Robust Recursive Least-Squares Wiener Filter for Linear Continuous-Time Uncertain Stochastic Systems

SEIICHI NAKAMORI
Professor Emeritus, Faculty of Education
Kagoshima University
1-20-6, Korimoto, Kagoshima 890-0065
JAPAN

Abstract: - For linear continuous-time systems with uncertainties in the system and observation matrices, an original robust RLS Wiener filter is designed in this study. The robust RLS Wiener filter does not assume norm-bounded uncertainty for the system and observation matrices, in contrast to the robust Kalman filter. In the design of the robust RLS Wiener filter, the degraded signal, affected by the uncertainties in the system and observation matrices, is modeled by an autoregressive (AR) model. The system and observation matrices for the degraded signal are formulated from the relationship between the AR model of the degraded signal and the state-space model. Estimation formulas for the system and observation matrices are proposed in Section 2. The robust filtering problem is introduced based on the minimization of the mean-square value of the filtering errors for the system states. The robust filtering estimate is given as an integral transformation of the degraded observations using the impulse response function. The integral equation that an optimal impulse response function satisfies is given in Section 3. Theorem 1 presents a robust RLS Wiener filtering algorithm starting from this integral equation. The proposed robust RLS Wiener filter outperforms the existing robust Kalman filter regarding estimate accuracy, as shown by a numerical simulation example.

Key-Words: - Robust RLS Wiener filter, degraded observations, stochastic systems with uncertainties, autoregressive model, continuous-time stochastic systems.

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1 Introduction

For both continuous-time and discrete-time uncertain stochastic systems, robust filters have been studied during the past few decades, e.g., [1]-[8]. For linear continuous-time stochastic systems with norm-bounded uncertainties in the system and observation matrices, robust Kalman filters [1], [2] are developed. In [3], norm-bounded uncertainties are assumed in the four matrices, including the input and observation noise matrices in the continuous-time state-space model. In [4], the filtering, prediction, and smoothing problems are considered for the system with uncertain matrices and known input. The discrete-time robust Kalman filter is investigated in [5] and [6]. In [7], robust Kalman filters are described for continuous and discrete-time stochastic systems with uncertainties. For linear discrete-time stochastic systems with uncertainties, the robust recursive least-squares (RLS) Wiener filter is proposed [8]. A specific characteristic is that the degraded signal affected by uncertain parameters is expressed in terms of an autoregressive (AR) model of finite order. Unlike the robust Kalman filter, the robust RLS Wiener filter does not use knowledge of norm-bounded uncertainties.

This paper designs a novel robust RLS Wiener filter for linear continuous-time stochastic systems with uncertainties in the system and observation matrices. This paper does not assume norm-bounded uncertainties for the system and observation matrices. The AR model [9] and the autoregressive moving average (ARMA) model [10] have been investigated in conjunction with modeling for continuous-time stochastic processes. In the design of the robust RLS Wiener filter, the degraded signal, influenced by the uncertainties in the system and observation matrices, is modeled by an AR model. The system and observation matrices for the degraded signal are formulated from the relationship between the AR model of the degraded signal and the state-space model, as shown in Section 2.

The vehicle tracking problem with model uncertainty is an example where the proposed robust RLS Wiener filter can be applied similarly to the robust Kalman filter. Also, the H-infinity tracking control algorithm in [11] is designed for linear, discrete-time stochastic systems with uncertain parameters. It includes a practical example of tracking control for an F-16 aircraft. In [11], the robust RLS Wiener filter of [12] and [13] estimates signal and state based on the separation principle between control and estimation. The H-infinity tracking control algorithm in [14] is designed for linear, continuous-time deterministic systems.

The estimates of the system and observation matrices are formulated in Section 2. Section 3 introduces a robust filtering problem. In Section 4, Theorem 1 presents the robust RLS Wiener filtering algorithm. Section 5 demonstrates a numerical simulation example of the robust RLS Wiener filter in comparison with the robust Kalman filter [1].

2 Nominal and Degraded State-Space Models and Degraded System Realization

Let (1) be a discrete-time state-space model of the linear stochastic system.

$$\begin{aligned} & y(t) = z(t) + v(t), z(t) = Hx(t), \\ & \frac{dx(t)}{dt} = Ax(t) + \Gamma w(t), x(0) = c, \\ & E[v(k)v^T(s)] = R\delta(t-s), R > 0 \\ & E[w(t)w^T(s)] = Q\delta(t-s), Q > 0 \\ & E[v(t)w^T(s)] = 0, E[x(0)w^T(t)] = 0. \end{aligned} \tag{1}$$

Here, $x(t) \in R^n$ is the state vector, and $z(t) \in R^m$ is the signal vector. The input noise $w(t) \in R^l$ and the observation noise $v(t)$ are mutually uncorrelated white Gaussian noise of mean zero. Γ is the $n \times l$ input matrix, and H is the $m \times n$ observation matrix. The auto-covariance functions for the input noise $w(t)$ and the observation noise $v(t)$ are given in (1). This paper considers the state and observation equations with uncertain parameters in (2).

$$\begin{aligned} & \breve{y}(t) = \breve{z}(t) + v(t), \\ & \breve{z}(t) = \overleftrightarrow{H}(t)\overleftrightarrow{x}(t), \overleftrightarrow{H}(t) = H + \Delta H(t), \\ & \frac{d\overleftrightarrow{x}(t)}{dt} = \overleftrightarrow{A}(t)\overleftrightarrow{x}(t) + \Gamma w(t), \\ & \overleftrightarrow{A}(t) = A + \Delta A(t), \overleftrightarrow{x}(0) = \overleftrightarrow{c}, \\ & E[v(t)w^T(s)] = 0, E[\Delta A(t)w^T(s)] = 0, \\ & E[\Delta C(t)v^T(s)] = 0, E[\overleftrightarrow{x}(0)w^T(t)] = 0, \\ & E[\overleftrightarrow{x}(0)v^T(t)] = 0 \end{aligned} \tag{2}$$

In (2), the degraded system matrix $\overleftrightarrow{A}(t)$ and the degraded observation matrix $\overleftrightarrow{C}(t)$ are introduced instead of the system matrix A and the observation matrix C in (1), respectively. Here, the matrix elements of $\Delta A(t)$ and $\Delta C(t)$ contain uncertain variables. The initial system state $\overleftrightarrow{x}(0)$ is a random vector that is uncorrelated with both system and measurement noise processes.

Assume that the degraded signal is expressed by $\breve{z}(t) = \breve{H}\breve{x}(t)$ with a state vector $\breve{x}(t)$ having n components.

$$\begin{aligned} & \breve{z}(t) = \breve{H}\breve{x}(t), \breve{z}(t) = \breve{x}_1(t) \\ & \breve{H} = [I_{m\times m} \quad 0 \quad 0 \quad \cdots \quad 0], \\ & \breve{x}(t) = \begin{bmatrix} \breve{x}_1(t) \\ \breve{x}_2(t) \\ \vdots \\ \breve{x}_{n-1}(t) \\ \breve{x}_n(t) \end{bmatrix}. \end{aligned} \tag{3}$$

Let $\breve{x}_1(t)$ satisfy a differential equation

$$\begin{aligned} \frac{d\breve{x}_1^n(t)}{dt^n} = & -\breve{a}_1 \frac{d\breve{x}_1^{n-1}(t)}{dt^n} - \breve{a}_2 \frac{d\breve{x}_1^{n-2}(t)}{dt^{n-2}} \cdots \\ & - \breve{a}_{n-1} \frac{d\breve{x}_1(t)}{dt} - \breve{a}_n \breve{x}_1(t) \\ & + \xi(t). \end{aligned} \tag{4}$$

(4) is transformed into the state differential equations.

$$\begin{aligned} & \frac{d\breve{x}(t)}{dt} = \breve{A}\breve{x}(t) + \breve{\Gamma}\xi(t), \\ & E[\xi(t)\xi^T(s)] = \breve{Q}\delta(t-s), \\ & \breve{A} = \begin{bmatrix} 0 & I_{m\times m} & 0 & \cdots & 0 \\ 0 & 0 & I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_{m\times m} \\ -\breve{a}_n & -\breve{a}_{n-1} & -\breve{a}_{n-2} & \cdots & -\breve{a}_1 \end{bmatrix}, \\ & \breve{\Gamma} = [0 \quad 0 \quad \cdots \quad 0 \quad I_{I_{m\times m}}]^T, \end{aligned} \tag{5}$$

$\xi(t)$ in (4) is the residual in approximating the degraded signal $\breve{z}(t)$. It is recommended that the order of the differential equation of (4) is n and the variance $\breve{Q}$ of the random residual $\xi(t)$ is set to zero from the viewpoint of least mean squares estimation for the system matrix $\breve{A}$. A numerical simulation example will verify these two suggestions. For $\breve{Q} = 0$, $\frac{d\breve{x}(t)}{dt} = \breve{A}\breve{x}(t)$ is valid. Hence, $\breve{A}$ satisfies $E\left[\frac{d\breve{x}(t)}{dt}\breve{x}^T(t)\right] = \breve{A}E[\breve{x}(t)\breve{x}^T(t)]$ and $\breve{A}$ is estimated by the relationship of (6).

$$\breve{A} = E\left[\frac{d\breve{x}(t)}{dt}\breve{x}^T(t)\right]E[\breve{x}(t)\breve{x}^T(t)]^{-1}.$$

$$\begin{aligned}&\breve{x}^T(t) = \\ &[\breve{x}_1(t) \quad \breve{x}_2(t) \quad \cdots \quad \breve{x}_{n-1}(t) \quad \breve{x}_n(t)], \\ &\breve{x}_1(t) = \breve{z}(t) \quad , \quad \breve{x}_2(t) = \frac{d\breve{z}(t)}{dt} \quad , \quad \cdots \quad , \\ &\breve{x}_{n-1}(t) = \frac{d^{n-2}\breve{z}(t)}{dt^{n-2}}, \breve{x}_n(t) = \frac{d^{n-1}\breve{z}(t)}{dt^{n-1}}\end{aligned} \quad (6)$$

Also, $\breve{H}$ is estimated by

$$\breve{H} = E[\breve{y}(t)\breve{x}^T(t)]E[\breve{x}(t)\breve{x}^T(t)]^{-1}. \quad (7)$$

3 Robust Filtering Problem

Let the filtering estimate $\hat{x}(t)$ of $x(t)$ be given by

$$\hat{x}(t) = \int_0^t h(t,s)\breve{y}(s)ds \quad (8)$$

as a linear transformation of the degraded observed value $\breve{y}(s)$. Here, $h(t,s)$ represents an impulse response function. Let us consider minimizing the mean-square value

$$J = E[(x(t) - \hat{x}(t))^T(x(t) - \hat{x}(t))] \quad (9)$$

of the filtering error $x(t) - \hat{x}(t)$. The filtering estimate $\hat{x}(t)$ to minimize the cost function J satisfies the relationship

$$x(t) - \hat{x}(t) \perp \breve{y}(s), 0 < s < t, \quad (10)$$

from the orthogonal projection lemma [16]. Therefore, we get an integral equation

$$\begin{aligned}&E[x(t)\breve{y}^T(s)] \\ &= \int_0^t h(t,\tau)E[\breve{y}(\tau)\breve{y}^T(s)]ds, 0 < s < t.\end{aligned} \quad (11)$$

Substituting the degraded observation equation in (2) into (11), (11) is transformed into

$$\begin{aligned}&h(t,s)R \\ &= K_{x\breve{y}}(t,s) - \int_0^t h(t,\tau)\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau. \\ &K_{x\breve{y}}(t,s) = E[x(t)\breve{y}^T(s)], \\ &K_{\breve{x}}(\tau,s) = E[\breve{x}(\tau)\breve{x}^T(s)].\end{aligned} \quad (12)$$

Starting with (12), the robust RLS Wiener filtering algorithm is derived. Assume that the cross-covariance function $K_{x\breve{y}}(t,s)$ of $x(t)$ with $\breve{y}(s)$ is expressed as

$$K_{x\breve{y}}(t,s) = \begin{cases}\alpha(t)\beta^T(s), 0 \le s \le t, \\ \gamma(t)\delta^T(s), 0 \le t \le s.\end{cases} \quad (13)$$

Let the covariance function $K_{\breve{x}}(t,s)$ of $\breve{x}(t)$ be expressed by

$$K_{\breve{x}}(t,s) = \begin{cases}\breve{C}(t)\breve{D}^T(s), 0 \le s \le t, \\ \breve{D}(t)\breve{C}^T(s), 0 \le t \le s.\end{cases} \quad (14)$$

Apart from the current approach, the Kalman filter is utilized for state estimation with accurate information on the state-space model. When dealing with systems that have uncertain parameters, the Kalman filter with artificial intelligence (AI) based on neural networks (NN) can be classified into four groups [17].

4 Robust RLS Wiener Filtering Algorithm

Theorem 1 presents the robust RLS Wiener filtering algorithm for $\hat{x}(t)$.

Theorem 1 For the nominal system (1), the robust RLS Wiener filtering algorithm with the degraded observed value $\breve{y}(t)$ in (2) consists of (15)-(20).

$$\begin{aligned}&\frac{d\hat{x}(t)}{dt} = A\hat{x}(t) + h(t,t)\left(\breve{y}(t) - \breve{H}\hat{\breve{x}}(t)\right), \\ &\hat{x}(0) = 0\end{aligned} \quad (15)$$

$h(t,t)$: Filter gain for $\hat{x}(t)$.

$$h(t,t) = \left(K_{x\breve{y}}(t,t) - S(t)\breve{H}^T\right)R^{-1}, R > 0 \quad (16)$$

$K_{x\breve{y}}(t,t)$: Cross-variance function of $x(t)$ with $\breve{y}(t)$.

$S(t)$: Cross-variance function of $\hat{x}(t)$ with $\hat{\breve{x}}(t)$, $E\left[\hat{x}(t)\hat{\breve{x}}^T(t)\right]$.

$$\begin{aligned}\frac{dS(t)}{dt} &= AS(t) + S(t)\breve{A}^T \\ &\quad + h(t,t)\breve{H}\big(K_{\breve{x}}(t,t) \\ &\quad - S_0(t)\big), \\ S(0) &= 0\end{aligned} \quad (17)$$

$\hat{\breve{x}}(t)$: Filtering estimate of $\breve{x}(t)$.

$$\begin{aligned}&\frac{d\hat{\breve{x}}(t)}{dt} = \breve{A}\hat{\breve{x}}(t) + \breve{h}(t,t)\left(\breve{y}(t) - \breve{H}\hat{\breve{x}}(t)\right), \\ &\hat{\breve{x}}(0) = 0\end{aligned} \quad (18)$$

$h(t,t)$: Filter gain for $\hat{\breve{x}}(t)$.

$$\breve{h}(t,t) = \left(K_{\breve{x}}(t,t) - S_0(t)\right)\breve{H}^T R^{-1}, R > 0 \quad (19)$$

$S_0(t)$: Variance function of $\hat{\breve{x}}(t)$, $E\left[\hat{\breve{x}}(t)\hat{\breve{x}}^T(t)\right]$.

$$\frac{dS_0(t)}{dt} = \breve{A}S_0(t) + S_0(t)\breve{A}^T + \breve{h}(t,t)\breve{H}\left(K_{\breve{x}}(t,t) - S_0(t)\right), \quad S_0(0) = 0 \quad (20)$$

For the robust RLS Wiener filter of Theorem 1 to be stable, R must be a positive definite matrix: $R > 0$. In addition, the asymptotic stability condition for the robust RLS Wiener filter is that all eigenvalues of the matrices A and $\breve{A} - \breve{h}(t,t)\breve{H}$ have negative real parts.

Proof of Theorem 1 is deferred to the Appendix.

4 A Numerical Simulation Example

Let the observation equation for $y(t)$ and the state differential equations for $x(t)$ be given by

$$\begin{aligned} &y(t) = z(t) + v(t), z(t) = Hx(t), \\ &H = [1 \quad 0], \\ &\frac{dx(t)}{dt} = Ax(t) + \Gamma w(t), \\ &x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, x(0) = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}, \\ &A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, a_1 = 4, a_2 = 3, \\ &\Gamma = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \\ &E[v(k)v(s)] = R\delta(t-s), \\ &E[w(t)w(s)] = Q\delta(t-s), Q = 1, \\ &E[v(t)w(s)] = 0. \end{aligned} \quad (21)$$

Let the observation equation for the degraded signal $\breve{z}(t)$, and the state differential equations for the degraded state $\overleftrightarrow{x}(t)$ be given by

$$\begin{aligned} &\breve{y}(t) = \breve{z}(t) + v(t), \breve{z}(t) = \overleftrightarrow{H}(t)\overleftrightarrow{x}(t), \\ &\frac{d\overleftrightarrow{x}(t)}{dt} = \overleftrightarrow{A}(t)\overleftrightarrow{x}(t) + \Gamma w(t), \\ &\overleftrightarrow{A}(t) = A + \Delta A(t), \overleftrightarrow{H}(t) = H + \Delta H(t), \\ &\Delta A(t) = \begin{bmatrix} 0 & 0 \\ -0.1 * rand & -0.1 * rand \end{bmatrix}, \\ &\Delta H(t) = [0.1 \quad 0], \\ &E[\overleftrightarrow{x}(0)w(t)] = 0. \end{aligned} \quad (22)$$

Here, $\Delta A(t)$ is the additional uncertain matrix to the system matrix A. "$rand$" represents a scalar random number chosen from a uniform distribution in the interval (0,1). From (3) and (5), let the observation equation for the degraded signal $\breve{z}(t)$ and the state differential equations for the degraded state $\breve{x}(t)$ be given by

$$\begin{aligned} &\breve{y}(t) = \breve{z}(t) + v(t), \breve{z}(t) = \breve{H}\breve{x}(t), \\ &\breve{z}(t) = \breve{x}_1(t), \\ &\breve{x}(t) = \begin{bmatrix} \breve{x}_1(t) \\ \breve{x}_2(t) \end{bmatrix}, \\ &\frac{d\breve{x}(t)}{dt} = \breve{A}\breve{x}(t) + \breve{\Gamma}\xi(t), \breve{\Gamma} = [0 \quad 1]^T, \\ &E[\xi(t)\xi(s)] = \breve{Q}\delta(t-s). \end{aligned} \quad (23)$$

It should be noted that the robust RLS Wiener filtering algorithm of Theorem 1 does not require information on the input noise variance $\breve{Q}$. Provided that $\breve{Q} = 0$, from (6), $\breve{A}$ is calculated by

$$\breve{A} = \begin{bmatrix} E\left[\frac{d\breve{z}(t)}{dt}\breve{z}(t)\right] & E\left[\frac{d\breve{z}(t)}{dt}\frac{d\breve{z}(t)}{dt}\right] \\ E\left[\frac{d^2\breve{z}(t)}{dt^2}\breve{z}(t)\right] & E\left[\frac{d^2\breve{z}(t)}{dt^2}\frac{d\breve{z}(t)}{dt}\right] \end{bmatrix} \times \begin{bmatrix} E[\breve{z}^2(t)] & E\left[\breve{z}(t)\frac{d\breve{z}(t)}{dt}\right] \\ E\left[\frac{d\breve{z}(t)}{dt}\breve{z}(t)\right] & E\left[\frac{d\breve{z}(t)}{dt}^2\right] \end{bmatrix}^{-1}. \quad (24)$$

The other expression for $\breve{A}$ is given by

$$\breve{A} = \begin{bmatrix} E\left[\frac{d\breve{z}(t)}{dt}\breve{z}(s)\right] & E\left[\frac{d\breve{z}(t)}{dt}\frac{d\breve{z}(s)}{ds}\right] \\ E\left[\frac{d^2\breve{z}(t)}{dt^2}\breve{z}(s)\right] & E\left[\frac{d^2\breve{z}(t)}{dt^2}\frac{d\breve{z}(s)}{ds}\right] \end{bmatrix} \times \begin{bmatrix} E[\breve{z}(t)\breve{z}(s)] & E\left[\breve{z}(t)\frac{d\breve{z}(s)}{ds}\right] \\ E\left[\frac{d\breve{z}(t)}{dt}\breve{z}(s)\right] & E\left[\frac{d\breve{z}(t)}{dt}\frac{d\breve{z}(s)}{ds}\right] \end{bmatrix}^{-1}. \quad (25)$$

In (25), $s = t - h$, $h = 0.001$, is one candidate. The estimates of $\breve{A}$ by (24) and (25) for $n = 2$ are an exact coincidence. (25) is based on the relationship $\frac{\partial K_{\breve{x}}(t,s)}{\partial t} = \breve{A}K_{\breve{x}}(t,s)$, $0 \le s < t$ [11].

For the data sampling interval $h = 0.001$, a four-point forward-difference formula with a truncation error of $O(h^2)$ is utilized in numerical differentiation to approximate the derivatives in (24). For computing the expectation $E\left[\frac{d\breve{z}(t)}{dt}\breve{z}(t)\right]$ as an example, $\frac{1}{T}\int_0^T \frac{d\breve{z}(t)}{dt}\breve{z}(t)dt$, $T = 2.0$, is approximately computed using Simpson's $\frac{1}{3}$ rule's numerical integration method. The integral step size in this case is 0.001. For the state vector $\breve{x}(t)$ with two components, the estimate of the system matrix $\breve{A}$ is calculated by (24) as

$$\begin{bmatrix} 2.380918 \times 10^{-19} & 9.999999 \times 10^{-1} \\ -3.20776 & -4.194539 \end{bmatrix}. \qquad (26)$$

For the state vector $\breve{x}(t)$ with three and four components, respectively, the estimates of the system matrices are calculated as (27) and (28).

$$\begin{bmatrix} -4.408317 \times 10^{-17} & 9.999999 \times 10^{-1} & 2.623525 \times 10^{-18} \\ -3.894717 \times 10^{-16} & 7.728085 \times 10^{-16} & 9.999999 \times 10^{-1} \\ -6.847911 \times 10^{3} & -8.989048 \times 10^{3} & -2.131369 \times 10^{3} \end{bmatrix} \qquad (27)$$

$$\begin{bmatrix} -8.919732 \times 10^{-17} & 1 & 8.299837 \times 10^{-18} & -1.417337 \times 10^{-21} \\ -2.607540 \times 10^{-16} & 2.135920 \times 10^{-16} & 9.999999 \times 10^{-1} & -4.299229 \times 10^{-20} \\ -9.375048 \times 10^{-13} & -2.060725 \times 10^{-12} & -3.707363 \times 10^{-13} & 9.999999 \times 10^{-1} \\ -4.0327561 \times 10^{7} & -5.292500 \times 10^{7} & -1.251926 \times 10^{7} & -4.814171 \times 10^{3} \end{bmatrix} \qquad (28)$$

The estimates of the system matrices $\breve{A}$ in (27) and (28) are unavailable, since the third row in (27) and the fourth row in (28) display large values of orders 10^3 and 10^7, respectively. Therefore, $n = 2$ is the appropriate order for the estimate of $\breve{A}$. Table 1 shows the estimates of $\breve{H}$ from (7) for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, and $N(0, 0.5^2)$. For the degraded signal $\breve{z}(t) = \breve{H}\breve{x}(t)$ in (23), the estimate of $\breve{H}$ is very close to $H = [1 \quad 0]$. Substituting A, $\breve{A}$, $\breve{H}$, $K_{x\breve{y}}(t,t)$, $K_{\breve{x}}(t,t)$ and the observed value $\breve{y}(t)$ into the robust RLS Wiener filtering algorithm of Theorem 1, the

Table 1. Estimates of $\breve{H}$ for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, and $N(0, 0.5^2)$.

White Gaussian observation noise	Estimates of $\breve{H}$
$N(0, 0.1^2)$	$[9.999999 \times 10^{-1} \quad -1.694109 \times 10^{-16}]$
$N(0, 0.3^2)$	$[1 \quad 2.227923 \times 10^{-16}]$
$N(0, 0.5^2)$	$[9.999999 \times 10^{-1} \quad 1.736875 \times 10^{-17}]$

filtering estimate $\hat{x}(t)$ of the state $x(t)$ is recursively computed. Fig. 1 illustrates the filtering estimate $\hat{x}_1(t)$ of the state variable $x_1(t)$ vs. t for the white Gaussian observation noise $N(0, 0.3^2)$.

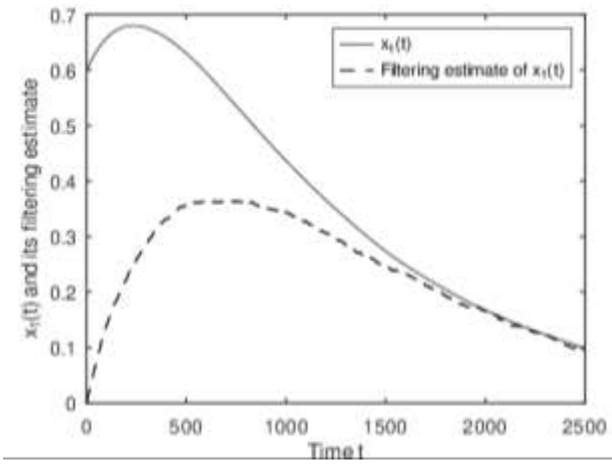

Fig. 1: Filtering estimate $\hat{x}_1(t)$ of the state variable $x_1(t)$ vs. t for the white Gaussian observation noise $N(0, 0.3^2)$.

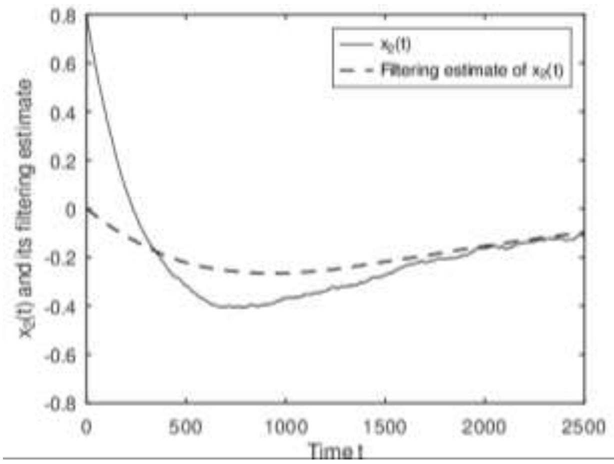

Fig. 2: Filtering estimate $\hat{x}_2(t)$ of the state variable $x_2(t)$ vs. t for the white Gaussian observation noise $N(0, 0.3^2)$.

As time t advances, the filtering estimate $\hat{x}_1(t)$ approaches the state $x_1(t)$ gradually. Fig. 2 illustrates the filtering estimate $\hat{x}_2(t)$ of the state variable $x_2(t)$ vs. t for the white Gaussian observation noise $N(0, 0.3^2)$. As time t advances, the filtering estimate $\hat{x}_2(t)$ approaches the state $x_2(t)$ gradually. Table 1 shows mean-square values (MSVs) of the filtering errors $x_1(t) - \hat{x}_1(t)$ and $x_2(t) - \hat{x}_2(t)$ for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, and $N(0, 0.5^2)$. Here, the MSVs are calculated by $\frac{1}{2500}\sum_{i=1}^{2500}\left(x_1(i \cdot h) - \hat{x}_1(i \cdot h)\right)^2$ and $\frac{1}{2500}\sum_{i=1}^{2500}\left(x_2(i \cdot h) - \hat{x}_2(i \cdot h)\right)^2$, $h = 0.001$, respectively.

Table 2. Mean-square values of the filtering errors $x_1(t) - \hat{x}_1(t)$ and $x_2(t) - \hat{x}_2(t)$ for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, and $N(0, 0.5^2)$.

White Gaussian observation noise	MSV of $x_1(t) - \hat{x}_1(t)$	MSV of $x_2(t) - \hat{x}_2(t)$
$N(0, 0.1^2)$	6.707662×10^{-2}	3.209815×10^{-2}
$N(0, 0.3^2)$	4.526449×10^{-2}	2.479447×10^{-2}
$N(0, 5^2)$	8.558077×10^{-2}	3.611747×10^{-2}

Table 3 shows the MSVs of the filtering errors $x_1(t) - \hat{x}_1(t)$ and $x_2(t) - \hat{x}_2(t)$ for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, and $N(0, 0.5^2)$ by the robust Kalman filter [1]. Based on the robust Kalman filter [1], we employ the following parameters: $D_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, $D_2 = [0 \quad 1]$, $\Delta(t) = \begin{bmatrix} 0 & 0 \\ 0 & -0.1 \end{bmatrix}$, $0 \le \Delta^T(t)\Delta(t) \le I$, a scaling parameter $\varepsilon = 1 > 0$.

Table 2 and Table 3 show that the proposed robust RLS Wiener filter is superior in estimation accuracy to the robust Kalman filter [1].

In the simulation example, the numerical integration computations were performed using a fourth-order Runge-Kutta-Gill method with a sampling interval of h=0.001.

Table 3. Mean-square values of the filtering errors $x_1(t) - \hat{x}_1(t)$ and $x_2(t) - \hat{x}_2(t)$ for the white Gaussian observation noises $N(0, 0.1^2)$, $N(0, 0.3^2)$, and $N(0, 0.5^2)$ [1].

White Gaussian observation noise	MSV of $x_1(t) - \hat{x}_1(t)$	MSV of $x_2(t) - \hat{x}_2(t)$
$N(0, 0.1^2)$	0.125156	7.160740×10^{-2}
$N(0, 0.3^2)$	0.128165	7.178225×10^{-2}
$N(0, 5^2)$	0.133048	7.222379×10^{-2}

4 Conclusion

This paper has proposed a novel robust RLS Wiener filter for linear continuous-time systems with uncertainties in the system and observation matrices. Under the uncertainties in the system and observation matrices, the degraded signal is fitted to the AR model of finite order. The AR model of the degraded signal is related to the state-space model, and the system and observation matrices for the degraded signal are formulated. Estimation formulas for the system and observation matrices were suggested in Section 2. Therefore, unlike the robust Kalman filter, the robust RLS Wiener filter of Theorem 1 does not need to assume norm-bounded uncertainties for the uncertain system and observation matrices. The robust RLS Wiener filter does not use the information of input matrix Γ and the input noise variance Q in (1). The robust filtering problem is introduced based on the minimization of the mean-square value of the filtering errors for the nominal system states. The robust filtering estimate is given as the integral transformation of the degraded observations using the impulse response function. The integral equation that the optimal impulse response function satisfies is given in Section 3. Theorem 1 presented the robust RLS Wiener filtering algorithm starting from this integral equation.

The numerical simulation example has shown that the robust RLS Wiener filter has better estimation accuracy than the robust Kalman filter.

The new design of an H-infinity tracking controller is desirable for linear continuous stochastic systems with uncertain parameters as a future challenge. By combining the robust RLS Wiener filter proposed in this paper with the new H-infinity tracking control algorithm, tracking control is implementable in linear continuous-time stochastic systems with uncertain parameters.

Appendix: Proof of Theorem 1

Substituting $K_{x\breve{y}}(t,s)=\alpha(t)\beta^T(s), 0\le s\le t,$ in (13) into (12), (12) is rewritten as

$$h(t,s)R = \alpha(t)\beta^T(s) - \int_0^t h(t,\tau)\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau. \quad \text{(A-1)}$$

Let us introduce an auxiliary function $J(t,s)$, which satisfies

$$J(t,s)R = \beta^T(s) - \int_0^t J(t,\tau)\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau. \quad \text{(A-2)}$$

From (A-1) and (A-2), $h(t,s)$ is given by

$$h(t,s)=\alpha(t)J(t,s). \quad \text{(A-3)}$$

Differentiating (A-2) with respect to t, we have

$$\frac{\partial J(t,s)}{\partial t}R = -J(t,t)\breve{H}K_{\breve{x}}(t,s)\breve{H}^T - \int_0^t \frac{\partial J(t,\tau)}{\partial t}\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau. \quad \text{(A-4)}$$

Introducing

$$L(t,s)R = \breve{D}^T(s)\breve{H}^T - \int_0^t L(t,\tau)\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau, \quad \text{(A-5)}$$

and using (14), $\frac{\partial J(t,s)}{\partial t}$ satisfies

$$\frac{\partial J(t,s)}{\partial t} = -J(t,t)\breve{H}\breve{C}(t)L(t,s). \quad \text{(A-6)}$$

Putting $s=t$ in (A-2), we have

$$J(t,t)R = \beta^T(t) - \int_0^t J(t,\tau)\breve{H}K_{\breve{x}}(\tau,t)\breve{H}^T d\tau. \quad \text{(A-7)}$$

Introducing

$$r(t)=\int_0^t J(t,\tau)\breve{H}\breve{D}(\tau)d\tau, \quad \text{(A-8)}$$

and using (14), we have

$$J(t,t)R=\beta^T(t)-r(t)\breve{C}^T(t)\breve{H}^T. \quad \text{(A-9)}$$

Differentiating (A-8) with respect to t, we have

$$\frac{dr(t)}{dt} = J(t,t)\breve{H}\breve{D}(t) + \int_0^t \frac{\partial J(t,\tau)}{\partial t}\breve{H}\breve{D}(\tau)d\tau. \quad \text{(A-10)}$$

Substituting (A-6) into (A-10), we have

$$\frac{dr(t)}{dt} = J(t,t)\breve{H}\left(\breve{D}(t) - \int_0^t \breve{C}(t)L(t,\tau)\breve{H}\breve{D}(\tau)d\tau\right) = J(t,t)\breve{H}(\breve{D}(t)-\breve{C}(t)m(t)), \quad r(0)=0. \quad \text{(A-11)}$$

Here, we introduced the function $m(t)$ given by

$$m(t)=\int_0^t L(t,\tau)\breve{H}\breve{D}(\tau)d\tau). \quad \text{(A-12)}$$

Differentiating (A-5) with respect to t, we have

$$\frac{\partial L(t,s)}{\partial t}R = -L(t,t)\breve{H}K_{\breve{x}}(t,s)\breve{H}^T - \int_0^t \frac{\partial L(t,\tau)}{\partial t}\breve{H}K_{\breve{x}}(\tau,s)\breve{H}^T d\tau. \quad \text{(A-13)}$$

From (A-5) and (A-13), we obtain

$$\frac{\partial L(t,s)}{\partial t} = -L(t,t)\breve{H}\breve{C}(t)L(t,s). \quad \text{(A-14)}$$

Differentiating (A-12) with respect to t, we have

$$\frac{dm(t)}{dt} = L(t,t)\breve{H}\breve{D}(t) + \int_0^t \frac{\partial L(t,\tau)}{\partial t}\breve{H}\breve{D}(\tau)d\tau). \quad \text{(A-15)}$$

Substituting (A-14) into (A-15), we obtain

$$\frac{dm(t)}{dt} = L(t,t)\breve{H}(\breve{D}(t) - \breve{C}(t)m(t)), \quad m(0) = 0. \quad \text{(A-16)}$$

From (8) and (A-3), the filtering estimate $\hat{x}(t)$ is written as

$$\hat{x}(t) = \alpha(t)\int_0^t J(t,s)\breve{y}(s)ds. \quad \text{(A-17)}$$

Differentiating (A-17) with respect to t, using (A-3) and (A-6) with $\frac{d\alpha(t)}{dt} = A\alpha(t)$ from (13), and introducing

$$\hat{\breve{x}}(t) = \breve{C}(t)\int_0^t L(t,s)\breve{y}(s)ds, \quad \text{(A-18)}$$

we have

$$\frac{d\hat{x}(t)}{dt} = \frac{d\alpha(t)}{dt}\int_0^t J(t,s)\breve{y}(s)ds + \alpha(t)\int_0^t \frac{\partial J(t,s)}{\partial t}\breve{y}(s)ds + \alpha(t)J(t,t)\breve{y}(t) = A\hat{x}(t) + h(t,t)\left(\breve{y}(t) - \breve{H}\hat{\breve{x}}(t)\right), \quad \text{(A-19)}$$

$$\hat{x}(0) = 0.$$

From (A-3) and (A-9), $h(t,t)$ is given by

$$h(t,t) = \left(\alpha(t)\beta^T(t) - \alpha(t)r(t)\breve{C}^T(t)\breve{H}^T\right)R^{-1}. \quad \text{(A-20)}$$

Using (13) and introducing a function $S(t) = \alpha(t)r(t)\breve{C}^T(t)$, $h(t,t)$ is written as

$$h(t,t) = \left(K_{x\breve{y}}(t,s) - S(t)\breve{H}^T\right)R^{-1}. \quad \text{(A-21)}$$

Let $\breve{h}(t,s)$ be given by

$$\breve{h}(t,s) = \breve{C}(t)L(t,s). \quad \text{(A-22)}$$

From (5) and (14), it is clear that

$$\frac{d\breve{C}(t)}{dt} = \breve{A}\breve{C}(t). \quad \text{(A-23)}$$

Differentiating (A-18) with respect to t and using (A-14), we obtain

$$\frac{d\hat{\breve{x}}(t)}{dt} = \breve{A}\hat{\breve{x}}(t) + \breve{C}(t)\int_0^t \frac{\partial L(t,s)}{\partial t}\breve{y}(s)ds + \breve{C}(t)L(t,t)\breve{y}(t) = \breve{A}\hat{\breve{x}}(t) + \breve{h}(t,t)\left(\breve{y}(t) - \breve{H}\hat{\breve{x}}(t)\right), \quad \hat{\breve{x}}(0) = 0. \quad \text{(A-24)}$$

Here, $\breve{h}(t,t)$ is given by

$$\breve{h}(t,t) = \breve{C}(t)L(t,t). \quad \text{(A-25)}$$

Differentiating $S(t)$ with respect to t, we have

$$\frac{dS(t)}{dt} = \frac{d\alpha(t)}{dt}r(t)\breve{C}^T(t) + \alpha(t)\frac{dr(t)}{dt}\breve{C}^T(t) + \alpha(t)r(t)\frac{d\breve{C}^T(t)}{dt}. \quad \text{(A-26)}$$

From, (13), (14), (A-3), and (A-11), (A-26) is rewritten as

$$\frac{dS(t)}{dt} = AS(t) + S(t)\breve{A}^T + h(t,t)\breve{H}(K_{\breve{x}}(t,t) - S_0(t)). \quad \text{(A-27)}$$

Here,

$$S_0(t) = \breve{C}(t)m(t)\breve{C}^T(t). \quad \text{(A-28)}$$

Differentiating (A-28) with respect to t, using (A-16), and introducing

$$h_0(t,t) = \breve{C}(t)L(t,t), \quad \text{(A-29)}$$

we obtain

$$\frac{dS_0(t)}{dt} = \breve{A}S_0(t) + S_0(t)\breve{A}^T + \breve{C}(t)L(t,t)\overline{\breve{H}(\breve{D}}(t) - \breve{C}(t)m(t))\breve{C}^T(t)$$
$$= \breve{A}S_0(t) + S_0(t)\breve{A}^T + h_0(t,t)\overline{\breve{H}(K_{\breve{x}}}(t,t) - S_0(t)), \quad \text{(A-30)}$$
$$S_0(0) = 0.$$

From (A-5), $L(t,t)$ satisfies

$$L(t,t)R = \breve{D}^T(t)\breve{H}^T - \int_0^t L(t,\tau)\breve{H}K_{\breve{x}}(\tau,t)\breve{H}^T d\tau. \quad \text{(A-31)}$$

From (14) and (A-12), (A-31) is rewritten as

$$L(t,t)R = \breve{D}^T(t)\breve{H}^T - m(t)\breve{C}^T(t)\breve{H}^T. \quad \text{(A-32)}$$

Hence, we obtain an expression for $h_0(t,t)$ as

$$h_0(t,t) = \left(K_{\breve{x}}(t,t) - S_0(t)\right)\breve{H}^T\breve{R}^{-1}. \quad \text{(A-33)}$$

(Q.E.D.)

References:

[1] L. Xie, Y. C. Soh, Robust Kalman filtering for uncertain systems, *Systems & Control Letters*, Vol.22, No.2, 1994, pp. 123–129.
[2] U. Shaked, C. E. De Souza, Robust minimum variance filtering, *IEEE Transactionsons on Signal Processing,* Vol.43, No.11, 1995, pp. 2474–2483.
[3] X. Yuanquing, H. Jingqing, Robust Kalman filtering for systems under norm bounded uncertainties in all system matrices and error covariance constraints, *Journal of Systems Science and Complexity*, Vol.18, No. 4, 2005, pp. 439–445.
[4] S. O. R. Moheimani, A. V. Savkin, I. R. Petersen, Robust filtering, prediction, smoothing and observability of uncertain systems, *IEEE Transactions on Circuits and Systems—I: Fundamental Theory and Applications*, Vol.45, No.4, 1998, pp. 446–457.
[5] X. Zhu, Y. C. Soh, L. Xie, Design and analysis of discrete-time robust Kalman filters, *Automatica*, Vol.38, No.6, 2002, pp. 1069–1077.
[6] M. Fu, C. E. deSouza, Z.-Q. Luo, Finite-horizon Kalman filter design, *IEEE Transactionsons on Signal Processing*, Vol.49, No.9, 2001, pp. 2103–2112.
[7] F. L. Lewis, L. Xie, D. Popa, *Optimal and Robust Estimation With an Introduction to Stochastic Control Theory*, Second Edition, CRC Press, Boca Raton, 2008.
[8] S. Nakamori, Robust RLS Wiener state estimators in linear discrete-time stochastic systems with uncertain parameters, *Computer Reviews Journal*, Vol. 2, No.1, 2019, pp. 18–33.
[9] A.C. Harvey, J. H. Stock, The Estimation of Higher-Order Continuous Time Autoregressive Models, *Econometric Theory*, Vol.1, No.1, 1985, pp. 97–117.
[10] P. J. Brockwell, O. Stramer, On the approximation of continuous time threshold ARMA processes, *Ann. Inst. Stat. Math.*, Vol.47, No.1, 1995, pp. 1–20.
[11] S. Nakamori, Linear H-Infinity Tracking Control in Discrete-Time Stochastic Systems with Uncertain Parameters, *WSEAS Transactions on Signal Processing*, Vol. 19, 2023, pp. 41-52.
[12] S. Nakamori, Robust RLS Wiener signal estimators for discrete-time stochastic systems with uncertain parameters, *Frontiers in Signal Processing*, Vol. 3, No. 1, 2019, pp. 1-18.
[13] S. Nakamori, Robust RLS Wiener state estimators in linear discrete-time stochastic systems with uncertain parameters, *Computer Reviews Journal*, Vol. 4, 2019, pp. 18-33
[14] S. Nakamori, New H-infinity Tracking Control Algorithm Based on Integral Equation Approach in Linear Continuous Time Systems, *International Journal of Mathematics, Statistics and Operations Research*, Vol. 1, No. 2, 2021, pp. 109-124.
[15] J. Gadewadikar, F. L. Lewis, M. Abu-Khalaf, Necessary and sufficient conditions for *H*-∞ static output-feedback control, *Journal of Guidance, Control, and Dynamics*, Vol. 29, No. 4, 2006, pp. 915-920.
[16] A. P. Sage, J. L. Melsa, *Estimation Theory with Applications to Communications and Control*, McGraw-Hill Book Company, New York,1971.
[17] S. Kim, I. Petrunin, H-S Shin, A review of Kalman filter with artificial intelligence technique, *Proceedings of the 2022 Integrated Communication, Navigation and Surveillance Conferenc*e, 2022, Dulles, USA.

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