,T)$$

Substituting (B-13) into (B-15), we obtain (B-16).

$$\frac{\partial\breve{q}_2(t,T)}{\partial t}=-\breve{\Phi}^T\breve{q}_2(t,T)$$
$$-\breve{H}^T R^{-1}(\breve{y}(t)-\breve{H}\hat{\breve{x}}(t,t))$$
$$+\breve{H}^T R^{-1}(\breve{K}(t,t)\breve{H}^T-S_{33}(t)\breve{H}^T)^T \quad \text{(B-16)}$$
$$\times\breve{q}_2(t,T),$$
$$\breve{q}_2(T,T)=0$$

(Q.E.D.)