Robust Estimators for Missing Observations in Linear Discrete-Time Stochastic Systems with Uncertainties

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Abstract: - As a first approach to estimating the signal and the state, Theorem 1 proposes recursive least-squares (RLS) Wiener fixed-point smoothing and filtering algorithms that are robust to missing measurements in linear discrete-time stochastic systems with uncertainties. The degraded quantity is given by multiplying the Bernoulli random variable by the degraded signal caused by the uncertainties in the system and observation matrices. The degraded quantity is observed with additional white observation noise. The probability that the degraded signal is present in the observation equation is assumed to be known. The design feature of the proposed robust estimators is the fitting of the degraded signal to a finite-order autoregressive (AR) model. Theorem 1 is transformed into Corollary 1, which expresses the covariance information in a semi-degenerate kernel form. The autocovariance function of the degraded state and the cross-covariance function between the nominal state and the degraded state is expressed in semi-degenerate kernel forms. Theorem 2 shows the robust RLS Wiener fixed-point and filtering algorithms for estimating the signal and state from degraded observations in the second method. The robust estimation algorithm of Theorem 2 has the advantage that, unlike Theorem 1 and the usual studies, it does not use information on the existence probability of the degraded signal. This is a unique feature of Theorem 2.

Key-Words: - Robust RLS Wiener fixed-point smoother, robust RLS Wiener filter, missing observations, discrete-time stochastic systems, uncertain parameters, degraded signal.

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1 Introduction

1.1 Brief Literature Review

In sensor network systems, missing measurements are often due to the limited bandwidth of the network. Missing measurements occur at random rates. The presence of the degraded signal in the observation equation is described by the Bernoulli random variable. It takes the value 1 or 0 with a known probability. This study aims to develop a new robust estimation technique for missing measurements.

A variety of estimation problems for systems with missing measurements have been studied in detail, [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]. For nonlinear stochastic systems without uncertainties, estimators for missing measurements have been developed, [10], [11]. A robust filter for nonlinear time-delayed stochastic systems with uncertainties was developed in, [12]. In, [13], a robust finite-horizon Kalman filter was presented for systems with norm-bounded parameter uncertainty and missing measurements with a random *N*-step observation delay. Missing probabilities are used for the robust Kalman filter. Robust fusion estimation problems with missing measurements have been studied in multisensor network systems, [2], [7], [14], [15], [16], [17], [18]. In, [7], [16], robust centralized fusion (CF) and weighted measurement fusion (WMF) Kalman estimators were designed for missing measurements in linear stochastic systems with uncertainties. In, [2], robust fusion Kalman estimators were proposed for stochastic systems with uncertainties in the system and input matrices. Observations are delayed by one randomly occurring step and missing observations occur.

In linear discrete-time stochastic systems with uncertainties, robust recursive least-squares (RLS) Wiener estimators are developed as follows: (1) Robust RLS Wiener fixed-point smoother and filter for signal estimation, [19], (2) Robust RLS Wiener finite impulse response (FIR) predictor, [20], (3) Centralized multisensor robust Chandrasekhar-type RLS Wiener filter, [21]. In, [22], robust RLS Wiener estimators, [19], were applied to the estimation problem for random delays, packet dropouts, and out-of-order packets in observed data when the system matrix and observation vector have uncertain parameters.

1.2 Current Study

As the first approach for estimating the signal and state, Theorem 1 proposes RLS Wiener fixed-point smoothing and filtering algorithms that are robust to missing measurements in linear discrete-time stochastic systems with uncertainties. The Bernoulli random variable is multiplied by the degraded signal due to the uncertainties in the system and observation matrices. It is assumed that the probability of the degraded signal being present in the observation equation is known. Theorem 1 presents the RLS Wiener fixed-point smoothing and filtering algorithms that estimate the signal and the state from the missing measurements using information on the probability and without using any information on the uncertainties. The design feature of the proposed robust estimators is to fit the degraded signal to an autoregressive (AR) model of a finite order. Theorem 1 is transformed into Corollary 1, which expresses the covariance information in a semi-degenerate kernel form. The autocovariance function of the degraded state and the cross-covariance function between the nominal state and the degraded state is expressed in semi-degenerate kernel forms. Theorem 2 shows the robust RLS Wiener fixed-point smoothing and filtering algorithms, [19], or estimating the signal and the state from degraded observations in the second method. A degraded quantity is given by multiplying the Bernoulli random variable by the degraded signal caused by the uncertainties in the system and observation matrices. The degraded quantity is observed with additional white observation noise. In contrast to Theorem 1 and other studies, e.g., [10], [11], [12], [13], the robust estimation algorithms of Theorem 2 have the advantage of not using information on the existence probability of the degraded signal. This is a unique feature of Theorem 2.

By the way, a combination of an unscented Kalman filter and a back-propagation neural network has been applied to GPS/SINS integrated navigation, [23]. In order to predict the traffic state of the entire network by modeling the dependencies of individual self and neighbors, a deep learning framework called Deep Kalman Filtering Network has been studied, [24]. In, [25], the extended estimators using covariance information are presented. Its estimation accuracy is superior to the Kalman filter neuro-computing and maximum a posteriori (MAP) estimation methods.

The numerical simulation example in Section 4 compares the estimation accuracies of the robust RLS Wiener estimators in Theorem 1 with those of the robust RLS Wiener estimators in Theorem 2. The estimation accuracies of the robust RLS Wiener estimators in Theorem 2 are superior to those of the robust RLS Wiener estimators in Theorem 1.

2 Recursive Least-Squares Fixed-Point Smoothing Problem

Let the state-space model for the signal $z(k)$ be given by (1).

$$\begin{aligned}
&y(k) = z(k) + v(k), z(k) = Hx(k),\\
&x(k+1) = \Phi x(k) + \Gamma w(k),\\
&E[x(k)v^T(s)] = 0, E[x(k)w^T(s)] = 0,\\
&E[v(k)w^T(s)] = 0,\\
&E[v(k)v^T(s)] = R\delta_K(k-s),\\
&E[w(k)w^T(s)] = Q\delta_K(k-s)
\end{aligned} \quad (1)$$

$y(k)$: $m \times 1$ observation vector; $z(k)$: $m \times 1$ signal vector; $x(k)$: $\mathrm{n} \times 1$ state vector; $v(k)$: $m \times 1$ white observation noise with mean zero; $w(k)$: $l \times 1$ input noise vector with mean zero; Φ: $\mathrm{n} \times n$ system matrix; H: $m \times n$ observation matrix.

$z(k)$ is the signal to be estimated. Here, the following assumptions are introduced.

(1) $v(k)$ is the white observation noise with the variance R. $w(k)$ is the white input noise with the variance Q. $\delta_K(k-s)$ denotes the Kronecker delta function. $v(k)$ and $w(k)$ have mean zeros.
(2) The state $x(k)$, the observation noise $v(k)$, and the input noise $w(k)$ are mutually independent.

Consider the degraded state-space model (2) with uncertainties in the system and observation matrices for the system (1).

$$\begin{aligned}
&\breve{y}(k) = \gamma(k)\breve{z}(k) + v(k),\\
&\breve{z}(k) = \bar{H}(k)\breve{x}(k), \bar{H}(k) = H + \Delta H(k),\\
&\breve{x}(k+1) = \bar{\Phi}(k)\breve{x}(k) + \Gamma w(k),\\
&\bar{\Phi}(k) = \Phi + \Delta\Phi(k),\\
&Pr[\gamma(k) = 1] = p(k)
\end{aligned} \quad (2)$$

$\breve{y}(k)$: $m \times 1$ degraded observation vector; $\breve{z}(k)$: $m \times 1$ degraded signal vector; $\breve{x}(k)$: $n \times 1$ degraded state vector; $\bar{\Phi}(k)$: $n \times n$ degraded system matrix; $\bar{H}(k)$: $m \times n$ degraded observation matrix;

$\Delta H(k)$: $m \times n$ uncertain matrix; $\Delta\Phi(k)$: $n \times n$ uncertain matrix

The degraded observed value $\breve{y}(k)$ is given as the sum of the degraded quantity $\gamma(k)\breve{z}(k)$ and the observation noise $v(k)$. The Bernoulli random variable $\gamma(k)$ in the observation equation has the probabilities $Pr[\gamma(k)=1]=p(k)$ and $Pr[\gamma(k)=0]=1-p(k)$. The probability that $\gamma(k)=1$ is $p(k)$. For $\gamma(k)=1$, the observation equation is given by $\breve{y}(k)=\breve{z}(k)+v(k)$. The probability that $\gamma(k)=0$ is $1-p(k)$. For $\gamma(k)=0$, the observed value $\breve{y}(k)$ consists only of the observation noise $v(k)$. The degraded system matrix $\bar{\Phi}(k)$ is given as a sum of the system matrix Φ and the uncertain matrix $\Delta\Phi(k)$. The degraded observation matrix $\bar{H}(k)$ is given as a sum of the system matrix H and the uncertain matrix $\Delta H(k)$. Assume that $\Delta H(k)$ and $\Delta\Phi(k)$ contain uncertain parameters, respectively.

The first objective of this study is to design the RLS Wiener fixed-point smoothing and filtering algorithms that estimate the signal from the observed value $\breve{y}(k)$ using information such as the probability $p(k)$ and without using any information on the uncertain matrices $\Delta\Phi(k)$ and $\Delta H(k)$.

Let the sequence of the degraded signal $\breve{z}(k)$ be fitted to a Nth-order AR model.

$$\begin{aligned}&\breve{z}(k)=-a_1\breve{z}(k-1)-a_2\breve{z}(k-2)\cdots\\&-a_N\breve{z}(k-N)+\breve{e}(k),\\&E[\breve{e}(k)\breve{e}^T(s)]=\breve{Q}\delta_K(k-s)\end{aligned} \tag{3}$$

Let $\breve{z}(k)$ be expressed as

$$\breve{z}(k)=\breve{H}\breve{x}(k),$$

$$\breve{x}(k)=\begin{bmatrix}\breve{x}_1(k)\\ \breve{x}_2(k)\\ \vdots\\ \breve{x}_{N-1}(k)\\ \breve{x}_N(k)\end{bmatrix}=\begin{bmatrix}\breve{z}(k)\\ \breve{z}(k+1)\\ \vdots\\ \breve{z}(k+N-2)\\ \breve{z}(k+N-1)\end{bmatrix}, \tag{4}$$

$$\breve{H}=[I_{m\times m}\quad 0\quad 0\quad \cdots\quad 0\quad 0].$$

From (3) and (4), the state equation for $\breve{x}(k)$ is given by

$$\begin{aligned}&\begin{bmatrix}\breve{x}_1(k+1)\\ \breve{x}_2(k+1)\\ \vdots\\ \breve{x}_{N-1}(k+1)\\ \breve{x}_N(k+1)\end{bmatrix}\\&=\begin{bmatrix}0 & I_{m\times m} & 0 & \cdots & 0\\ 0 & 0 & I_{m\times m} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & I_{m\times m}\\ -\breve{a}_N & -\breve{a}_{N-1} & -\breve{a}_{N-2} & \cdots & -\breve{a}_1\end{bmatrix}\\&\times\begin{bmatrix}\breve{x}_1(k)\\ \breve{x}_2(k)\\ \vdots\\ \breve{x}_{N-1}(k)\\ \breve{x}_N(k)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ I_{m\times m}\end{bmatrix}\zeta(k),\\&\zeta(k)=\breve{e}(k+N),\\&E[\zeta(k)\zeta^T(s)]=\breve{Q}\delta_K(k-s),\end{aligned} \tag{5}$$

Let $\breve{K}(k,s)$ be the autocovariance function of the state $\breve{x}(k)$. $\breve{K}(k,s)$ has the wide-sense stationarity (WSS) $\breve{K}(k,s)=\breve{K}(k-s)$, [26]. $\breve{K}(k,s)$ is expressed in the semi-degenerate functional form as follows:

$$\begin{aligned}&\breve{K}(k,s)=\begin{cases}A(k)B^T(s), 0\le s\le k,\\ B(k)A^T(s), 0\le k\le s,\end{cases}\\&A(k)=\breve{\Phi}^k, B^T(s)=\breve{\Phi}^{-s}\breve{K}(s,s).\end{aligned} \tag{6}$$

Here $\breve{\Phi}$ denotes the system matrix of the state equation (5). $\breve{\Phi}$ is given by

$$\begin{aligned}&\breve{\Phi}\\&=\begin{bmatrix}0 & I_{m\times m} & 0 & \cdots & 0\\ 0 & 0 & I_{m\times m} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & I_{m\times m}\\ -\breve{a}_N & -\breve{a}_{N-1} & -\breve{a}_{N-2} & \cdots & -\breve{a}_1\end{bmatrix}.\end{aligned} \tag{7}$$

From the relation $K_{\breve{z}}(k,s)=K_{\breve{z}}(k-s)=E[\breve{z}(k)\breve{z}^T(s)]$ in wide-sense stationary stochastic systems, [26], and (4), the autovariance function $\breve{K}(k,k)$ of the state $\breve{x}(k)$ becomes

$$\begin{aligned}&\breve{K}(k,k)=E\left[\begin{bmatrix}\breve{z}(k)\\ \breve{z}(k+1)\\ \vdots\\ \breve{z}(k+N-2)\\ \breve{z}(k+N-1)\end{bmatrix}\right.\\&\times[\breve{z}^T(k)\quad \breve{z}^T(k+1)\quad \cdots\\&\left.\breve{z}^T(k+N-2)\quad \breve{z}^T(k+N-1)]\right].\end{aligned} \tag{8}$$

Then

$$\breve{K}(k,k)=\begin{bmatrix}K_{\breve{z}}(0) & K_{\breve{z}}(-1) & \cdots & K_{\breve{z}}(-N+2) & K_{\breve{z}}(-N+1)\\ K_{\breve{z}}(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(-N+3) & K_{\breve{z}}(-N+2)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ K_{\breve{z}}(N-2) & K_{\breve{z}}(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(-1)\\ K_{\breve{z}}(N-1) & K_{\breve{z}}(N-2) & \cdots & K_{\breve{z}}(1) & K_{\breve{z}}(0)\end{bmatrix}.$$

The Yule-Walker equations for the AR parameters are the following:

$$\hat{K}(k,k)\begin{bmatrix} \breve{a}_1^T \\ \breve{a}_2^T \\ \vdots \\ \breve{a}_{N-1}^T \\ \breve{a}_N^T \end{bmatrix} = -\begin{bmatrix} K_{\breve{z}}^T(1) \\ K_{\breve{z}}^T(2) \\ \vdots \\ K_{\breve{z}}^T(N-1) \\ K_{\breve{z}}^T(N) \end{bmatrix}. \quad (9)$$

Here,

$$\hat{K}(k,k) = \begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(1) & \cdots & K_{\breve{z}}(N-2) & K_{\breve{z}}(N-1) \\ K_{\breve{z}}^T(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(N-3) & K_{\breve{z}}(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}^T(N-2) & K_{\breve{z}}^T(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(1) \\ K_{\breve{z}}^T(N-1) & K_{\breve{z}}^T(N-2) & \cdots & K_{\breve{z}}^T(1) & K_{\breve{z}}(0) \end{bmatrix}.$$

Let $K_{x\breve{y}}(k,s)$ denote the cross-covariance function between the state $x(k)$ and the observed value $\breve{y}(s)$. $K_{x\breve{y}}(k,s)$ satisfies the relation $K_{x\breve{y}}(k,s) = K_{x\breve{y}}(k-s) = E[x(k)\breve{y}^T(s)]$ in wide-sense stationary stochastic systems, [26]. Let $K_{x\breve{x}}(k,s)$ denote the cross-covariance function between the state $x(k)$ and the degraded state $\breve{x}(s)$. $K_{x\breve{x}}(k,s)$ is expressed in the following functional form:

$$K_{x\breve{x}}(k,s) = \alpha(k)\beta^T(s), 0 \le s \le k, \\ \alpha(k) = \Phi^k, \beta^T(s) = \Phi^{-s}K_{x\breve{x}}(s,s). \quad (10)$$

From (1), Φ represents the system matrix for the state $x(k)$.

Let the fixed-point smoothing estimate $\hat{x}(k,L)$ of the state $x(k)$ at the fixed point k be given by

$$\hat{x}(k,L) = \sum_{i=1}^{L} h(k,i,L)\breve{y}(i), \quad (11)$$

using the observed values $\{\breve{y}(i), 1 \le i \le L\}$. In (11), $h(k,i,L)$ denotes a time-varying impulse response function. Consider the estimation problem of minimizing the mean square value (MSV)

$$J = E[||x(k) - \hat{x}(k,L)||^2] \quad (12)$$

of fixed-point smoothing errors. By the orthogonal projection lemma, [26],

$$x(k) - \sum_{i=1}^{L} h(k,i,L)\breve{y}(i) \perp \breve{y}(s), \quad (13)$$

$$1 \le s \le L,$$

the impulse response function satisfies the Wiener-Hopf equation

$$E[x(k)\breve{y}^T(s)] \\ = \sum_{i=1}^{L} h(k,i,L)E[\breve{y}(i)\breve{y}^T(s)]. \quad (14)$$

Here, '⊥' denotes the orthogonality notation. From (1), (2), (4), (8), and the relation $E[x(k)\breve{y}^T(s)] = K_{x\breve{x}}(k,s)\breve{H}^T p(s) = K_{x\breve{z}}(k,s)p(s)$, we get

$$h(k,s,L)R \\ = K_{x\breve{x}}(k,s)\breve{H}^T p(s) \\ - \sum_{i=1}^{L} h(k,i,L)\breve{H}p(i)\breve{K}(i,s)\breve{H}^T p(s). \quad (15)$$

Here, $K_{x\breve{z}}(k,s) = E[x(k)\breve{z}^T(s)]$ is the cross-covariance function between the state $x(k)$ and the degraded signal $\breve{z}(s)$. Clearly, $E[x(k)\breve{z}^T(s)] = E[x(k)\breve{x}^T(s)]\breve{H}^T$.

3 Robust RLS Wiener Fixed-Point Smoothing and Filtering Algorithms

In (2), the degraded observation $\breve{y}(k)$ is given as the sum of the degraded quantity $\gamma(k)\breve{z}(k)$ and the observation noise $v(k)$. The Bernoulli random variable $\gamma(k)$ in the observation equation has the probabilities $Pr[\gamma(k) = 1] = p(k)$ and $Pr[\gamma(k) = 0] = 1 - p(k)$. Theorem 1 assumes that $p(k)$ is known. The degraded signal $\breve{z}(k)$ in (2) is affected by the uncertain matrices $\Delta H(k)$ and $\Delta\Phi(k)$. The sequence of $\breve{z}(k)$ is fitted to the Nth-order AR model (3). The AR model corresponds to the state-space model (5) for $\breve{z}(k)$. The model parameters are calculated using the Yule-Walker equation (9). The observation matrix $\breve{H}$ and the system matrix $\breve{\Phi}$ do not use any information about $\Delta H(k)$ and $\Delta\Phi(k)$. $\breve{H}$ and $\breve{\Phi}$ are used in the robust RLS Wiener algorithms for the fixed-point smoothing estimate $\hat{z}(k,L)$ at the fixed point k and the filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$ in Theorem 1.

Based on the linear estimation problem for the state $x(k)$ in Section 2, Theorem 1 proposes the robust RLS Wiener fixed-point smoothing and filtering algorithms.

Theorem 1 Let Φ and H denote the system and observation matrices, respectively, for the signal $z(k)$ in the state-space model (1). In the state-space model (2), the system and observation matrices

$\bar{\Phi}(k)$ and $\bar{H}(k)$ contain the uncertain matrices $\Delta\Phi$ and ΔH, respectively. $\breve{\Phi}$ and $\breve{H}$ denote the system and observation matrices, respectively, when the degraded signal process in $\breve{z}(k)$ is fitted to the AR model (3) of order N. Let $\breve{K}(k,k)$ be the variance of the state $\breve{x}(k)$ for the degraded signal $\breve{z}(k)$ and $K_{x\breve{x}}(k,k)$ the cross-variance function between the state $x(k)$ and the degraded state $\breve{x}(k)$. In the observation equation (2) for $\breve{y}(k)$, the presence of the degraded signal $\breve{z}(k)$ depends on the values of the Bernoulli random variable $\gamma(k)$. Let R be the variance of the white observation noise $v(k)$. Then the robust RLS Wiener algorithms for the fixed-point smoothing estimate $\hat{z}(k,L)$ at the fixed point k and the filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$ consist of (16)-(26) in linear discrete-time stochastic systems with uncertainties.

Fixed-point smoothing estimate of the signal $z(k)$ at the fixed point k: $\hat{z}(k,L)$

$$\hat{z}(k,L) = H\hat{x}(k,L) \tag{16}$$

Fixed-point smoothing estimate of the state $x(k)$ at the fixed point k: $\hat{x}(k,L)$

$$\begin{aligned}&\hat{x}(k,L) = \hat{x}(k,L-1) + h(k,L,L)(\breve{y}(L)\\&-p(L)\breve{H}\breve{\Phi}\hat{\breve{x}}(L-1,L-1)),\\&\hat{x}(k,L)|_{L=k} = \hat{x}(k,k)\end{aligned} \tag{17}$$

Smoother gain for $\hat{x}(k,L)$ in (17): $h(k,L,L)$

$$\begin{aligned}&h(k,L,L)\\&= [K_{x\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T p(L)\\&-q(k,L-1)\breve{\Phi}^T\breve{H}^T p(L)]\\&\times\{R + p(L)\breve{H}[\breve{K}(L,L) -\\&\breve{\Phi}S_0(L-1)\breve{\Phi}^T p(L)]\breve{H}^T\}^{-1}\end{aligned} \tag{18}$$

$$\begin{aligned}&q(k,L) = q(k,L-1)\breve{\Phi}^T\\&+h(k,L,L)p(L)\breve{H}\\&\times\left[\breve{K}(L,L) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T\right],\\&q(k,k) = S(k)\end{aligned} \tag{19}$$

Filtering estimate of the signal $z(k)$: $\hat{z}(k,k)$

$$\hat{z}(k,k) = H\hat{x}(k,k) \tag{20}$$

Filtering estimate of the state $x(k)$: $\hat{x}(k,k)$

$$\begin{aligned}&\hat{x}(k,k) = \Phi\hat{x}(k-1,k-1)\\&+G(k)(\breve{y}(k)\\&-p(k)\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)),\\&\hat{x}(0,0) = 0\end{aligned} \tag{21}$$

Filter gain for $\hat{x}(k,k)$ in (21): $G(k)$

$$\begin{aligned}&G(k) = [K_{x\breve{y}}(k,k)\\&-\Phi S(k-1)\breve{\Phi}^T\breve{H}^T p(k)]\\&\times\{R + p(k)\breve{H}[\breve{K}(k,k)\\&-\breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T p(k)\}^{-1},\\&K_{x\breve{y}}(k,k) = K_{x\breve{x}}(k,k)\breve{H}^T p(k)\\&= K_{x\breve{z}}(k,k)p(k)\end{aligned} \tag{22}$$

Filtering estimate of the degraded state $\breve{x}(k)$: $\hat{\breve{x}}(k,k)$

$$\begin{aligned}&\hat{\breve{x}}(k,k) = \breve{\Phi}\hat{\breve{x}}(k-1,k-1) + g(k)\\&\times\left(\breve{y}(k) - p(k)\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)\right),\\&\hat{\breve{x}}(0,0) = 0\end{aligned} \tag{23}$$

Filter gain for $\hat{\breve{x}}(k,k)$ in (23): $g(k)$

$$\begin{aligned}&g(k) = [\breve{K}(k,k)\breve{H}^T p(k)\\&-\breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T p(k)]\\&\times\{R + p(k)\breve{H}[\breve{K}(k,k)\\&-\breve{\Phi}S_0(L-1)\breve{\Phi}^T]\breve{H}^T p(k)\}^{-1}\end{aligned} \tag{24}$$

Autovariance function of $\hat{\breve{x}}(k,k)$: $S_0(k) = E[\hat{\breve{x}}(k,k)\hat{\breve{x}}^T(k,k)]$

$$\begin{aligned}&S_0(k) = \breve{\Phi}S_0(k-1)\breve{\Phi}^T\\&+g(k)p(k)\breve{H}\\&\times\left[\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T\right],\\&S_0(0) = 0\end{aligned} \tag{25}$$

Cross-variance function between $\hat{x}(k,k)$ and $\hat{\breve{x}}(k,k)$: $S(k) = E[\hat{x}(k,k)\hat{\breve{x}}^T(k,k)]$

$$\begin{aligned}&S(k) = \Phi S(k-1)\breve{\Phi}^T\\&+G(k)p(k)\breve{H}\\&\times\left[\breve{K}(k,k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^T\right],\\&S(0) = 0\end{aligned} \tag{26}$$

Proof of Theorem 1 is deferred to the Appendix.

The conditions for the stability of the fixed-point smoothing and filtering algorithms of Theorem 1 are as follows:

1. All the eigenvalues of the matrix Φ lie within the unit circle.
2. All the eigenvalues of the matrix $\breve{\Phi} - g(k)p(k)\breve{H}\breve{\Phi}$ lie within the unit circle.
3. $R + p(k)\breve{H}(\breve{K}(k,k) - \breve{\Phi}S_0(L-1)\breve{\Phi}^T)\breve{H}^T p(k)$ is a positive definite matrix, and its inverse exists.

Instead of Theorem 1, Corollary 1 presents robust RLS Wiener fixed-point smoothing and filtering algorithms using covariance information.

Corollary 1 Let the autocovariance function $\breve{K}(k,s)$ of the state $\breve{x}(k)$ be given by (6). Let the cross-covariance function between the state $x(k)$ and the degraded state $\breve{x}(s)$ be given by (10). Let the state-space model for the signal $z(k)$ be given by (1). Let the degraded state and observation equations containing the uncertain matrices $\Delta\Phi$ and ΔH be given by (2). In the observation equation (2) for $\breve{y}(k)$, the presence of the degraded signal $\breve{z}(k)$ depends on the values of the Bernoulli random variable $\gamma(k)$. Let the variance of the white observation noise be R. Using the covariance information, the robust RLS Wiener algorithms for the fixed-point smoothing estimate $\hat{z}(k,L)$ at the fixed point k and the filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$ consist of (27)-(39) in linear discrete-time stochastic systems with uncertainties.

Fixed-point smoothing estimate of the signal $z(k)$ at the fixed point k: $\hat{z}(k,L)$

$$\hat{z}(k,L) = H\hat{x}(k,L) \tag{27}$$

Fixed-point smoothing estimate of the state $x(k)$ at the fixed point k: $\hat{x}(k,L)$

$$\begin{aligned}&\hat{x}(k,L) = \hat{x}(k,L-1)\\&+h(k,L,L)(\breve{y}(L)\\&-p(L)\breve{H}A(L)e_0(L-1))\end{aligned} \tag{28}$$

Smoother gain for $\hat{x}(k,L)$ in (28): $h(k,L,L)$

$$\begin{aligned}&h(k,L,L)\\&=[\alpha(k)\beta^T(k)(A^T)^{L-k}\breve{H}^T p(L)\\&-P(k,L-1)A^T(L)\breve{H}^T p(L)]\\&\times[R+p(L)\breve{H}[B(L)\\&-A(L)r_0(L-1)]A^T(L)\breve{H}^T p(L)]^{-1}\end{aligned} \tag{29}$$

$$\begin{aligned}&P(k,L) = P(k,L-1)\\&+h(k,L,L)p(L)\breve{H}[B(L)\\&-A(L)r_0(L-1)],\\&P(k,k) = \alpha(k)r(k)\end{aligned} \tag{30}$$

Filtering estimate of $z(k)$: $\hat{z}(k,k)$

$$\hat{z}(k,k) = H\hat{x}(k,k) \tag{31}$$

Filtering estimate of $x(k)$: $\hat{x}(k,k)$

$$\hat{x}(k,k) = \alpha(k)e(k) \tag{32}$$

$$\begin{aligned}&e(k) = e(k-1)+J(k,k)(\breve{y}(k)\\&-p(k)\breve{H}A(k)e_0(k-1)),\\&e(0)=0\end{aligned} \tag{33}$$

$$\begin{aligned}&J(k,k) = [\beta^T(k)\breve{H}^T p(k)\\&-r(k-1)A^T(k)\breve{H}p(k)]\\&\times\{R+p(k)\breve{H}[B(k)\\&-A^k r_0(k-1)]A^T(k)\breve{H}^T p(k)\}^{-1}\end{aligned} \tag{34}$$

$$\begin{aligned}&r(k) = r(k-1)+J(k,k)p(k)\breve{H}(B(k)\\&-A(k)r_0(k-1)),\\&r(0)=0.\end{aligned} \tag{35}$$

Filtering estimate of $\breve{x}(k)$: $\hat{\breve{x}}(k,k)$

$$\hat{\breve{x}}(k,k) = A(k)e_0(k) \tag{36}$$

$$\begin{aligned}&e_0(k) = e_0(k-1)+J_0(k,k)(\breve{y}(k)\\&-p(k)\breve{H}A(k)e_0(k-1)),\\&e_0(0)=0\end{aligned} \tag{37}$$

$$\begin{aligned}&J_0(k,k) = [B^T(k)\breve{H}^T p(k)\\&-r_0(k-1)A^T(k)\breve{H}^T p(k)]\\&\times\{R+p(k)\breve{H}[B(k)\\&-A^k r_0(k-1)]A^T(k)\breve{H}^T p(k)\}^{-1}\end{aligned} \tag{38}$$

$$\begin{aligned}&r_0(k) = r_0(k-1)\\&+J_0(k,k)p(k)\breve{H}(B(k)\\&-A(k)r_0(k-1)),\\&r_0(0)=0\end{aligned} \tag{39}$$

Proof

(28) is obtained from (6), (A-30), and (A-48). (29) is obtained from (6), (10), (A-16), (A-41), and (A-42). (30) is obtained from (A-42). From (10), (A-9), (A-13), and (A-40), $P(k,k)$ is obtained. From (A-27) and (A-28), (32) is obtained. From (6), (A-30), and (A-31) we get (33). The initial condition $e(0)=0$ is clear from (A-28). From (6), (10), and (A-18) we get (34). From (6) and (A-17) we get (35). From (A-13) the initial condition $r(0)=0$ is clear. From (6) and (A-32) we get (36). From (6), (A-32), and (A-33) we get (37). From (6), (A-19), and (A-23), (38) is obtained. (A-21) is equivalent to (39). The initial condition $r_0(0)=0$ is clear from (A-16).

(Q.E.D.)

Note that the robust fixed-point smoother and the filter in Theorem 1 use information about the existence probability $p(k)$ of $\gamma(k)$ and the degraded signal $\breve{z}(k)$. Suppose that the degraded quantity $\breve{\breve{z}}(k)$ is defined as the multiplication of the Bernoulli random variable $\gamma(k)$ by the degraded signal $\breve{z}(k)$. The observation equation for $\breve{y}(k)$ and the state equation for $\bar{x}(k)$ in (2) are rewritten as

$$\begin{aligned}&\breve{y}(k) = \breve{\breve{z}}(k)+v(k),\\&\breve{\breve{z}}(k) = \gamma(k)\breve{z}(k),\\&\breve{z}(k) = \breve{H}(k)\bar{x}(k), \breve{H}(k) = H+\Delta H(k),\\&\bar{x}(k+1) = \bar{\Phi}(k)\bar{x}(k)+\Gamma w(k),\\&\bar{\Phi}(k) = \Phi+\Delta\Phi(k),\\&E[v(k)v^T(s)] = R\delta_K(k-s),\\&E[w(k)w^T(s)] = Q\delta_K(k-s),\end{aligned} \tag{40}$$

in linear discrete-time stochastic systems with uncertainties. Assume that the sequence of the degraded signal $\tilde{\overset{\frown}{z}}(k)$ is fitted to the Nth-order AR model.

$$\begin{aligned}\tilde{\overset{\frown}{z}}(k) &= -\tilde{\overset{\frown}{a}}_1\tilde{\overset{\frown}{z}}(k-1)-\tilde{\overset{\frown}{a}}_2\tilde{\overset{\frown}{z}}(k-2)\\ &\cdots-\tilde{\overset{\frown}{a}}_N\tilde{\overset{\frown}{z}}(k-N)+\tilde{\overset{\frown}{e}}(k), \quad (41)\\ E[\varepsilon(k)\varepsilon^T(s)] &= \tilde{\overset{\frown}{Q}}\,\delta_K(k-s)\end{aligned}$$

Suppose that $\tilde{\overset{\frown}{z}}(k)$ is represented by

$$\tilde{\overset{\frown}{z}}(k)=\breve{H}\tilde{\overset{\frown}{x}}(k),$$

$$\tilde{\overset{\frown}{x}}(k)=\begin{bmatrix}\tilde{\overset{\frown}{x}}_1(k)\\ \tilde{\overset{\frown}{x}}_2(k)\\ \vdots\\ \tilde{\overset{\frown}{x}}_{N-1}(k)\\ \tilde{\overset{\frown}{x}}_N(k)\end{bmatrix}=\begin{bmatrix}\tilde{\overset{\frown}{z}}(k)\\ \tilde{\overset{\frown}{z}}(k+1)\\ \vdots\\ \tilde{\overset{\frown}{z}}(k+N-2)\\ \tilde{\overset{\frown}{z}}(k+N-1)\end{bmatrix}, \quad (42)$$

$$\breve{H}=[I_{m\times m}\quad 0\quad 0\quad \cdots\quad 0\quad 0].$$

From (41) and (42), the state equation for the degraded state $\tilde{\overset{\frown}{x}}(k)$ becomes

$$\begin{bmatrix}\tilde{\overset{\frown}{x}}_1(k+1)\\ \tilde{\overset{\frown}{x}}_2(k+1)\\ \vdots\\ \tilde{\overset{\frown}{x}}_{N-1}(k+1)\\ \tilde{\overset{\frown}{x}}_N(k+1)\end{bmatrix}=\begin{bmatrix}0 & I_{m\times m} & 0 & \cdots & 0\\ 0 & 0 & I_{m\times m} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & I_{m\times m}\\ -\tilde{\overset{\frown}{a}}_N & -\tilde{\overset{\frown}{a}}_{N-1} & -\tilde{\overset{\frown}{a}}_{N-2} & \cdots & -\tilde{\overset{\frown}{a}}_1\end{bmatrix}\begin{bmatrix}\tilde{\overset{\frown}{x}}_1(k)\\ \tilde{\overset{\frown}{x}}_2(k)\\ \vdots\\ \tilde{\overset{\frown}{x}}_{N-1}(k)\\ \tilde{\overset{\frown}{x}}_N(k)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \vdots\\ 0\\ I_{m\times m}\end{bmatrix}\tilde{\overset{\frown}{\zeta}}(k),\ \tilde{\overset{\frown}{\zeta}}(k)=\tilde{\overset{\frown}{\mathrm{e}}}(k+N), \quad (43)$$

$$E\left[\tilde{\overset{\frown}{\mathrm{e}}}(k)\tilde{\overset{\frown}{\mathrm{e}}}^T(s)\right]=\tilde{\overset{\frown}{Q}}\,\delta_K(k-s).$$

The relation $\tilde{\overset{\frown}{K}}(k,s)=\tilde{\overset{\frown}{K}}(k-s)$ holds for the autocovariance function of the state $\tilde{\overset{\frown}{x}}(k)$ in wide-sense stationary stochastic systems, [26]. Let $\tilde{\overset{\frown}{K}}(k,s)$ be expressed in the following semi-degenerate functional form:

$$\begin{aligned}&\tilde{\overset{\frown}{K}}(k,s)\\ &=\begin{cases}\tilde{\overset{\frown}{A}}(k)\tilde{\overset{\frown}{B}}^T(s), 0\le s\le k,\\ \tilde{\overset{\frown}{B}}(k)\tilde{\overset{\frown}{A}}^T(s), 0\le k\le s,\end{cases} \quad (44)\\ &\tilde{\overset{\frown}{A}}(k)=\tilde{\overset{\frown}{\Phi}}^k,\\ &\tilde{\overset{\frown}{B}}^T(s)=\tilde{\overset{\frown}{\Phi}}^{-s}\tilde{\overset{\frown}{K}}(s,s).\end{aligned}$$

Here, $\tilde{\overset{\frown}{\Phi}}$ is the state transition matrix for the state $\tilde{\overset{\frown}{x}}(k)$. The system matrix $\tilde{\overset{\frown}{\Phi}}$ in the state equation (43) is given by

$$\tilde{\overset{\frown}{\Phi}}=\begin{bmatrix}0 & I_{m\times m} & 0 & \cdots & 0\\ 0 & 0 & I_{m\times m} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & I_{m\times m}\\ -\tilde{\overset{\frown}{a}}_N & -\tilde{\overset{\frown}{a}}_{N-1} & -\tilde{\overset{\frown}{a}}_{N-2} & \cdots & -\tilde{\overset{\frown}{a}}_1\end{bmatrix}. \quad (45)$$

By letting $K_{\tilde{\overset{\frown}{z}}}(k,s)=K_{\tilde{\overset{\frown}{z}}}(k-s)=E\left[\tilde{\overset{\frown}{z}}(k)\tilde{\overset{\frown}{z}}^T(s)\right]$, the autovariance function $\tilde{\overset{\frown}{K}}(k,k)$ of the state $\tilde{\overset{\frown}{x}}(k)$ is given by

$$\tilde{\overset{\frown}{K}}(k,k)=E\left[\begin{bmatrix}\tilde{\overset{\frown}{z}}(k)\\ \tilde{\overset{\frown}{z}}(k+1)\\ \vdots\\ \tilde{\overset{\frown}{z}}(k+N-2)\\ \tilde{\overset{\frown}{z}}(k+N-1)\end{bmatrix}\times\left[\tilde{\overset{\frown}{z}}^T(k)\quad \tilde{\overset{\frown}{z}}^T(k+1)\quad\cdots\quad \tilde{\overset{\frown}{z}}^T(k+N-2)\quad \tilde{\overset{\frown}{z}}^T(k+N-1)\right]\right]. \quad (46)$$

Then

$$\tilde{\overset{\frown}{K}}(k,k)=\begin{bmatrix}K_{\tilde{\overset{\frown}{z}}}(0) & K_{\tilde{\overset{\frown}{z}}}(-1) & \cdots & K_{\tilde{\overset{\frown}{z}}}(-N+2) & K_{\tilde{\overset{\frown}{z}}}(-N+1)\\ K_{\tilde{\overset{\frown}{z}}}(1) & K_{\tilde{\overset{\frown}{z}}}(0) & \cdots & K_{\tilde{\overset{\frown}{z}}}(-N+3) & K_{\tilde{\overset{\frown}{z}}}(-N+2)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ K_{\tilde{\overset{\frown}{z}}}(N-2) & K_{\tilde{\overset{\frown}{z}}}(N-3) & \cdots & K_{\tilde{\overset{\frown}{z}}}(0) & K_{\tilde{\overset{\frown}{z}}}(-1)\\ K_{\tilde{\overset{\frown}{z}}}(N-1) & K_{\tilde{\overset{\frown}{z}}}(N-2) & \cdots & K_{\tilde{\overset{\frown}{z}}}(1) & K_{\tilde{\overset{\frown}{z}}}(0)\end{bmatrix}.$$

Using $K_{\tilde{\overset{\frown}{z}}}(k-s)$, the Yule-Walker equations for the AR parameters are given by

$$\tilde{\overset{\frown}{K}}(k,k)\begin{bmatrix}\tilde{\overset{\frown}{a}}_1^T\\ \tilde{\overset{\frown}{a}}_2^T\\ \vdots\\ \tilde{\overset{\frown}{a}}_{N-1}^T\\ \tilde{\overset{\frown}{a}}_N^T\end{bmatrix}=-\begin{bmatrix}K_{\tilde{\overset{\frown}{z}}}^T(1)\\ K_{\tilde{\overset{\frown}{z}}}^T(2)\\ \vdots\\ K_{\tilde{\overset{\frown}{z}}}^T(N-1)\\ K_{\tilde{\overset{\frown}{z}}}^T(N)\end{bmatrix}. \quad (47)$$

Here,

$$\tilde{\overset{\frown}{K}}(k,k)=\begin{bmatrix}K_{\tilde{\overset{\frown}{z}}}(0) & K_{\tilde{\overset{\frown}{z}}}(1) & \cdots\\ K_{\tilde{\overset{\frown}{z}}}(1) & K_{\tilde{\overset{\frown}{z}}}(0) & \cdots\\ \vdots & \vdots & \ddots\\ K_{\tilde{\overset{\frown}{z}}}(N-2) & K_{\tilde{\overset{\frown}{z}}}(N-3) & \cdots\\ K_{\tilde{\overset{\frown}{z}}}(N-1) & K_{\tilde{\overset{\frown}{z}}}(N-2) & \cdots\end{bmatrix}$$

$$\begin{matrix} K_{\overset{\frown}{z}}(N-2) & K_{\overset{\frown}{z}}(N-1) \\ K_{\overset{\frown}{z}}(N-3) & K_{\overset{\frown}{z}}(N-2) \\ \vdots & \vdots \\ K_{\overset{\frown}{z}}(0) & K_{\overset{\frown}{z}}(1) \\ K_{\overset{\frown}{z}}(1) & K_{\overset{\frown}{z}}(0) \end{matrix}\Bigg].$$

Let $K_{x\overset{\frown}{x}}(k,s)$ represent the cross-covariance function between the state $x(k)$ and the degraded state $\overset{\frown}{x}(s)$ in wide-sense stationary stochastic systems. Assume that $K_{x\overset{\frown}{x}}(k,s)$ has the following functional form:

$$\begin{aligned} &K_{x\overset{\frown}{x}}(k,s) = \alpha(k)\overset{\frown}{\beta}^T(s), \\ &0 \le s \le k, \\ &\alpha(k) = \Phi^k, \\ &\overset{\frown}{\beta}^T(s) = \Phi^{-s}K_{x\overset{\frown}{x}}(s,s). \end{aligned} \quad (48)$$

Here, Φ is the system matrix for the state $x(k)$.

Theorem 2 Let the state-space model containing the uncertain matrices $\Delta\Phi$ and ΔH be given by (40). Let Φ and H be the system and observation matrices for the signal process in (1) for $z(k)$, respectively. When the sequence of the degraded signal $\overset{\frown}{z}(k)$ is fitted to the AR model (41) of order N and represented in the state-space model, $\overset{\frown}{\Phi}$ and $\breve{H}$ stand for the system matrix and the observation matrix, respectively. Let the variance $\overset{\frown}{K}(k,k)$ of the state $\overset{\frown}{x}(k)$ for the degraded signal $\overset{\frown}{z}(k)$ and the cross-variance function $K_{x\overset{\frown}{x}}(k,k)$ between the state $x(k)$ for the signal $z(k)$ and the state $\overset{\frown}{x}(k)$ for the degraded signal $\overset{\frown}{z}(k)$ be given. Let the variance of the white observation noise $v(k)$ be R. Then, the robust RLS Wiener algorithms for the fixed-point smoothing estimate $\hat{z}(k,L)$ at the fixed point k and the filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$ consist of (49)-(59) in linear discrete-time stochastic systems with uncertainties.

Fixed-point smoothing estimate of the signal $z(k)$ at the fixed point k: $\hat{z}(k,L)$

$$\hat{z}(k,L) = H\hat{x}(k,L) \quad (49)$$

Fixed-point smoothing estimate of the state $x(k)$ at the fixed point k: $\hat{x}(k,L)$

$$\begin{aligned} &\hat{x}(k,L) = \hat{x}(k,L-1) + h(k,L,L)(\breve{y}(L) \\ &-\breve{H}\overset{\frown}{\Phi}\hat{\overset{\frown}{x}}(L-1,L-1)), \\ &\hat{x}(k,L)|_{L=k} = \hat{x}(k,k) \end{aligned} \quad (50)$$

Smoother gain for $\hat{x}(k,L)$ in (50): $h(k,L,L)$

$$\begin{aligned} &h(k,L,L) = [K_{x\overset{\frown}{x}}(k,k)(\overset{\frown}{\Phi}^T)^{L-k}\breve{H}^T \\ &-q(k,L-1)\overset{\frown}{\Phi}^T\breve{H}^T] \\ &\times\{R + \breve{H}[\overset{\frown}{K}(L,L) \\ &-\overset{\frown}{\Phi}S_0(L-1)\overset{\frown}{\Phi}^T]\breve{H}^T\}^{-1} \end{aligned} \quad (51)$$

$$\begin{aligned} &q(k,L) = q(k,L-1)\overset{\frown}{\Phi}^T \\ &+h(k,L,L)\breve{H}, \\ &\times\left[\overset{\frown}{K}(L,L) - \overset{\frown}{\Phi}S_0(L-1)\overset{\frown}{\Phi}^T\right] \\ &q(k,k) = S(k) \end{aligned} \quad (52)$$

Filtering estimate of the signal $z(k)$: $\hat{z}(k,k)$

$$\hat{z}(k,k) = H\hat{x}(k,k) \quad (53)$$

Filtering estimate of the state $x(k)$: $\hat{x}(k,k)$

$$\begin{aligned} &\hat{x}(k,k) = \Phi\hat{x}(k-1,k-1) + \\ &G(k)\left(\breve{y}(k) - \breve{H}\overset{\frown}{\Phi}\hat{\overset{\frown}{x}}(k-1,k-1)\right), \\ &\hat{x}(0,0) = 0 \end{aligned} \quad (54)$$

Filter gain for $\hat{x}(k,k)$ in (54): $G(k)$

$$\begin{aligned} &G(k) = [K_{x\breve{y}}(k,k) \\ &-\Phi S(k-1)\overset{\frown}{\Phi}^T\breve{H}^T p(k)] \\ &\times\{R + \breve{H}[\overset{\frown}{K}(k,k) \\ &-\overset{\frown}{\Phi}S_0(L-1)\overset{\frown}{\Phi}^T]\breve{H}^T\}^{-1}, \\ &K_{x\breve{y}}(k,k) = K_{x\overset{\frown}{x}}(k,k)\breve{H}^T \\ &= K_{x\overset{\frown}{z}}(k,k) \end{aligned} \quad (55)$$

$K_{x\overset{\frown}{z}}(k,k)$ represents the cross-variance function between $x(k)$ and the degraded signal $\overset{\frown}{z}(k)$.

Filtering estimate of $\overset{\frown}{x}(k)$: $\hat{\overset{\frown}{x}}(k,k)$

$$\begin{aligned} &\hat{\overset{\frown}{x}}(k,k) = \overset{\frown}{\Phi}\hat{\overset{\frown}{x}}(k-1,k-1) \\ &+g(k)\left(\breve{y}(k) - \breve{H}\overset{\frown}{\Phi}\hat{\overset{\frown}{x}}(k-1,k-1)\right), \\ &\hat{\overset{\frown}{x}}(0,0) = 0 \end{aligned} \quad (56)$$

Filter gain for $\hat{\overset{\frown}{x}}(k,k)$ in (56): $g(k)$

$$\begin{aligned} &g(k) = [\overset{\frown}{K}(k,k)\breve{H}^T \\ &-\overset{\frown}{\Phi}S_0(k-1)\overset{\frown}{\Phi}^T\breve{H}^T] \\ &\times\{R + \breve{H}[\overset{\frown}{K}(k,k) \\ &-\overset{\frown}{\Phi}S_0(L-1)\overset{\frown}{\Phi}^T]\breve{H}^T\}^{-1} \end{aligned} \quad (57)$$

Autovariance function of $\hat{\overset{\frown}{x}}(k,k)$: $S_0(k) = E[\hat{\overset{\frown}{x}}(k,k)\hat{\overset{\frown}{x}}^T(k,k)]$

$$\begin{aligned} &S_0(k) = \overset{\frown}{\Phi}S_0(k-1)\overset{\frown}{\Phi}^T \\ &+g(k)\breve{H}[\overset{\frown}{K}(k,k) - \overset{\frown}{\Phi}S_0(k-1)\overset{\frown}{\Phi}^T], \\ &S_0(0) = 0 \end{aligned} \quad (58)$$

Cross-variance function of $\hat{x}(k,k)$ with $\hat{\overset{\frown}{x}}(k,k)$: $S(k) = E[\hat{x}(k,k)\hat{\overset{\frown}{x}}^T(k,k)]$

$$\begin{aligned} &S(k) = \Phi S(k-1)\overset{\frown}{\Phi}^T \\ &+G(k)\breve{H}[\overset{\frown}{K}(k,k) - \overset{\frown}{\Phi}S_0(k-1)\overset{\frown}{\Phi}^T], \\ &S(0) = 0 \end{aligned} \quad (59)$$

See, [19], for the proof of Theorem 2.

The conditions for the stability of the fixed-point smoothing and filtering and algorithms of Theorem 2 are as follows:

1 All eigenvalues of the matrix Φ lie within the unit circle.

2 All eigenvalues of the matrix $\breve{\breve{\Phi}} - g(k)\breve{H}\breve{\breve{\Phi}}$ are inside the unit circle.

3 $R + \breve{H}(\breve{\breve{K}}(k,k) - \breve{\breve{\Phi}} S_0(L-1)\breve{\breve{\Phi}}^T)\breve{H}^T$ is a positive definite matrix, and its inverse exists.

4 Filtering Error Variance Function of Signal in Theorem 1

This section presents the filtering error variance function $\tilde{P}_z(k)$ for the filtering estimate $\hat{z}(k,k)$ in Theorem 1. Let the autocovariance function $K(k,s)$ of the state $x(k)$ be expressed by

$$K(k,s) = \begin{cases} A_x(k)B_x^T(s), 0 \le s \le k, \\ B_x(k)A_x^T(s), 0 \le k \le s, \end{cases}$$
$$A_x(k) = \alpha(k) = \Phi^k,$$
$$B_x^T(s) = \Phi^{-s}K(s,s). \quad (60)$$

The filtering error variance function for the filtering estimate $\hat{z}(k,k)$ is given by

$$\tilde{P}_z(k) = H[K(k,k) - E[\hat{x}(k,k)\hat{x}^T(k,k)]H^T = H[K(k,k) - E[x(k)\hat{x}^T(k,k)]H^T. \quad (61)$$

From (10) and (A-27), and introducing a function,

$$r_s(k) = \sum_{i=1}^{k} J(k,i)\beta(i), \quad (62)$$

(61) is rewritten as

$$\tilde{P}_z(k) = H(K(k,k) - \alpha(k)r_s^T(k)(\alpha^T)^k)H^T. \quad (63)$$

Subtracting $r_s(k-1)$ from $r_s(k)$, using (A-11) and introducing a function

$$\overline{r}_0(k) = \sum_{i=1}^{k} J_0(k,i)\beta(i), \quad (64)$$

we have

$$r_s(k) = r_s(k-1) + J(k,k)(\beta(k) - p(k)\breve{H}\breve{\Phi}^k\overline{r}_0(k-1)),$$
$$r_s(0) = 0. \quad (65)$$

Subtracting $\overline{r}_0(k-1)$ from $\overline{r}_0(k)$ and using (A-5), we have

$$\overline{r}_0(k) = \overline{r}_0(k-1) + J_0(k,k)(\beta(k) - p(k)\breve{H}\breve{\Phi}^k\overline{r}_0(k-1)),$$
$$r_s(0) = 0. \quad (66)$$

Therefore, the filtering error variance function $\tilde{P}_z(k)$ is calculated by (63) with (34), (35), (38), (39), (65), and (66) recursively.

Since $\tilde{P}_z(k)$ is the semidefinite function, the filtering variance function $HE[\hat{x}(k,k)\hat{x}^T(k,k)]H^T = H\Phi^k r_s^T(k)(\Phi^T)^k H^T$ is upper bounded by $HK(k,k)H^T$ and lower bounded by the zero matrix as follows:

$$0 \le HE[\hat{x}(k,k)\hat{x}^T(k,k)]H^T \le HK(k,k)H^T. \quad (67)$$

This shows the existence of a robust filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$.

5 A Numerical Simulation Example

Suppose that the scalar observation and state equations for the state $x(k)$ are given by

$$y(k) = z(k) + v(k), z(k) = Hx(k),$$
$$H = [1 \quad 0], x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix},$$
$$x(k+1) = \Phi x(k) + \Gamma w(k),$$
$$\Phi = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix},$$
$$a_1 = -0.1, a_2 = -0.8, \Gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$E[v(k)v(s)] = R\delta_K(k-s),$$
$$E[w(k)w(s)] = Q\delta_K(k-s), Q = 0.5^2. \quad (68)$$

In (68), the signal process for $z(k)$ is represented by a second-order AR model. Suppose that the state-space model containing the uncertain matrices $\Delta H(k)$ and $\Delta\Phi(k)$ is given by

$$\breve{y}(k) = \gamma(k)\breve{z}(k) + v(k),$$
$$\breve{z}(k) = \bar{H}(k)\breve{x}(k), \breve{x}(k) = \begin{bmatrix} \breve{x}_1(k) \\ \breve{x}_2(k) \end{bmatrix},$$
$$\bar{H}(k) = H + \Delta H(k) = [1 + \Delta_3(k) \quad 0],$$
$$\Delta H(k) = [\Delta_3(k) \quad 0], \Delta_3(k) = 0.05\zeta(k),$$
$$\breve{x}(k+1) = \bar{\Phi}(k)\breve{x}(k) + \Gamma w(k),$$
$$\bar{\Phi}(k) = \Phi + \Delta\Phi(k),$$
$$\Delta\Phi(k) = \begin{bmatrix} 0 & 0 \\ \Delta_2(k) & \Delta_1(k) \end{bmatrix},$$
$$\Delta_1(k) = 0.01\zeta(k), \Delta_2(k) = -0.1\zeta(k),$$
$$\Pr\{\gamma(k) = 1\} = 0.9, \quad (69)$$

In linear discrete-time stochastic systems. The observed value $\breve{y}(k)$ is given by the sum of the degraded quantity $\gamma(k)\breve{z}(k)$ and the observation noise $v(k)$. It should be noted that the matrices $\Delta H(k)$ and $\Delta\Phi(k)$ are uncertain. ζ(k) in (69) denotes the random variable generated by the "rand" command in MATLAB or Octave. Let the probability that $\gamma(k) = 1$ be $p(k) = 0.9$. $\Delta_1(k)$, $\Delta_2(k)$, and $\Delta_3(k)$ consist of the mean values and zero mean stochastic variables, respectively. The task is to recursively estimate the signal $z(k)$ from the observed value $\breve{y}(k)$. Suppose that $\breve{z}(k)$ is fitted to the Nth -order AR model:

$$\begin{aligned} &\breve{z}(k) \\ &= -\breve{a}_1\breve{z}(k-1) - \breve{a}_2\breve{z}(k-2) - \cdots \\ &-\breve{a}_N\breve{z}(k-N) + \breve{e}(k), \\ &E[\breve{e}(k)\breve{e}(s)] = \breve{Q}\delta_K(k-s), N = 10. \end{aligned} \quad (70)$$

From (4), for the scalar observation equation in (69), $\breve{z}(k)$ is given by

$$\begin{aligned} &\breve{z}(k) = \breve{H}\breve{x}(k), \\ &\breve{H} = [1 \quad 0 \quad 0 \quad \cdots \quad 0 \quad 0], \\ &\breve{H}\text{: m} \times N \text{ vector.} \end{aligned} \quad (71)$$

The state equation for $\breve{x}(k)$ is given by (5). In this example, this equation corresponds to the case $m = 1$. The autocovariance function $\breve{K}(k,s)$ of the state $\breve{x}(k)$ is expressed in the form of the semi-degenerate function in (6) and has the property $\breve{K}(k,s) = \breve{K}(k-s)$ in the wide sense of stationary stochastic systems. In (6), Φ is the system matrix for the state $\breve{x}(k)$. $\breve{\Phi}$ is given by (7). $\breve{K}(k,k)$ of the state $\breve{x}(k)$ is described as follows:

$$\breve{K}(k,k) = \begin{bmatrix} K_{\breve{z}}(0) & K_{\breve{z}}(1) & \cdots & K_{\breve{z}}(N-2) & K_{\breve{z}}(N-1) \\ K_{\breve{z}}^T(1) & K_{\breve{z}}(0) & \cdots & K_{\breve{z}}(N-3) & K_{\breve{z}}(N-2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{\breve{z}}^T(N-2) & K_{\breve{z}}^T(N-3) & \cdots & K_{\breve{z}}(0) & K_{\breve{z}}(1) \\ K_{\breve{z}}^T(N-1) & K_{\breve{z}}^T(N-2) & \cdots & K_{\breve{z}}^T(1) & K_{\breve{z}}(0) \end{bmatrix}. \quad (72)$$

Suppose that $K_{z\breve{z}}(k,s) = E[z(k)\breve{z}(s)]$ represents the cross-covariance function between the signal $z(k)$ and the degraded signal $\breve{z}(s)$. From (4) and (68), the cross-covariance function $K_{x\breve{x}}(k,s)$ is given by

$$\begin{aligned} &K_{x\breve{x}}(k,s) = \Phi^{k-s}K_{x\breve{x}}(s,s), 0 \le s \le k, \\ &K_{x\breve{x}}(k,k) = E\left[\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}[\breve{z}(k) \quad \breve{z}(k+1) \quad \cdots \quad \breve{z}(k+N-2) \quad \breve{z}(k+N-1)]\right] \\ &= E\left[\begin{bmatrix} z(k) \\ z(k+1) \end{bmatrix}[\breve{z}(k) \quad \breve{z}(k+1) \quad \cdots \quad \breve{z}(k+N-2) \quad \breve{z}(k+N-1)]\right] \\ &= \begin{bmatrix} K_{z\breve{z}}(k,k) & K_{z\breve{z}}(k,k+1) & \cdots & K_{z\breve{z}}(k,k+N-2) & K_{z\breve{z}}(k,k+N-1) \\ K_{z\breve{z}}(k+1,k) & K_{z\breve{z}}(k+1,k+1) & \cdots & K_{z\breve{z}}(k+1,k+N-2) & K_{z\breve{z}}(k+1,k+N-1) \end{bmatrix}. \end{aligned} \quad (73)$$

The Yule-Walker equations (9) calculate the AR parameters $\breve{a}_1, \breve{a}_2, \cdots, \breve{a}_{N-1}, \breve{a}_N$ in (70). Substituting H, $\breve{H}$, Φ, $\breve{\Phi}$, $K_{x\breve{x}}(k,k)$, $\breve{K}(k,k) = \breve{K}(L,L)$, R and $p(k)$ into the robust RLS Wiener estimation algorithms of Theorem 1, the fixed-point smoothing and filtering estimates are recursively computed. In evaluating $\breve{\Phi}$ in (7), $\breve{K}(k,k)$ in (72), and $K_{x\breve{x}}(k,k)$ in (73), 2,000 data sets of signal and degraded signal are used, respectively. The computation of $K_{x\breve{y}}(k,k)$ in (22) uses 2,000 data sets of signal and observation, respectively. Figure 1 illustrates the fixed-point smoothing estimate $\hat{z}(k,k+5)$ and the filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$ by Theorem 1 vs. k for the white Gaussian observation noise $N(0,0.3^2)$ in the case of the AR model order $N = 10$. Figure 2 illustrates the mean square values of the filtering errors $z(k) - \hat{z}(k,k)$ and the fixed-point smoothing errors $z(k) - \hat{z}(k,k+Lag)$ vs. Lag, $0 \le Lag \le 5$, by Theorem 1 for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$, $N(0,0.5^2)$, and $N(0,0.7^2)$ in the case of the AR model order $N = 10$. Figure 3 illustrates the fixed-point smoothing estimate $\hat{z}(k,k+5)$ and the filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$ by Theorem 2, [19], vs. k for the white Gaussian observation noise $N(0,0.3^2)$ in the case of the AR model order $N = 10$. Figure 4 illustrates the mean square values of the filtering errors $z(k) - \hat{z}(k,k)$ and the fixed-point smoothing errors $z(k) - \hat{z}(k,k+Lag)$ vs. Lag, $0 \le Lag \le 5$, by Theorem 2 for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$, $N(0,0.5^2)$, and $N(0,0.7^2)$ in the case of the AR model order $N = 10$. As shown in Figure 2 and Figure 4, the estimation accuracies of the RLS Wiener filter and fixed-point smoother of Theorem 2 are superior to those of Theorem 1 for each observation noise. In Figure 4, the MSV decreases as Lag increases. This shows the smoothing effect of the RLS Wiener fixed-point smoother by Theorem 2. Figure 2 shows the smoothing effect by Theorem 1 only for the observation noise $N(0,0.7^2)$. In Figure 2 and Figure 4, the MSVs of the fixed-point smoothing and filtering errors are evaluated by

$\sum_{i=1}^{2000}(z(k) - \hat{z}(k,k+Lag))^2/2000, 1 \le Lag \le 5$, and $\sum_{i=1}^{2000}(z(k) - \hat{z}(k,k))^2/2000$, respectively.

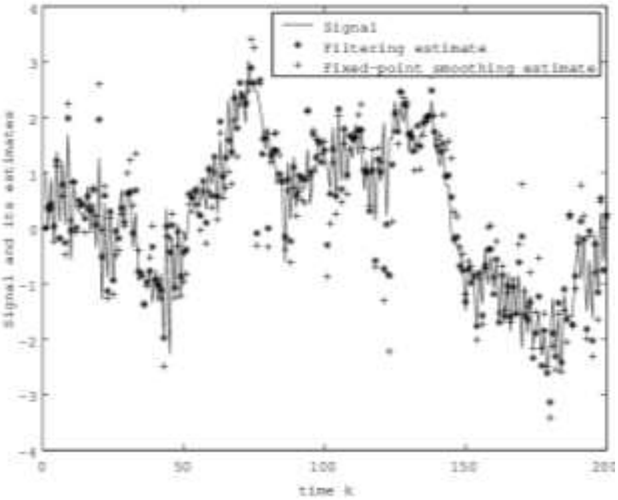

Fig. 1: Fixed-point smoothing estimate $\hat{z}(k,k+5)$ and filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$ by Theorem 1 vs. k for the white Gaussian observation noise $N(0,0.3^2)$ in the case of the AR model order $N=10$.

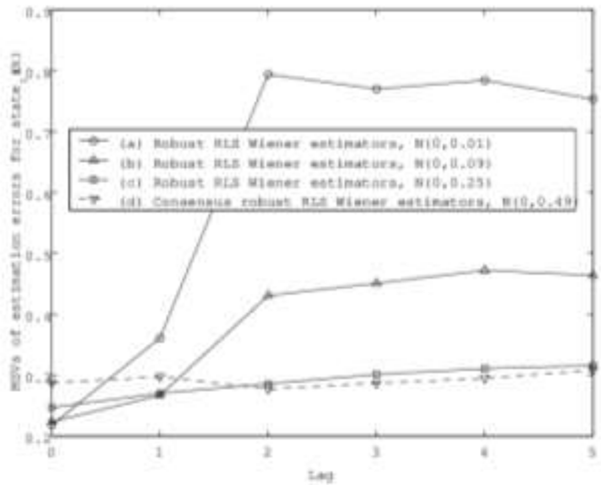

Fig. 2: MSVs of the filtering errors $z(k)-\hat{z}(k,k)$ and the fixed-point smoothing errors $z(k)-\hat{z}(k,k+Lag)$ vs. Lag, $0 \le Lag \le 5$, by Theorem 1 for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$, $N(0,0.5^2)$, and $N(0,0.7^2)$ in the case of the AR model order $N=10$.

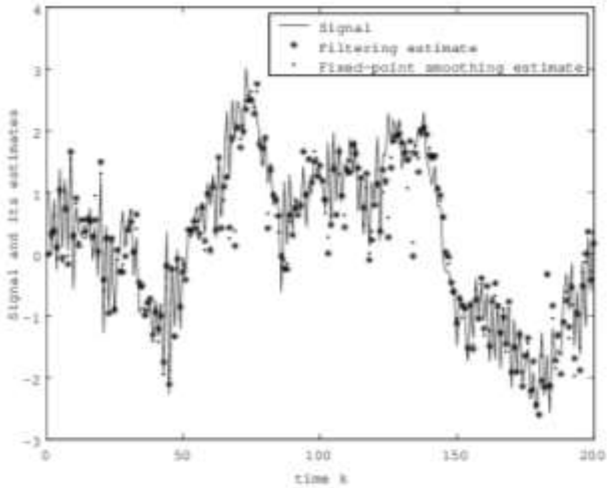

Fig. 3: Fixed-point smoothing estimate $\hat{z}(k,k+5)$ and filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$ by Theorem 2, [19], vs. k for the white Gaussian observation noise $N(0,0.3^2)$ in the case of the AR model order $N=10$.

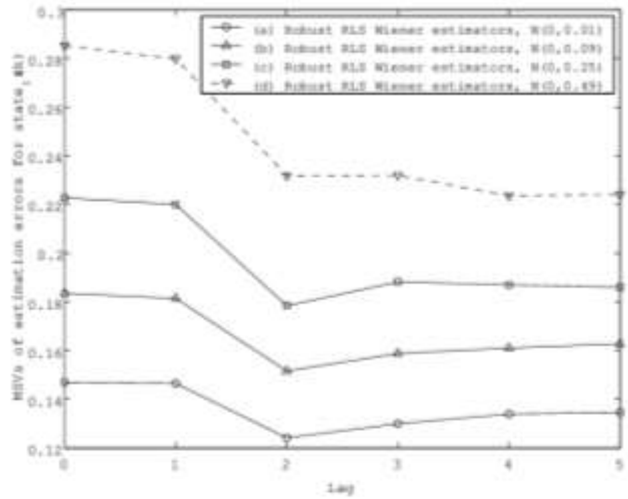

Fig. 4: MSVs of the filtering errors $z(k)-\hat{z}(k,k)$ and the fixed-point smoothing errors $z(k)-\hat{z}(k,k+Lag)$ vs. Lag, $0 \le Lag \le 5$, by Theorem 2, [19], for the white Gaussian observation noises $N(0,0.1^2)$, $N(0,0.3^2)$, $N(0,0.5^2)$ and $N(0,0.7^2)$ in the case of the AR model order $N=10$.

6 Conclusion

Theorem 1 proposed the robust RLS Wiener fixed-point smoother and filter for missing measurements in linear discrete-time stochastic systems with uncertainties. In (2), the degraded observation $\breve{y}(k)$ is given as the sum of the degraded quantity $\gamma(k)\breve{z}(k)$ and the observation noise $v(k)$. The Bernoulli random variable $\gamma(k)$ in the observation equation has the probabilities $Pr[\gamma(k)=1]=p(k)$ and $Pr[\gamma(k)=0]=1-p(k)$. Theorem 1 assumes that $p(k)$ is known. The degraded signal $\breve{z}(k)$ in (2) is affected by the uncertain matrices $\Delta H(k)$ and $\Delta\Phi(k)$. The sequence of $\breve{z}(k)$ is fitted to the Nth-order AR model (3). The AR model corresponds to the state-space model (5) for $\breve{z}(k)$. The model parameters are calculated using the Yule-Walker equation (9). The observation matrix $\breve{H}$ and the system matrix $\breve{\Phi}$ do not use any information about $\Delta H(k)$ and $\Delta\Phi(k)$. $\breve{H}$ and $\breve{\Phi}$ are used in the robust RLS Wiener algorithms for the fixed-point smoothing estimate $\hat{z}(k,L)$ at the fixed point k and the filtering estimate $\hat{z}(k,k)$ of the signal $z(k)$ in Theorem 1. The design feature of the proposed robust estimators is to fit the degraded signal to a finite-order AR model. Theorem 1 is transformed into Corollary 1, which expresses the covariance information in the semi-degenerate kernel form. Second, Theorem 2 showed the robust RLS Wiener fixed-point smoother and filter, [19]. The robust estimation algorithm of Theorem 2 has the advantage that, unlike Theorem 1 and conventional studies, it does not use information on the existence probability $p(k)$ of the degraded signal.

As shown in Figure 2 and Figure 4, the estimation accuracies of the RLS Wiener filter and fixed-point smoother of Theorem 2 are superior to those of Theorem 1 for each observation noise. As shown in Figure 4, the MSV decreases as the Lag increases. This shows the smoothing effect of the RLS Wiener fixed-point smoother by Theorem 2.

Extending this study to robust fusion estimation problems with missing measurements is future work in multisensor network systems. The current study is based on the least-squares estimation method. The neural network-aided Kalman filter is known. A combination of the current study with neural networks is also left for future work at this time.

APPENDIX

proof of Theorem 1

The impulse response function $h(k,s,L)$ satisfies (15). Subtracting $h(k,s,L-1)R$ from $h(k,s,L)R$, we have

$$\begin{aligned}&(h(k,s,L)-h(k,s,L-1))R\\&=-h(k,L,L)p(L)\breve{H}\breve{K}(L,s)\breve{H}^Tp(s)\\&-\sum_{i=1}^{L-1}(h(k,i,L)-h(k,i,L-1))\\&\times p(i)\breve{H}\breve{K}(i,s)\breve{H}^Tp(s).\end{aligned} \quad \text{(A-1)}$$

Introducing

$$\begin{aligned}&J_0(L,s)R\\&=\breve{\Phi}^{-s}\breve{K}(s,s)\breve{H}^Tp(s)\\&-\sum_{i=1}^{L}J_0(L,i)p(i)\breve{H}\breve{K}(i,s)\breve{H}^Tp(s),\end{aligned} \quad \text{(A-2)}$$

we obtain

$$\begin{aligned}&h(k,s,L)-h(k,s,L-1)\\&=-h(k,L,L)p(L)\breve{H}\breve{\Phi}^LJ_0(s,L-1).\end{aligned} \quad \text{(A-3)}$$

Subtracting $J_0(s,L-1)R$ from $J_0(s,L)R$, we get

$$\begin{aligned}&(J_0(L,s)-J_0(L-1,s))R\\&=-J_0(L,L)p(L)\breve{H}\breve{K}(L,s)\breve{H}^Tp(s)\\&-\sum_{i=1}^{L-1}(J_0(L,i)-J_0(L-1,i))\\&\times p(i)\breve{H}\breve{K}(i,s)\breve{H}^Tp(s).\end{aligned} \quad \text{(A-4)}$$

From (A-2) and (A-4), we obtain

$$\begin{aligned}&J_0(L,s)-J_0(L-1,s)\\&=-J_0(L,L)p(L)\breve{H}\breve{\Phi}^LJ_0(L-1,s).\end{aligned} \quad \text{(A-5)}$$

The filtering estimate is given by

$$\hat{x}(k,k)=\sum_{i=1}^{k}h(k,i,k)\breve{y}(i). \quad \text{(A-6)}$$

From (15), the impulse response function $h(k,s,k)$ satisfies

$$\begin{aligned}&h(k,s,k)R\\&=K_{x\breve{x}}(k,s)\breve{H}^Tp(s)\\&-\sum_{i=1}^{k}h(k,i,k)p(i)\breve{H}\breve{K}(i,s)\breve{H}^Tp(s).\end{aligned} \quad \text{(A-7)}$$

Introducing

$$\begin{aligned}&J(k,s)R\\&=\Phi^{-s}K_{x\breve{x}}(s,s)\breve{H}^Tp(s)\\&-\sum_{i=1}^{k}J(k,i)\breve{H}p(i)\breve{K}(i,s)\breve{H}^Tp(s),\end{aligned} \quad \text{(A-8)}$$

we obtain

$$h(k,s,k)=\Phi^kJ(k,s). \quad \text{(A-9)}$$

Subtracting $J(k-1,s)R$ from $J(k,s)R$, we have

$$\begin{aligned}&(J(k,s)-J(k-1,s))R\\&=-J(k,k)p(k)\breve{H}\breve{K}(k,s)\breve{H}^Tp(s)\\&-\sum_{i=1}^{k-1}(J(k,i)-J(k-1,i))\\&\times p(i)\breve{H}\breve{K}(i,s)\breve{H}^Tp(s).\end{aligned} \quad \text{(A-10)}$$

From (A-2) and (A-10), we obtain

$$\begin{aligned}&J(k,s)-J(k-1,s)\\&=-J(k,k)p(k)\breve{H}\breve{\Phi}^kJ_0(k-1,s).\end{aligned} \quad \text{(A-11)}$$

From (A-8), $J(k,k)$ satisfies

$$\begin{aligned}&J(k,k)R\\&=\Phi^{-k}K_{x\breve{x}}(k,k)\breve{H}^Tp(k)\\&-\sum_{i=1}^{k}J(k,i)p(i)\breve{H}\breve{K}(i,k)\breve{H}^Tp(k).\end{aligned} \quad \text{(A-12)}$$

Using (6) and introducing

$$r(k)=\sum_{i=1}^{k}J(k,i)p(i)\breve{H}B(i), \quad \text{(A-13)}$$

we obtain

$$J(k,k)R = \Phi^{-k}K_{x\breve{x}}(k,k)\breve{H}^{T}p(k) - r(k)A^{T}(k)\breve{H}^{T}p(k). \quad \text{(A-14)}$$

Subtracting $r(k-1)$ from $r(k)$, we have

$$r(k)-r(k-1) = J(k,k)p(k)\breve{H}B(k) + \sum_{i=1}^{k-1}(J(k,i)-J(k-1,i))p(i)\breve{H}B(i). \quad \text{(A-15)}$$

Introducing

$$r_0(k)=\sum_{i=1}^{k}J_0(k,i)p(i)\breve{H}B(i), \quad \text{(A-16)}$$

from (A-11), we obtain

$$r(k)-r(k-1) = J(k,k)p(k)\breve{H}B(k) - J(k,k)p(k)\breve{H}\breve{\Phi}^{k}r_0(k-1), \quad r(0)=0. \quad \text{(A-17)}$$

Substituting (A-17) into (A-14), we have

$$J(k,k)=[\Phi^{-k}K_{x\breve{x}}(k,k)\breve{H}^{T}p(k) - r(k-1)A^{T}(k)\breve{H}^{T}p(k)] \times [R+p(k)\breve{H}\breve{K}(k,k)\breve{H}^{T}p(k) - p(k)\breve{H}\breve{\Phi}^{k}r_0(k-1)(\breve{\Phi}^{T})^{k}\breve{H}^{T}p(k)]^{-1}. \quad \text{(A-18)}$$

Introducing a function

$$S_0(k)=A(k)r_0(k)A^{T}(k), \quad A(k)=\breve{\Phi}^{k}, \quad \text{(A-19)}$$

we rewrite (A-18) as

$$J(k,k)=[\Phi^{-k}K_{x\breve{x}}(k,k)\breve{H}^{T}p(k) - r(k-1)A^{T}(k)\breve{H}^{T}p(k)] \times [R+p(k)\breve{H}(\breve{K}(k,k)\breve{H}^{T}p(k) - p(k)\breve{H}\breve{\Phi}S_0(k-1)\breve{\Phi}^{T}\breve{H}^{T}p(k)]^{-1}. \quad \text{(A-20)}$$

Subtracting $r_0(k-1)$ from $r_0(k)$ and using (A-5), we obtain

$$r_0(k)-r_0(k-1) = J_0(k,k)p(k)\breve{H}B(k) + \sum_{i=1}^{k-1}(J_0(k,i)-J_0(k-1,i)) \times p(i)\breve{H}B(i) = J_0(k,k)p(k)\breve{H} \times \big(B(k)-A(k)r_0(k-1)\big), \quad r_0(0)=0. \quad \text{(A-21)}$$

From (A-2), $J_0(k,k)$ satisfies

$$J_0(k,k)R=\breve{\Phi}^{-k}\breve{K}(k,k)\breve{H}^{T}p(k) - \sum_{i=1}^{k}J_0(k,i)p(i)\breve{H}\breve{K}(i,k)\breve{H}^{T}p(k).$$

From (6) and (A-16), it follows that

$$J_0(k,k)R = \breve{\Phi}^{-k}\breve{K}(k,k)\breve{H}^{T}p(k) - r_0(k)A^{T}(k)\breve{H}^{T}p(k). \quad \text{(A-22)}$$

Substituting (A-21) into (A-22), we obtain an expression for $J_0(k,k)$ as

$$J_0(k,k)=[B^{T}(k)\breve{H}^{T}p(k) - r_0(k-1)A^{T}(k)\breve{H}^{T}p(k)][R + p(k)\breve{H}\breve{K}(k,k)\breve{H}^{T}p(k) - p(k)\breve{H}\breve{\Phi}S_0(k-1)\breve{\Phi}^{T}\breve{H}^{T}p(k)]^{-1}. \quad \text{(A-23)}$$

From (A-19) and (A-21), it follows that

$$S_0(k)=A(k)[r_0(k-1) + J_0(k,k)p(k)\breve{H} \times \big(B(k)-A(k)r_0(k-1)\big)]A^{T}(k) = \breve{\Phi}S_0(k-1)\breve{\Phi}^{T} + A(k)J_0(k,k)p(k) \times \breve{H}\big(\breve{K}(k,k)-\breve{\Phi}S_0(k-1)\breve{\Phi}^{T}\big), \quad S(0)=0. \quad \text{(A-24)}$$

Let us introduce a function

$$g(k)=A(k)J_0(k,k). \quad \text{(A-25)}$$

From (A-23) and (A-25), it follows that

$$g(k)=[\breve{K}(k,k)\breve{H}^{T}p(k) - \breve{\Phi}S_0(k-1)\breve{\Phi}^{T}(k)\breve{H}^{T}p(k)] [R+p(k)\breve{H}\breve{K}(k,k)\breve{H}^{T}p(k) - p(k)\breve{H}\breve{\Phi}S_0(k-1)\breve{\Phi}^{T}\breve{H}^{T}p(k)]^{-1}. \quad \text{(A-26)}$$

Now, from (A-6) and (A-9), the filtering estimate $\hat{x}(k,k)$ of $x(k)$ is given by

$$\hat{x}(k,k)=\Phi^{k}\sum_{i=1}^{k}J(k,i)\breve{y}(i). \quad \text{(A-27)}$$

Introducing a function

$$e(k)=\sum_{i=1}^{k}J(k,i)\breve{y}(i), \quad \text{(A-28)}$$

the filtering estimate is expressed as

$$\hat{x}(k,k)=\Phi^{k}e(k). \quad \text{(A-29)}$$

Subtracting $e(k-1)$ from $e(k)$, using (A-5) and (A-11), and introducing a function

$$e_0(k)=\sum_{i=1}^{k}J_0(k,i)\breve{y}(i), \quad \text{(A-30)}$$

we obtain

$$e(k)-e(k-1)=J(k,k)(\breve{y}(k) - p(k)\breve{H}\breve{\Phi}^{k}\sum_{i=1}^{k-1}J_0(k-1,i)\breve{y}(i)) = J(k,k)(y(k)-p(k)\breve{H}\breve{\Phi}e_0(k-1), \quad e_0(0)=0. \quad \text{(A-31)}$$

Let us introduce a function

$$\hat{\breve{x}}(k,k)=\breve{\Phi}^{k}e_0(k), \quad \text{(A-32)}$$

which represents the filtering estimate of $\breve{x}(k)$. Subtracting $e_0(k-1)$ from $e_0(k)$ and using (A-5), we obtain

$$e_0(k) - e_0(k-1) = J_0(k,k)(\breve{y}(k) -p(k)\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)), \quad e_0(0) = 0. \tag{A-33}$$

Substituting (A-31) into (A-29), we have

$$\begin{aligned}\hat{x}(k,k) &= \Phi\hat{x}(k-1,k-1) +\Phi^k J(k,k) \times (\breve{y}(k) - p(k)\breve{H}\breve{\Phi}^k e_0(k-1)) \\ &= \Phi\hat{x}(k-1,k-1) +G(k) \times (\breve{y}(k) - p(k)\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)), \\ G(k) &= \Phi^k J(k,k), \\ \hat{x}(0,0) &= 0.\end{aligned} \tag{A-34}$$

From (A-18), and by introducing a function

$$S(k) = \Phi^k r(k)(\breve{\Phi}^T)^k, \tag{A-35}$$

$G(k)$ is expressed as

$$\begin{aligned}G(k) &= [K_{x\breve{x}}(k,k)\breve{H}^T p(k) -\Phi S(k-1)\breve{\Phi}^T \breve{H}p(k)] \\ &\times [R + p(k)\breve{H}\breve{K}(k,k)\breve{H}^T p(k) -p(k)\breve{H}\breve{\Phi}S_0(k-1)\breve{\Phi}^T\breve{H}^T p(k)]^{-1}.\end{aligned} \tag{A-36}$$

From (A-32) and (A-33), it follows that

$$\begin{aligned}\hat{\breve{x}}(k,k) &= \breve{\Phi}^k e_0(k-1) +\breve{\Phi}^k J_0(k,k)(\breve{y}(k) -p(k)\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)) \\ &= \breve{\Phi}\hat{\breve{x}}(k-1,k-1) + g(k)(\breve{y}(k) -p(k)\breve{H}\breve{\Phi}\hat{\breve{x}}(k-1,k-1)), \\ \hat{\breve{x}}(0,0) &= 0.\end{aligned} \tag{A-37}$$

From (A-17) and (A-35), it follows that

$$\begin{aligned}S(k) &= \Phi^k r(k-1)(\breve{\Phi}^T)^k +G(k)p(k)(\breve{H}B(k) -\breve{H}\breve{\Phi}^k r_0(k-1))(\breve{\Phi}^T)^k \\ &= \Phi S(k-1)\breve{\Phi}^T + G(k)p(k)\breve{H}(\breve{K}(k,k) -\breve{\Phi}S_0(k-1)\breve{\Phi}^T), \\ S(0) &= 0.\end{aligned} \tag{A-38}$$

From (15), $h(k,L,L)$ satisfies

$$\begin{aligned}h(k,L,L)R &= K_{x\breve{x}}(k,L)\breve{H}^T p(L) - \sum_{i=1}^{L} h(k,i,L)p(i)\breve{H}\breve{K}(i,L)\breve{H}^T p(L) \\ &= K_{x\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T p(L) - \sum_{i=1}^{L} h(k,i,L)p(i)\breve{H}B(i)A^T(L) \times \breve{H}^T p(L).\end{aligned} \tag{A-39}$$

Introducing a function

$$P(k,L) = \sum_{i=1}^{L} h(k,i,L)p(i)\breve{H}B(i), \tag{A-40}$$

we have an expression for $h(k,L,L)R$ as

$$h(k,L,L)R = K_{x\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T p(L) - p(k,L)(\breve{\Phi}^T)^L\breve{H}^T p(L). \tag{A-41}$$

Subtracting $P(k,L-1)$ from $P(k,L)$, from (A-3) and (A-16), we have

$$\begin{aligned}P(k,L) - P(k,L-1) &= h(k,L,L)p(L)\breve{H}B(L) + \sum_{i=1}^{L-1}(h(k,i,L) - h(k,i,L-1)) \times p(i)\breve{H}B(i) \\ &= h(k,L,L)p(L)\breve{H}B(L) -h(k,L,L)p(L)\breve{H}\breve{\Phi}^L \times \sum_{i=1}^{L-1} J_0(L,i)\breve{H}B(i) \\ &= h(k,L,L)p(L)\breve{H}B(L) -h(k,L,L)p(L)\breve{H}\breve{\Phi}^L r_0(L-1).\end{aligned} \tag{A-42}$$

Let us introduce a function

$$q(k,L) = P(k,L)(\breve{\Phi}^T)^L. \tag{A-43}$$

From (A-19), (A-42), and (A-43), it follows that

$$\begin{aligned}q(k,L) &= q(k,L-1)\breve{\Phi}^T + h(k,L,L)(p(L)\breve{H}\breve{K}(L,L) - p(L)\breve{H}\breve{\Phi}S_0(L-1)\breve{\Phi}^T).\end{aligned} \tag{A-44}$$

From (A-41), it is clear that

$$h(k,L,L)R = K_{x\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T p(L) - q(k,L)\breve{H}^T p(L). \tag{A-45}$$

Substituting (A-44) into (A-45), we have

$$\begin{aligned}h(k,L,L) &= [K_{x\breve{x}}(k,k)(\breve{\Phi}^T)^{L-k}\breve{H}^T p(L) -q(k,L-1)\breve{\Phi}^T\breve{H}^T p(L)] \\ &\times [R + p(L)\breve{H}(\breve{K}(L,L) -\breve{\Phi}S_0(L-1)\breve{\Phi}^T)\breve{H}^T p(L)]^{-1}.\end{aligned} \tag{A-46}$$

From (A-9), (A-13), (A-35), (A-40), and (A-43), the initial condition $q(k,k)$ in (A-44) for $q(k,L)$ at $L = k$ is given by

$$\begin{aligned}q(k,k) &= P(k,k)(\breve{\Phi}^T)^k \\ &= \sum_{i=1}^{k} h(k,i,k)p(i)\breve{H}B(i)(\breve{\Phi}^T)^k \\ &= \Phi^k \sum_{i=1}^{k} J(i,k)p(i)\breve{H}B(i)(\breve{\Phi}^T)^k \\ &= \Phi^k r(k)(\breve{\Phi}^T)^k \\ &= S(k).\end{aligned} \tag{A-47}$$

The fixed-point smoothing estimate is given by (11). Subtracting $\hat{x}(k,L-1)$ from $\hat{x}(k,L)$ and using (A-3), (A-30), and (A-32), we have

$$
\begin{aligned}
&\hat{x}(k,L) = \hat{x}(k,L-1) \\
&+h(k,L,L)(\breve{y}(L) \\
&-p(L)\breve{H}\breve{\Phi}^L \sum_{i=1}^{L-1} J_0\,(L-1,i)\breve{y}(i)) \\
&= \hat{x}(k,L-1) + h(k,L,L)(\breve{y}(L) \\
&-p(L)\breve{H}\breve{\Phi}^L e_0(L-1)) \\
&= \hat{x}(k,L-1) + h(k,L,L)(\breve{y}(L) \\
&-p(L)\breve{H}\breve{\Phi}\hat{\breve{x}}(L-1,L-1)), \\
&\hat{x}(k,L)|_{L=k} = \hat{x}(k,k).
\end{aligned} \tag{A-48}
$$

(Q.E.D.)

References:

[1] D. Lou, L. Liu, S. Fang, J. Hu, D. Zhang, H. Liang, An adaptive unscented Kalman filter for needle steering with missing measurements, *2023 IEEE International Conference on Advanced Robotics and Mechatronics*, ICARM 2023, 2023, pp.1089-1095.

[2] C. Ran, Z. Deng, Robust fusion Kalman estimators for networked mixed uncertain systems with random one-step measurement delays, missing measurements, multiplicative noises and uncertain noise variances, *Inf. Sci.*, Vol.534, 2020, pp. 27-52.

[3] G. Tao, W. Liu, X. Zhang, Robust centralized fusion Kalman predictor for uncertain descriptor system with missing measurements, *16th IEEE International Conference on Control & Automation*, ICCA 2020, pp. 253-259.

[4] K. Ma, L. Xu, H. Fan, Hybrid Kalman filtering algorithm with stochastic nonlinearities and multiple missing measurements, I*EEE Access*, Vol.7, 2019, pp. 84717-84726.

[5] Y. Xu, T. Shen, X. Chen, L. Bu, N. Feng, Predictive adaptive Kalman filter and Its application to INS/UWB-integrated human localization with missing UWB-based measurements, *Int. J. Autom. Comput.*, 2019, Vol.16, No.5, pp. 604-613.

[6] Y. Zhao, C. Yang, Information fusion robust guaranteed cost Kalman estimators with uncertain noise variances and missing measurements, *Int. J. Syst. Sci.*, 2019, Vol. 50, No.15, pp. 2853-2869.

[7] W. Liu, X. Wang, Z. Deng, Robust centralized and weighted measurement fusion Kalman predictors with multiplicative noises, uncertain noise variances, and missing measurements, *Circuits Syst. Signal Process.*, 2018, Vol.37, No.2, pp. 770-809.

[8] Y. Sun, Y. Wang, X. Wu, Y. Hu, Robust extended fractional Kalman filter for nonlinear fractional system with missing measurements, *J. Frankl. Inst.*, 2018, Vol.355, No.1, pp. 361-380.

[9] Z. Deng, Z. Yang, Robust weighted fusion Kalman estimators for systems with uncertain-variance multiplicative and additive noises and missing measurements, *20th IEEE International Conference on Information Fusion*, FUSION 2017, 2017, pp. 1-8.

[10] J. Hu, Z. Wang, S. Liu, H. Gao, A variance-constrained approach to recursive state estimation for time-varying complex networks with missing measurements, *Automatica*, Vol.64, 2016, pp. 155–162.

[11] J. Hu, Z. Wang, F. E. Alsaadi, T. Hayat, Event-based filtering for time-varying nonlinear systems subject to multiple missing measurements with uncertain missing probabilities, *Inf. Fusion*, Vol.38, 2017, pp. 74-83.

[12] Y. Liu, F. E. Alsaadi, X. Yin, Y. Wang, Robust H_∞ filtering for discrete nonlinear delayed stochastic systems with missing measurements and randomly occurring nonlinearities, *International Journal of General Systems*, Vol.44, No.2, 2015, pp. 169-181.

[13] H. Rezaei, R. M. Esfanjani, M. H. Sedaaghi, Improved robust finite-horizon Kalman filtering for uncertain networked time-varying systems, *Inf. Sci.*, Vol.293, 2015, pp. 263–274.

[14] C. Pang, S. Sun, Fusion predictors for multisensor stochastic uncertain systems with missing measurements and unknown measurement disturbances, *IEEE Sens. J.*, Vol.15, No.8, 2015, pp. 4346-4354.

[15] X. Wang, W. Liu, Z. Deng, Robust weighted fusion Kalman estimators for systems with multiplicative noises, missing measurements and uncertain-variance linearly correlated white noises, *Aerosp. Sci. Technol.*, Vol. 68, 2017, pp. 331–344.

[16] W. Liu, X. Wang, Z. Deng, Robust centralized and weighted measurement fusion Kalman estimators for uncertain multisensor systems with linearly correlated white noises, *Inf. Fusion*, Vol.35, 2017, pp. 11–25.

[17] W. Liu, X. Wang, Z. Deng, Robust centralized and weighted measurement fusion white noise deconvolution estimators for multisensor systems with mixed uncertainties,

Int. J. Adapt. Control Signal Process, Vol.32, 2018, pp. 185–212.

[18] C. Yang, Z. Yang, Z. Deng, Robust weighted state fusion Kalman estimators for networked systems with mixed uncertainties, *Inf. Fusion*, Vol.45, 2019, pp. 246-265.

[19] S. Nakamori, Robust RLS Wiener signal estimators for discrete-time stochastic systems with uncertain parameters, *Frontiers in Signal Processing*, Vol.3, No.1, 2019, pp. 1–18.

[20] S. Nakamori, Robust recursive least-squares finite impulse response predictor in linear discrete-time stochastic systems with uncertain parameters, *WSEAS Transactions on Systems*, Vol.19, 2020, pp. 86–101.

[21] S. Nakamori, Centralized robust multi-sensor Chandrasekhar-type recursive least-squares Wiener filter in linear discrete-time stochastic systems with uncertain parameters, *Jordan Journal of Electrical Engineering*, Vol.7, No.3, 2021, pp. 289-303.

[22] S. Nakamori, Numerical simulation of robust recursive least-squares Wiener estimators for observations with random delays and packet dropouts in systems with uncertainties, *Computer Reviews Journal*, Vol.7, 2020, pp. 29–40.

[23] S. Li, C. Cai, Research on strong tracking UKF algorithm of integrated navigation based on BP neural network, *Proceedings of the 4th International Conference on Computer Science and Application Engineering*, CSAE '20, Association for Computing Machinery, 2020, pp. 1-5.

[24] F. Chen, Z. Chen, S. Biswas, S. Lei, N. Ramakrishnan, C. Lu, Graph Convolutional Networks with Kalman Filtering for Traffic Prediction, *Proceedings of the 28th International Conference on Advances in Geographic Information Systems*, SIGSPATIAL '20, Association for Computing Machinery, 2020, p. 135–138.

[25] S. Nakamori, Design of estimators using covariance information in discrete-time stochastic systems with nonlinear observation mechanism, *IEICE Trans. Fundamentals*, Vol.E-82, 1999, pp. 1292–1304.

[26] A. P. Sage and J. L. Melsa, *Estimation Theory with Applications to Communications and Control*. New York: McGraw-Hill, 1971.

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