Linear H-Infinity Tracking Control in Discrete-Time Stochastic Systems with Uncertain Parameters

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Abstract: In linear discrete-time stochastic systems with uncertain parameters, this study proposes an H-infinity tracking control strategy based on an H-infinity tracking controller and a robust recursive least-squares Wiener filter. A linear H-infinity tracking control algorithm for quantity $u(k)$, whose components are the control and exogenous inputs, was proposed for discrete-time deterministic systems without input and observation noise. Based on the separation principle between control and estimation, this study presents equations for $u(k)$ in linear discrete-time stochastic systems with uncertain parameters as a counterpart to the equations in deterministic systems. The H-infinity tracking control algorithm in linear discrete-time stochastic systems with uncertain parameters is derived in the same manner as the H-infinity tracking control algorithm in linear discrete-time deterministic systems. The filtering estimate $\hat{\breve{x}}(k)$ of the degraded system state $\breve{x}(k)$ is used to calculate the estimate $\hat{u}(k)$ of $u(k)$. The robust RLS Wiener filter calculates the filtering estimate $\hat{\breve{x}}(k)$ of the system state $\breve{x}(k)$ for degraded stochastic systems with uncertain parameters. With knowledge of the estimate $\hat{u}(k-1)$ of $u(k-1)$, the degraded observed value $\breve{y}(k)$, and the filtering estimate $\hat{\breve{x}}(k-1)$ of the degraded state $\breve{x}(k-1)$, $\hat{\breve{x}}(k)$ is updated from $\hat{\breve{x}}(k-1)$.

Key-Words: - H-infinity tracking control, control input, exogenous input, robust recursive least-squares Wiener filter, discrete-time stochastic systems with uncertain parameters.

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1 Introduction

Linear quadratic Gaussian (LQG) control has been studied in e.g., [1], [2], [3], [4], [5], [6]. In addition, LQG tracking control problems have also been investigated, [7], [8], [9], [10], [11]. A real-time transcale LQG tracking control technique for discrete-time stochastic systems was presented in [11], and was based on wavelet packet decomposition (WPD). The system in this scenario excluded unknown parameters. An output feedback controller was developed for discrete-time stochastic systems with uncertainties and missing measurements, [12]. The parameter uncertainties were norm-bounded. The probability that the missing data will occur presupposes that it is known. Using linear matrix inequalities (LMIs) solves this problem. In [13], a robust controller based on a disturbance observer was proposed for linear continuous-time uncertain systems with a time delay. The LMI solution determines observer parameters. It deals with state feedback control. The H-infinity controller was designed in [14], for a state-space model with uncertain parameters in linear continuous-time stochastic systems. A robust controller has been developed for linear discrete-time uncertain systems, [15]. In [15], a state feedback controller is designed using the LMI technique, and a low-order disturbance observer is presented. The LMI approach is presented for state feedback quadratic stabilization in linear continuous-time uncertain systems in subsection 5. 3 of [16]. For linear discrete-time uncertain stochastic systems, a combined H2/Passivity controller was developed, [17]. Some sufficient conditions are converted into LMIs using the Lyapunov theory. A repeated-tracking controller for stochastic time-varying delay systems was designed in [18].

An H-infinity tracking control technique was proposed in [19], for deterministic systems without input and observation disturbances. Robust recursive least-squares (RLS) Wiener filter and fixed-point smoother were proposed in [20], [21], for linear discrete-time stochastic systems with uncertain parameters in the system and observation matrices. Signal estimation was the purpose of the estimators in [20]. The robust RLS Wiener estimators in [21], estimate the nondegraded nominal system state rather than the degraded system state when utilizing degraded observations. In this case, the estimators employ the system and observation matrices from the original system. In linear discrete-time systems with norm-bounded uncertainties in the system and input matrices, a robust filter estimates the degraded state, [22]. A robust Kalman filter, [23], was designed for the linear discrete-time state-space model with multiplicative noise and norm-constrained time-varying uncertainties both in the system and observation matrices. In [24], a robust Kalman filter was proposed for linear discrete-time uncertain systems with norm-bounded uncertainties in the system and observation matrices. Recently, an H-infinity tracking control method using the robust RLS Wiener filter in [21], was developed to track the nondegraded nominal signal to the desired value in linear discrete-time uncertain systems with uncertainties in the system and observation matrices, [25]. As seen from [22], [23], [24], these robust filters estimate the degraded state rather than the nondegraded nominal system state. From this fact, this paper aims to newly design an H-infinity tracking controller for the degraded signal to track the desired value for linear discrete-time stochastic systems with uncertain parameters in Theorem 1. For this purpose, Theorem 2 proposes a new robust RLS Wiener filter to estimate the degraded state. It is assumed herein that uncertainties exist in the system and observation matrices. No norm-bounded uncertainties are assumed for the uncertain matrices. The uncertain system and observation matrices are estimated by (18) and (19), respectively. Using the uncertain system and observation matrix estimates, the robust RLS Wiener filter of Theorem 2 recursively calculates the filtering estimate of the degraded system state. Based on the separation principle of control and estimation, it is shown in Section 2 that $u(k)$ satisfies (10)-(12) for linear discrete-time stochastic systems with uncertainties, corresponding to the deterministic systems in [19]. Here, $u(k)$ consists of vector components, control input, and exogenous input. The filtering estimate $\hat{\overleftrightarrow{x}}(k)$ of the degraded system state $\overleftrightarrow{x}(k)$ is used to calculate the estimate $\hat{u}(k)$ of $u(k)$. The robust RLS Wiener filter in Theorem 2 computes the filtering estimate $\hat{\overleftrightarrow{x}}(k)$ of the degraded system state $\overleftrightarrow{x}(k)$ for the degraded state-space model with uncertainties. Information on the estimate $\hat{u}(k-1)$ of $u(k-1)$, the degraded observed value $\breve{y}(k)$, and the filtering estimate $\hat{\overleftrightarrow{x}}(k-1)$ of the degraded state $\overleftrightarrow{x}(k-1)$ is used to update $\hat{\overleftrightarrow{x}}(k)$ from $\hat{\overleftrightarrow{x}}(k-1)$. The estimate $\hat{u}(k)$ of $u(k)$ in Theorem 1 uses the filtering estimate $\hat{\overleftrightarrow{x}}(k)$ of the degraded state $\overleftrightarrow{x}(k)$ by the robust RLS Wiener filter in Theorem 2.

In Section 4, the first numerical simulation example compares the tracking control accuracy of the H-infinity tracking controller of Theorem 1 and the robust RLS Wiener filter of Theorem 2 with that of the H-infinity tracking controller of Theorem 1 and the RLS Wiener filter, [26], or the robust Kalman filter, [24]. Compared to the combinations of the H-infinity tracking controller of Theorem 1 with either the RLS Wiener filter or the robust Kalman filter, the combination of the H-infinity tracking controller of Theorem 1 with the robust RLS Wiener filter of Theorem 2 provides superior tracking control accuracy. The second simulation example demonstrates F16 aircraft tracking control in terms of tracking accuracy.

2 H-Infinity Linear Tracking Control Problem

Let (1) represent the discrete-time state-space model in linear stochastic systems.

$$\begin{aligned}
&y(k) = z(k) + v(k), z(k) = Cx(k),\\
&x(k+1) = Ax(k) + Gu(k) + \Gamma w(k),\\
&G = [G_1 \quad G_2], u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix},\\
&x(0) = c, E[v(k)v^T(s)] = V\delta_K(k-s),\\
&E[w(k)w^T(s)] = W\delta_K(k-s),\\
&E[v(k)w^T(s)] = 0, E[x(0)w^T(k)] = 0.
\end{aligned} \quad (1)$$

Here, $x(k) \in R^n$ is the state vector, $u(k) \in R^m$ is the input vector, and $z(k) \in R^l$ is the signal vector. $u_1(k) \in R^{m_1}$ and $u_2(k) \in R^{m_2}$, $m_1 + m_2 = m$, are the control and exogenous input vectors, respectively. The input noise $w(k) \in R^p$ and the observation noise $v(k) \in R^l$ are mutually uncorrelated with zero mean white Gaussian noise. Γ is the $n \times p$ input matrix, and C is the $l \times n$ observation matrix. The auto-covariance functions for the input noise $w(k)$ and the observation noise $v(k)$ are given in (1). This paper considers the state and observation equations with uncertain parameters in (2).

$$\begin{aligned}
&\breve{y}(k) = \check{z}(k) + v(k),\\
&\check{z}(k) = \overleftrightarrow{C}(k)\overleftrightarrow{x}(k), \overleftrightarrow{C}(k) = C + \Delta C(k),\\
&\overleftrightarrow{x}(k+1) = \overleftrightarrow{A}(k)\overleftrightarrow{x}(k) + Gu(k) + \Gamma w(k),\\
&\overleftrightarrow{A}(k) = A + \Delta A(k), \overleftrightarrow{x}(0) = \overleftrightarrow{c},\\
&E[v(k)w^T(s)] = 0, E[\Delta A(k)w^T(s)] = 0,\\
&E[\Delta C(k)w^T(s)] = 0, E[\overleftrightarrow{x}(0)w^T(k)] = 0
\end{aligned} \quad (2)$$

In (2), the degraded system matrix $\overleftrightarrow{A}(k)$ and the degraded observation matrix $\overleftrightarrow{C}(k)$ are introduced instead of the system matrix A and the observation matrix C in (1), respectively. Here, the matrix elements of $\Delta A(k)$ and $\Delta C(k)$ consist of unknown variables. The initial system state $\overleftrightarrow{x}(0)$ is a random vector that is uncorrelated with both system and measurement noise processes. Let $\tilde{z}(k)$ represent the performance output, [27]. The expected value of $\|\tilde{z}(k)\|_2^2$ is given by (3).

$$E[\|\tilde{z}(k)\|_2^2] = E[(\eta(k) - \check{z}(k))^T Q(k)(\eta(k) - \check{z}(k))] + E[u_1^T(k)\tilde{R}(k)u_1(k)] \quad (3)$$

Here, $\eta(k)$ is the desired value, and $Q(k)$ and $\tilde{R}(k)$ are symmetric positive-definite matrices. As in [19], the H-infinity optimal tracking control problem is to find the control input $u_1(k)$ and the exogenous input $u_2(k)$ when γ is at its minimum value in the disturbance attenuation condition (4). $\gamma > 0$ is referred to as the constant-disturbance attenuation level.

$$\sum_{k=0}^{L} E[(\eta(k) - \check{z}(k))^T Q(k)(\eta(k) - \check{z}(k))] + \sum_{k=0}^{L} E[u_1^T(k)\tilde{R}(k)u_1(k)] \le \gamma^2 \sum_{k=0}^{L} E[u_2^T(k)u_2(k)] \quad (4)$$

The H-infinity tracking control problem for a finite horizon equivalently transforms into a two-person zero-sum linear quadratic dynamic game, [28], [29]. Namely, given γ^2, we investigate the minimax problem, which minimizes the value function $J(x, u_1, u_2)$ for the control input $u_1(k)$ and maximize $J(x, u_1, u_2)$ for the exogenous input $u_2(k)$.

$$J(x, u_1, u_2) = \sum_{k=0}^{L} E[(\eta(k) - \check{z}(k))^T Q(k)(\eta(k) - \check{z}(k)) + u_1^T(k)\tilde{R}(k)u_1(k) - \gamma^2 u_2^T(k)u_2(k)] \quad (5)$$

We assume that the vector $u(k)$ contains the components of the control input $u_1(k)$ and the exogenous input $u_2(k)$. Introducing $R(k) = \begin{bmatrix} \tilde{R}(k) & 0 \\ 0 & -\gamma^2 I_{m_2 \times m_2} \end{bmatrix}$ transforms (5) into (6).

$$J(x, u_1, u_2) = \sum_{k=0}^{L} [(\eta(k) - \check{z}(k))^T Q(k)(\eta(k) - \check{z}(k)) + u^T(k)R(k)u(k)] \quad (6)$$

In the value function (6) the discount factor is 1. $\overleftrightarrow{x}(k)$ is expressed as

$$\begin{aligned} &\overleftrightarrow{x}(k) = \overleftrightarrow{\Phi}(k,0)\overleftrightarrow{c} + \sum_{i=0}^{L} 1(k-i-1)\overleftrightarrow{\Phi}(k,i+1)(Gu(i) + \Gamma w(i)), \\ &1(\alpha) = \begin{cases} 1, 0 \le \alpha, \\ 0, \alpha < 0, \end{cases} \\ &\overleftrightarrow{\Phi}(k,s) = \begin{cases} \overleftrightarrow{A}(k-1)\overleftrightarrow{A}(k-2)\cdots\overleftrightarrow{A}(s), & 0 \le s < k, \\ I, & k = s. \end{cases} \end{aligned} \quad (7)$$

Here, $\overleftrightarrow{\Phi}(k,s)$ represents the state-transition matrix for the system matrix $\overleftrightarrow{A}(k)$, and $1(\alpha)$ represents the discrete-time unit step sequence. Substituting (7) into (6), we have

$$\begin{aligned} J(x, u_1, u_2) &= \sum_{k=0}^{L} E[(\eta(k) - \overleftrightarrow{C}(k)\overleftrightarrow{\Phi}(k,0)\overleftrightarrow{c} - \sum_{i=0}^{L} 1(k-i-1)\overleftrightarrow{C}(k)\overleftrightarrow{\Phi}(k,i+1)(Gu(i) + \Gamma w(i)))^T Q(k)(\eta(k) - \overleftrightarrow{C}(k)\overleftrightarrow{\Phi}(k,0)\overleftrightarrow{c} \\ &- \sum_{i=0}^{L} 1(k-i-1)\overleftrightarrow{C}(k)\overleftrightarrow{\Phi}(k,i+1)(Gu(i) + \Gamma w(i))) + u^T(k)R(k)u(k)]. \end{aligned} \quad (8)$$

According to the calculus of variations, [19], the necessary condition for $u(k)$ to minimize the value function (8) for $u_1(k)$ and maximize (8) for $u_2(k)$, is given by (9).

$$\begin{aligned} &R(k)u(k) + \sum_{i=0}^{L}\sum_{j=0}^{L} 1(i-k-1) \times 1(i-j-1)G^T\overleftrightarrow{\Phi}^T(i,k+1)\overleftrightarrow{C}^T(i)Q(i)\overleftrightarrow{C}(i) \times \overleftrightarrow{\Phi}(i,j+1)Gu(j) \\ &= \sum_{i=0}^{L} 1(i-k-1)G^T\overleftrightarrow{\Phi}^T(i,k+1)\overleftrightarrow{C}^T(i)Q(i) \times (\eta(i) - \overleftrightarrow{C}(i)\overleftrightarrow{\Phi}(i,0)\overleftrightarrow{c}) \end{aligned} \quad (9)$$

If we introduce

$$K(k,j) = \begin{cases} \sum_{i=k+1}^{L} G^T\overleftrightarrow{\Phi}^T(i,k+1)\overleftrightarrow{C}^T(i)Q(i)\overleftrightarrow{C}(i)\overleftrightarrow{\Phi}(i,j+1), & 0 \le j \le k \le L, \\ \sum_{i=j+1}^{L} G^T\overleftrightarrow{\Phi}^T(i,k+1)\overleftrightarrow{C}^T(i)Q(i)\overleftrightarrow{C}(i)\overleftrightarrow{\Phi}(i,j+1), & 0 \le k \le j \le L, \end{cases} \quad (10)$$

and

$$m(k+1) = -\sum_{i=k+1}^{L} G^T\overleftrightarrow{\Phi}^T(i,k+1)\overleftrightarrow{C}^T(i)Q(i) \quad (11)$$

$$\times \left(\overleftrightarrow{C}(i) \overleftrightarrow{\Phi}(i,0) \overleftrightarrow{c} - \eta(i) \right),$$

the optimal $u(k)$ satisfies

$$R(k)u(k) + \sum_{j=0}^{L} K(k,j)Gu(j) \\ = m(k+1). \tag{12}$$

The sufficient condition for the value function $J(x, u_1, u_2)$ to be minimal for $u_1(k)$ and maximal for $u_2(k)$ is given by $R(k)\delta_K(k-s) + K(k,s)G > 0$, [19].

Note that (10)-(12) of the H-infinity tracking control problem for the uncertain systems (2) can be obtained similarly to the equations in [19], for the H-infinity tracking control problem in linear deterministic systems. Theorem 1 presents the H-infinity tracking control algorithm derived from (10)-(12). In Section 2 of [11], the filtering estimate used in the LQG tracking control algorithm was calculated using the Kalman filter. Thus, the separation principle of control and estimation is valid for the LQG tracking control problem. In [11], the filtering estimate is calculated using (11) and (12), which include the term related to the control input. In addition, the filter gain was calculated in the Kalman filter, [11]. The separation principle of control and estimation is valid for the degraded uncertain systems (2). Theorem 1 proposes the H-infinity tracking-control algorithm. The estimate $\hat{u}(k)$ of $u(k)$ uses the filtering estimate $\hat{\overleftrightarrow{x}}(k)$ of the degraded state $\overleftrightarrow{x}(k)$. The robust RLS Wiener filter in Theorem 2 computes the filtering estimate $\hat{\overleftrightarrow{x}}(k)$ of the degraded state $\overleftrightarrow{x}(k)$ using the degraded observed value $\breve{y}(k)$. Theorems 1 and 2 utilize the time-invariant estimates $\hat{\overleftrightarrow{A}}$ and $\hat{\overleftrightarrow{C}}$ for the unknown time-varying system and observation matrices $\overleftrightarrow{A}(k)$ and $\overleftrightarrow{C}(k)$, respectively.

3 H-Infinity Tracking Control Algorithm and Robust RLS Wiener Filter in Stochastic Systems with Uncertainties

Fig.1 illustrates the structure of the H-infinity tracking controller and the robust RLS Wiener filter. Theorem 1 presents the H-infinity tracking control algorithm for estimating the control input $u_1(k)$ and the exogenous input $u_2(k)$. Using the filtering estimate $\hat{\overleftrightarrow{x}}(k)$ instead of $\overleftrightarrow{x}(k)$ the estimates of the control input $u_1(k)$ and the exogenous input $u_2(k)$ are denoted by $\hat{u}_1(k)$ and $\hat{u}_2(k)$, respectively. The robust RLS Wiener filter in Theorem 2 calculates the filtering estimate $\hat{\overleftrightarrow{x}}(k)$ of the state $\overleftrightarrow{x}(k)$ by using the degraded observed value $\breve{y}(k)$.

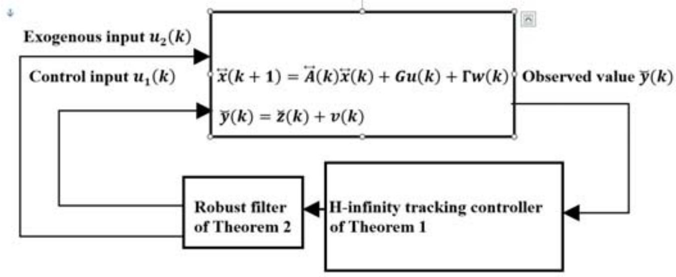

Fig. 1: Structure of H-infinity tracking controller of Theorem 1 and robust recursive least-squares Wiener filter of Theorem 2.

Theorem 1 Let $\eta(k)$ denote the desired value and $R(k)$ be expressed as $R(k) = \begin{bmatrix} \tilde{R}(k) & 0 \\ 0 & -\gamma^2 I_{m_2 \times m_2} \end{bmatrix}$. Let $u(k)$ have the components of the control input $u_1(k)$ and the exogenous input $u_2(k)$ as

$$u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}, \tag{13}$$

then the estimate $\hat{u}(k)$ of $u(k)$ is calculated using (14)-(19). In (14), $\hat{u}_1(k)$ is the estimate of the control input $u_1(k)$ and $\hat{u}_2(k)$ is the estimate of the exogenous input $u_2(k)$.

$$\hat{u}(k) = \begin{bmatrix} \hat{u}_1(k) \\ \hat{u}_2(k) \end{bmatrix} \tag{14}$$

$$\begin{aligned} &\hat{u}(k) \\ &= R^{-1}(k)G^T\{(\hat{\overleftrightarrow{A}}^T)^{-1}[\hat{\overleftrightarrow{A}}^T P(k+1) \\ &\times \left(I - GR^{-1}(k)G^T P(k+1)\right)^{-1}\hat{\overleftrightarrow{A}} \\ &+\hat{\overleftrightarrow{C}}^T Q(k)\hat{\overleftrightarrow{C}}] - \hat{\overleftrightarrow{C}}^T Q(k)\hat{\overleftrightarrow{C}}\}\hat{\overleftrightarrow{x}}(k) \\ &+R^{-1}(k)G^T(\hat{\overleftrightarrow{A}}^T)^{-1}\{\hat{\overleftrightarrow{A}}^T P(k+1) \\ &\times (I - GR^{-1}(k)G^T P(k+1))^{-1} \\ &\times GR^{-1}(k)G^T\xi(k+1) + \hat{\overleftrightarrow{A}}^T\xi(k+1) \\ &-\hat{\overleftrightarrow{C}}^T Q(k)\eta(k)\} \\ &+ R^{-1}(k)G^T(\hat{\overleftrightarrow{A}}^T)^{-1}\hat{\overleftrightarrow{C}}^T Q(k)\eta(k) \end{aligned} \tag{15}$$

Here, $\hat{\overleftrightarrow{A}}$ and $\hat{\overleftrightarrow{C}}$ represent the time-invariant estimates of the unknown time-variant system matrix $\overleftrightarrow{A}(k)$ and $\overleftrightarrow{C}(k)$ with uncertain matrix elements, respectively.

$$\begin{aligned}&P(k)=\hat{\overleftrightarrow{A}}^TP(k+1)\\&\times(I-GR^{-1}(k)G^TP(k+1))^{-1}\hat{\overleftrightarrow{A}}\\&-\hat{\overleftrightarrow{C}}^TQ(k)\hat{\overleftrightarrow{C}},P(L+1)=0\end{aligned}\tag{16}$$

$$\begin{aligned}&\xi(k)=\hat{\overleftrightarrow{A}}^TP(k+1)\\&\times(I-GR^{-1}(k)G^TP(k+1))^{-1}\\&\times GR^{-1}(k)G^T\xi(k+1)+\hat{\overleftrightarrow{A}}^T\xi(k+1)\\&+\hat{\overleftrightarrow{C}}^TQ(k)\eta(k),\xi(L+1)=0\end{aligned}\tag{17}$$

The estimate $\hat{\overleftrightarrow{A}}$ of the uncertain system matrix $\overleftrightarrow{A}(k)$ satisfies

$$\begin{aligned}&\hat{\overleftrightarrow{A}}=K_{\overleftrightarrow{x}}(1)K_{\overleftrightarrow{x}}^{-1}(0),\\&K_{\overleftrightarrow{x}}(1)=E[\overleftrightarrow{x}(k+1)\overleftrightarrow{x}^T(k)],\\&K_{\overleftrightarrow{x}}(0)=E[\overleftrightarrow{x}(k)\overleftrightarrow{x}^T(k)].\end{aligned}\tag{18}$$

The estimate $\hat{\overleftrightarrow{C}}$ of the uncertain observation matrix $\overleftrightarrow{C}(k)$ is given by

$$\begin{aligned}&\hat{\overleftrightarrow{C}}=E[\breve{z}(k)\overleftrightarrow{x}^T(k)](E[\overleftrightarrow{x}(k)\overleftrightarrow{x}^T(k)])^{-1}\\&\text{or}\\&\hat{\overleftrightarrow{C}}=E[\breve{y}(k)\overleftrightarrow{x}^T(k)](E[\overleftrightarrow{x}(k)\overleftrightarrow{x}^T(k)])^{-1}.\end{aligned}\tag{19}$$

In (15), we utilized the filtering estimate $\hat{\overleftrightarrow{x}}(k)$ for the state $\overleftrightarrow{x}(k)$. $\hat{\overleftrightarrow{x}}(k)$ is computed by the robust RLS Wiener filtering algorithm of Theorem 2 with the degraded observed value $\breve{y}(k)$. $P(k)$ and $\xi(k)$ are computed using (16) and (17) from time $k=\mathrm{L}+1$ in the reverse direction of time until steady-state values $\bar{P}$ and $\bar{\xi}$ are reached, respectively. The estimate $\hat{u}(k)$ of $u(k)$ is calculated by (15) using $\bar{P}$ and $\bar{\xi}$. In (15), $P(k+1)$ and $\xi(k+1)$ are replaced with their stationary values $\bar{P}$ and $\bar{\xi}$, respectively.

Now, let's introduce the robust RLS Wiener filter in Theorem 2. Suppose that the sequence of the degraded signal $\breve{z}(k)$ is fitted to an AR model of the order N.

$$\begin{aligned}&\breve{z}(k)=-\breve{a}_1\breve{z}(k-1)-\breve{a}_2\breve{z}(k-2)\\&\cdots-\breve{a}_N\breve{z}(k-N)+\breve{e}(k),\\&E[\breve{e}(k)\breve{e}^T(s)]=\breve{Q}\delta_K(k-s)\end{aligned}\tag{20}$$

$\breve{z}(k)$ is expressed using the state vector $\breve{x}(k)$ as follows.

$$\begin{aligned}&\breve{z}(k)=\breve{C}\breve{x}(k),\\&\breve{C}=[I_{l\times l}\quad 0\quad 0\quad\cdots\quad 0\quad 0],\\&\breve{x}(k)=\begin{bmatrix}\breve{x}_1(k)\\\breve{x}_2(k)\\\vdots\\\breve{x}_{N-1}(k)\\\breve{x}_N(k)\end{bmatrix}=\begin{bmatrix}\breve{z}(k)\\\breve{z}(k+1)\\\vdots\\\breve{z}(k+N-2)\\\breve{z}(k+N-1)\end{bmatrix}\end{aligned}\tag{21}$$

Therefore, the state equation for the state vector $\breve{x}(k)$ is given by

$$\begin{aligned}&\breve{x}(k+1)=\breve{A}\breve{x}(k)+\breve{\Gamma}\zeta(k),\\&E[\zeta(k)\zeta^T(s)]=\breve{Q}\delta_K(k-s),\\&\breve{A}=\begin{bmatrix}0&I_{l\times l}&0&\cdots&0\\0&0&I_{l\times l}&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&I_{l\times l}\\-\breve{a}_N&-\breve{a}_{N-1}&-\breve{a}_{N-2}&\cdots&-\breve{a}_1\end{bmatrix},\\&\breve{\Gamma}=[0\quad 0\quad\cdots\quad 0\quad I_{l\times l}]^T,\\&\zeta(k)=\breve{e}(k+N).\end{aligned}\tag{22}$$

The auto-covariance function $\breve{K}(k,s)$ of the state vector $\breve{x}(k)$ is expressed in the semi-degenerate functional form of

$$\begin{aligned}&\breve{K}(k,s)=\begin{cases}\Psi(k)\Xi^T(s),0\le s\le k,\\\Xi(k)\Psi^T(s),0\le k\le s,\end{cases}\\&\Psi(k)=\breve{A}^k,\Xi^T(s)=\breve{A}^{-s}\breve{K}(s,s).\end{aligned}\tag{23}$$

The wide-sense stationarity of the auto-covariance function $\breve{K}(k,s)=E[\breve{z}(k)\breve{z}^T(s)]$ for the degraded signal $\breve{z}(k)$ shows that the auto-variance function $\breve{K}(k,k)$ of $\breve{x}(k)$ satisfies (24).

$$\begin{aligned}&\breve{K}(k,k)=E\left[\begin{bmatrix}\breve{z}(k)\\\breve{z}(k+1)\\\vdots\\\breve{z}(k+N-1)\end{bmatrix}\right.\\&\left.\times[\breve{z}^T(k)\quad\breve{z}^T(k+1)\quad\cdots\quad\breve{z}^T(k+N-1)]\right]\\&=\begin{bmatrix}K_{\breve{z}}(0)&K_{\breve{z}}(-1)&\cdots&K_{\breve{z}}(-N+1)\\K_{\breve{z}}(1)&K_{\breve{z}}(0)&\cdots&K_{\breve{z}}(-N+2)\\\vdots&\vdots&\ddots&\vdots\\K_{\breve{z}}(N-2)&K_{\breve{z}}(N-3)&\cdots&K_{\breve{z}}(-1)\\K_{\breve{z}}(N-1)&K_{\breve{z}}(N-2)&\cdots&K_{\breve{z}}(0)\end{bmatrix}\end{aligned}\tag{24}$$

Using $K_{\breve{z}}(i)$, $0\le i\le N$, the Yule-Walker equation for the AR parameters $\breve{a}_i$, $1\le i\le N$, is given by

$$\begin{aligned}&\breve{K}(k,k)\begin{bmatrix}\breve{a}_1^T\\\breve{a}_2^T\\\vdots\\\breve{a}_{N-1}^T\\\breve{a}_N^T\end{bmatrix}=-\begin{bmatrix}K_{\breve{z}}^T(1)\\K_{\breve{z}}^T(2)\\\vdots\\K_{\breve{z}}^T(N-1)\\K_{\breve{z}}^T(N)\end{bmatrix},\\&\breve{K}(k,k)=\begin{bmatrix}K_{\breve{z}}(0)&K_{\breve{z}}(1)&\cdots&K_{\breve{z}}(N-1)\\K_{\breve{z}}^T(1)&K_{\breve{z}}(0)&\cdots&K_{\breve{z}}(N-2)\\\vdots&\vdots&\ddots&\vdots\\K_{\breve{z}}^T(N-2)&K_{\breve{z}}^T(N-3)&\cdots&K_{\breve{z}}(1)\\K_{\breve{z}}^T(N-1)&K_{\breve{z}}^T(N-2)&\cdots&K_{\breve{z}}(0)\end{bmatrix}.\end{aligned}\tag{25}$$

Let $K_{\overleftrightarrow{x}\breve{x}}(k,s)=E[\overleftrightarrow{x}(k)\breve{x}^T(s)]$ represent the cross-covariance function of $\overleftrightarrow{x}(k)$ with $\breve{x}(s)$. $K_{\overleftrightarrow{x}\breve{x}}(k,s)$ has the expression

$$\begin{aligned}&K_{\overleftrightarrow{x}\breve{x}}(k,s)=\alpha(k)\beta^T(s),0\le s\le k,\\&\alpha(k)=\overleftrightarrow{\Phi}(k,0),\\&\beta^T(s)=\overleftrightarrow{\Phi}^{-1}(s,0)K_{\overleftrightarrow{x}\breve{x}}(s,s),\end{aligned}\tag{26}$$

with the state-transition matrix $\overleftrightarrow{\Phi}(k,s)$ of the unknown system matrix $\overleftrightarrow{A}(k)$ for $\overleftrightarrow{x}(k)$ in (2).

Based on the above preliminaries, Theorem 2 presents the robust RLS Wiener filtering algorithm for the filtering estimate $\hat{\overleftrightarrow{x}}(k)$ of $\overleftrightarrow{x}(k)$ with the degraded observed value $\breve{y}(k)$ in (2). Note that the filter equation for $\hat{\overleftrightarrow{x}}(k)$ contains the term $G\hat{u}(k-1)$ on the right-hand side of (28). The benefits of the technique are that the uncertain observation and system matrices, $\breve{C}$ and $\breve{A}$, for the degraded signal $\breve{z}(k)$ used in the H infinity RLS Wiener filtering algorithm of Theorem 2 are given by (21) and (22) respectively, based on the autoregressive model in equation (20).

Theorem 2 Suppose that, in discrete-time stochastic systems with the uncertain system matrix $\overleftrightarrow{A}(k)$ and the uncertain observation matrix $\overleftrightarrow{C}(k)$, the linear state-space model of the state $\overleftrightarrow{x}(k)$ is given by (2). Suppose the degraded signal $\breve{z}(k)$ fits the AR model of order N. Let the variance $\breve{K}(k,k)$ of state $\breve{x}(k)$ with regards to the degraded signal $\breve{z}(k)$ be expressed as (24). Let V be the variance of the white Gaussian observation noise $v(k)$. Then (27)-(33) constitute the robust RLS Wiener filtering algorithm for the filtering estimate $\hat{\overleftrightarrow{x}}(k)$ of $\overleftrightarrow{x}(k)$.

Filtering estimate of the degraded signal $\breve{z}(k)$: $\hat{\breve{z}}(k)$

$$\hat{\breve{z}}(k) = \hat{\breve{C}}\hat{\overleftrightarrow{x}}(k) \quad (27)$$

Filtering estimate of $\overleftrightarrow{x}(k)$: $\hat{\overleftrightarrow{x}}(k)$

$$\begin{aligned} \hat{\overleftrightarrow{x}}(k) &= \hat{\breve{A}}\hat{\overleftrightarrow{x}}(k-1) + G\hat{u}(k-1) \\ &+\Theta(k)(\breve{y}(k) - \breve{C}\breve{A}\hat{\breve{x}}(k-1)), \\ \hat{\overleftrightarrow{x}}(0) &= 0 \end{aligned} \quad (28)$$

Filter gain for $\hat{\overleftrightarrow{x}}(k)$ in (28): $\Theta(k)$

$$\begin{aligned} \Theta(k) &= [K_{\overleftrightarrow{x}\breve{z}}(k,k) - \hat{\breve{A}}S(k-1)\breve{A}^T\breve{C}^T] \\ &\times \{V + \breve{C}[\breve{K}(k,k) - \breve{A}S_0(L-1)\breve{A}^T]\breve{C}^T\}^{-1}, \\ K_{\overleftrightarrow{x}\breve{z}}(k,k) &= E[\overleftrightarrow{x}(k)\breve{z}^T(k)] \\ \text{or } K_{\overleftrightarrow{x}\breve{z}}(k,k) &= E[\overleftrightarrow{x}(k)\breve{y}^T(k)] \end{aligned} \quad (29)$$

Filtering estimate of $\breve{x}(k)$: $\hat{\breve{x}}(k)$

$$\begin{aligned} \hat{\breve{x}}(k) &= \breve{A}\hat{\breve{x}}(k-1) \\ &+g(k)(\breve{y}(k) - \breve{C}\breve{A}\hat{\breve{x}}(k-1)), \\ \hat{\breve{x}}(0) &= 0 \end{aligned} \quad (30)$$

Filter gain for $\hat{\breve{x}}(k)$ in (30): $g(k)$

$$\begin{aligned} g(k) &= [\breve{K}(k,k)\breve{C}^T - \breve{A}S_0(k-1)\breve{A}^T\breve{C}^T] \\ &\times \{V + \breve{C}[\breve{K}(k,k) - \breve{A}S_0(L-1)\breve{A}^T]\breve{C}^T\}^{-1} \end{aligned} \quad (31)$$

Auto-variance function of $\hat{\breve{x}}(k)$: $S_0(k) = E[\hat{\breve{x}}(k)\hat{\breve{x}}^T(k)]$

$$\begin{aligned} S_0(k) &= \breve{A}S_0(k-1)\breve{A}^T \\ &+g(k)\breve{C}[\breve{K}(k,k) - \breve{A}S_0(k-1)\breve{A}^T], \\ S_0(0) &= 0 \end{aligned} \quad (32)$$

Cross-variance function of $\hat{\overleftrightarrow{x}}(k)$ with $\hat{\breve{x}}(k)$: $S(k) = E[\hat{\overleftrightarrow{x}}(k)\hat{\breve{x}}^T(k)]$

$$\begin{aligned} S(k) &= \hat{\breve{A}}S(k-1)\breve{A}^T \\ &+\Theta(k)\breve{C}[\breve{K}(k,k) - \breve{A}S_0(k-1)\breve{A}^T], \\ S(0) &= 0 \end{aligned} \quad (33)$$

Here, the estimates $\hat{\overleftrightarrow{A}}$ and $\hat{\overleftrightarrow{C}}$ of the uncertain matrices $\overleftrightarrow{A}(k)$ and $\overleftrightarrow{C}$ are given by (18) and (19), respectively. In (28), the term $G\hat{u}(k-1)$ is inserted. $\hat{u}(k-1)$ is gained from (15), where $\hat{u}(k)$ is calculated by the tracking control algorithm of Theorem 1. The equation for $\hat{u}(k)$ uses the filtering estimate $\hat{\overleftrightarrow{x}}(k)$, which is updated from $\hat{\overleftrightarrow{x}}(k-1)$ in (28).

Proof Theorem 2 is derived by modifying the robust RLS Wiener filter in [21], for estimating $x(k)$ to the estimation of $\overleftrightarrow{x}(k)$.

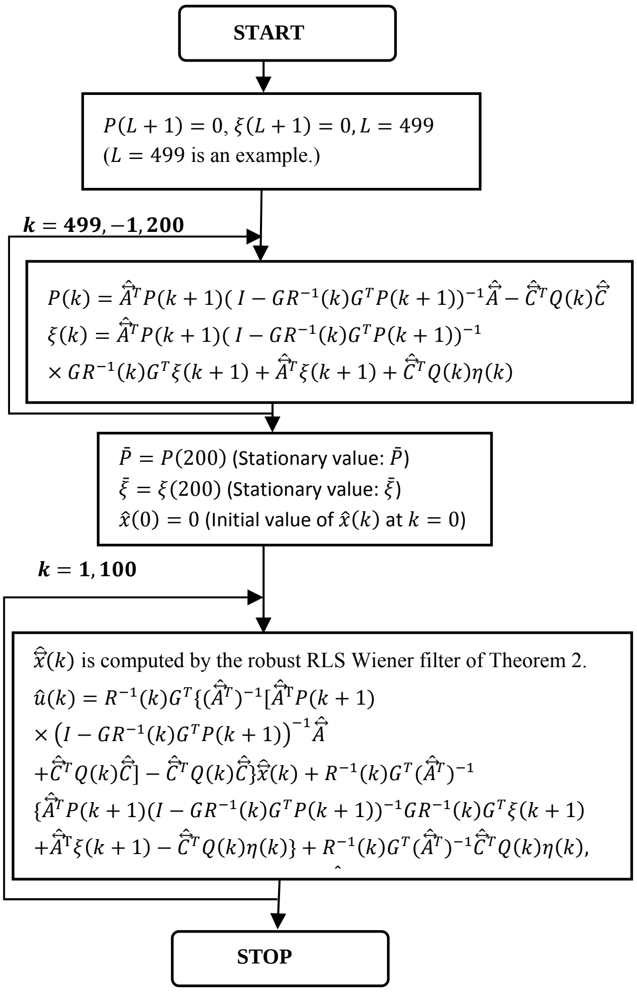

Fig. 2: Flowchart created by combining the H-infinity tracking controller of Theorem 1 with the robust RLS Wiener filter of Theorem 2.

Fig. 2 shows a flowchart combining the H-infinity tracking controller of Theorem 1 and the robust RLS Wiener filter of Theorem 2.

Section 4 presents numerical simulation examples for the tracking control characteristics of the H-infinity tracking controller using the estimate $\hat{\overleftrightarrow{x}}(k)$ of $\overleftrightarrow{x}(k)$ by the robust RLS Wiener filter of Theorem 2 in comparison with the RLS Wiener filter, [26], and the robust Kalman filter, [24].

4 Numerical Simulation Examples

EXAMPLE 1

Consider the observation and multi-input state equations given by

$$
\begin{aligned}
&y(k) = z(k) + v(k), z(k) = Cx(k),\\
&C = [0.95 \quad -0.4], x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix},\\
&x(k+1) = \mathrm{A}x(k) + Gu(k) + \Gamma w(k),\\
&u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}, \mathrm{A} = \begin{bmatrix} 0.05 & 0.95 \\ -0.98 & 0.2 \end{bmatrix},\\
&\mathrm{G} = \begin{bmatrix} 0.952 & 0 \\ 0.2 & 1 \end{bmatrix}, \Gamma = \begin{bmatrix} 0.952 \\ 0.2 \end{bmatrix},\\
&E[v(k)v(s)] = V\delta_K(k-s),\\
&E[w(k)w(s)] = 0.5^2\delta_K(k-s).
\end{aligned} \tag{34}
$$

The linear time-invariant (LTI) system in (34) is observable and controllable. In (34), $u_1(k)$ is the control input and $u_2(k)$ is the exogenous input. Consider that the degraded observed value $\check{y}(k)$ and the degraded signal $\check{z}(k)$ are generated by the observation and state equations in (35).

$$
\begin{aligned}
&\check{y}(k) = \check{z}(k) + v(k),\\
&\check{z}(k) = \overleftrightarrow{C}(k)\overleftrightarrow{x}(k), \overleftrightarrow{C}(k) = C + \Delta\mathrm{C}(k),\\
&\overleftrightarrow{x}(k+1) = \overleftrightarrow{A}(k)\overleftrightarrow{x}(k) + Gu(k) + \Gamma w(k),\\
&\overleftrightarrow{x}(k) = \begin{bmatrix} \overleftrightarrow{x}_1(k) \\ \overleftrightarrow{x}_2(k) \end{bmatrix}, \overleftrightarrow{A}(k) = A + \Delta\mathrm{A}(k),\\
&\Delta\mathrm{C}(k) = [0.3 * rand \quad 0],\\
&\Delta\mathrm{A}(k) = \begin{bmatrix} 0.1 * rand & 0 \\ 0 & 0.2 * rand \end{bmatrix}
\end{aligned} \tag{35}
$$

Here, " $rand$ " is a MATLAB or Octave function representing random numbers uniformly distributed in the interval (0,1). In [24], the uncertain matrices $\Delta\mathrm{A}(k)$ and $\Delta\mathrm{C}(k)$ are subject to the norm-bounded uncertainty conditions. The robust RLS Wiener filtering algorithm of Theorem 2 does not directly use knowledge of the uncertain matrices $\Delta\mathrm{A}(k)$ and $\Delta\mathrm{C}(k)$. $\overleftrightarrow{A}(k)$ is estimated according to the relationship $\hat{\overleftrightarrow{A}} = K_{\overleftrightarrow{x}}(1)K_{\overleftrightarrow{x}}^{-1}(0)$, where $K_{\overleftrightarrow{x}}(1) = E[\overleftrightarrow{x}(k+1)\overleftrightarrow{x}^T(k)]$, $K_{\overleftrightarrow{x}}(0) = E[\overleftrightarrow{x}(k)\overleftrightarrow{x}^T(k)]$. The expectation is approximated in terms of 351 $\overleftrightarrow{x}(k)$ data. That is, $K_{\overleftrightarrow{x}}(1) \cong \frac{1}{350}\sum_{k=1}^{350}\overleftrightarrow{x}(k+1)\overleftrightarrow{x}^T(k)$, $K_{\overleftrightarrow{x}}(0) \cong \frac{1}{350}\sum_{k=1}^{350}\overleftrightarrow{x}(k)\overleftrightarrow{x}^T(k)$. In addition, $K_{\overleftrightarrow{x}\check{z}}(k,k)$ in (29) is approximated as $K_{\overleftrightarrow{x}\check{z}}(k,k) \cong \frac{1}{350}\sum_{k=1}^{350}\overleftrightarrow{x}(k)\,\check{z}(k)$. The estimate $\hat{\overleftrightarrow{C}}$ of $\overleftrightarrow{C}(k)$ is given by $\hat{\overleftrightarrow{C}} = E[\check{y}(k)\overleftrightarrow{x}^T(k)](E[\overleftrightarrow{x}(k)\overleftrightarrow{x}^T(k)])^{-1}$. Here, $E[\check{y}(k)\overleftrightarrow{x}^T(k)]$ is approximated by $\frac{1}{2000}\sum_{k=1}^{2000}\check{y}(k)\overleftrightarrow{x}^T(k)$. The robust RLS Wiener filter in Theorem 2 computes the filtering estimate $\hat{\overleftrightarrow{x}}(k)$ in (28) to obtain the estimate $\hat{u}(k)$ of $u(k)$ in (15). In this instance, the AR model of order N=10 in (20) is applied to a sequence of uncertain signal $\check{z}(k)$. Fig. 3 illustrates the degraded signal $\check{z}(k) = \overleftrightarrow{C}(k)\overleftrightarrow{x}(k)$ and its filtering estimate $\hat{\check{z}}(k) = \hat{\overleftrightarrow{C}}\hat{\overleftrightarrow{x}}(k)$ vs. k for the white Gaussian observation noise $N(0, 0.3^2)$, provided that the desired value $\eta(k) = 10$, $\gamma = 10$, $\tilde{R} = 0.0001$ and $Q(k) = 1$. From Fig. 3, it can be seen that the sequence of filtering estimates $\hat{\check{z}}(k)$ is closer to the desired value of 10 than the degraded signal $\check{z}(k)$. Fig. 4 illustrates the estimate $\hat{u}_1(k)$ of the control input $u_1(k)$ vs. k for the white Gaussian observation noise $N(0, 0.3^2)$, provided that $\eta(k) = 10$, $\gamma = 10$, $\tilde{R} = 0.0001$ and $Q(k) = 1$. Fig. 5 illustrates the estimate $\hat{u}_2(k)$ of the exogenous input $u_2(k)$ vs. k for the white Gaussian observation noise $N(0, 0.3^2)$, provided that $\eta(k) = 10$, $\gamma = 10$, $\tilde{R} = 0.0001$ and $Q(k) = 1$. From Fig. 4 and 5, it follows that fluctuations in the sequence of the estimate $\hat{u}_2(k)$ of the exogenous input $u_2(k)$ are much smaller than those in the sequence of the estimate $\hat{u}_1(k)$ of the control input $u_1(k)$. Table 1 shows the mean square values (MSVs) of the tracking errors $\eta(k) - \check{z}(k)$, $\check{z}(k) = \overleftrightarrow{C}(k)\overleftrightarrow{x}(k)$ and $\eta(k) - \hat{\check{z}}(k)$, $\hat{\check{z}}(k) = \hat{\overleftrightarrow{C}}\hat{\overleftrightarrow{x}}(k)$, $1 \le k \le 1200$, by the H-infinity tracking controller of Theorem 1 and the robust RLS Wiener filter of Theorem 2 for $\gamma = 10$ and $\gamma = 0.01$, provided that $\eta(k) = 10$, $\tilde{R} = 0.0001$ and $Q(k) = 1$. In this case, the observation noise is subject to $N(0, 0.1^2)$, $N(0, 0.3^2)$, $N(0, 0.5^2)$, $N(0,1)$ and $N(0, 5^2)$. The MSV of the tracking errors $\eta(k) - \hat{\check{z}}(k)$ is less than that of the tracking errors $\eta(k) - \check{z}(k)$ for each observation noise. This indicates that the filtering estimate $\hat{\check{z}}(k)$ accurately tracks the desired value in comparison with $\check{z}(k)$. As the variance of the white Gaussian observation noise increases, the MSVs of the tracking errors $\eta(k) - \check{z}(k)$ and $\eta(k) - \hat{\check{z}}(k)$ become small gradually. For $\gamma = 10$ and $\gamma = 0.01$, the MSVs of the tracking errors $\eta(k) - \check{z}(k)$ are almost the same for each observation noise. Similarly, for $\gamma = 10$ and $\gamma = 0.01$, the MSVs of the tracking errors $\eta(k) -$

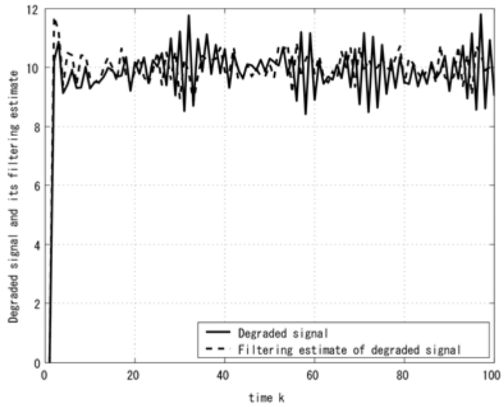

Fig. 3: Degraded signal $\breve{z}(k)=\overleftrightarrow{C}(k)\overleftrightarrow{x}(k)$ and its filtering estimate $\hat{\breve{z}}(k)=\hat{\overleftrightarrow{C}}\hat{\overleftrightarrow{x}}(k)$ vs. k for white Gaussian observation noise $N(0,0.3^2)$, provided that the desired value $\eta(k)=10$, $\gamma=10$, $\tilde{R}=0.0001$ and $Q(k)=1$.

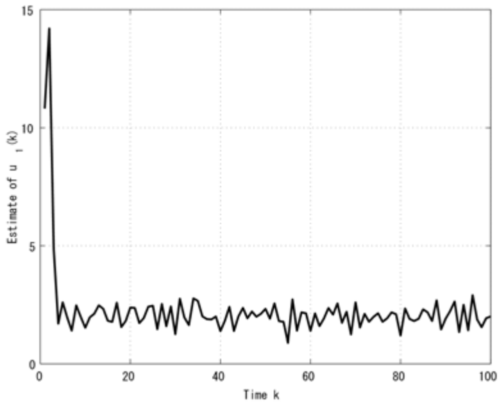

Fig. 4: Estimate $\hat{u}_1(k)$ of control input $u_1(k)$ vs. k for white Gaussian observation noise $N(0,0.3^2)$, provided that $\eta(k)=10, \gamma=10, \tilde{R}=0.0001$ and $Q(k)=1$.

$\hat{\breve{z}}(k)$ are almost the same for each observation noise. Table 2 shows the MSVs of the tracking errors $\eta(k)-\breve{z}(k)$, $\breve{z}(k)=\overleftrightarrow{C}(k)\overleftrightarrow{x}(k)$ and $\eta(k)-\hat{\breve{z}}(k)$, $\hat{\breve{z}}(k)=\hat{\overleftrightarrow{C}}\hat{\overleftrightarrow{x}}(k)$, $1\le k\le 1200$, by the H-infinity tracking controller of Theorem 1 and the RLS Wiener filter, [26], for $\gamma=10$ and $\gamma=0.01$, provided that $\eta(k)=10$, $\tilde{R}=0.0001$ and $Q(k)=1$. From Tables 1 and 2, the tracking controller of Theorem 1 combined with the robust RLS Wiener filter of Theorem 2 is superior in tracking control accuracy to the tracking controller of Theorem 1 combined with the RLS Wiener filter, [26], for each observation noise. Table 3 shows the MSVs of the tracking errors $\eta(k)-\breve{z}(k)$ and $\eta(k)-\hat{\breve{z}}(k)$ by the H-infinity tracking controller of Theorem 1 and the robust Kalman filter, [24], for the observation noise $N(0,0.1^2)$, $N(0,0.3^2)$, $N(0,0.5^2)$, $N(0,1)$ and

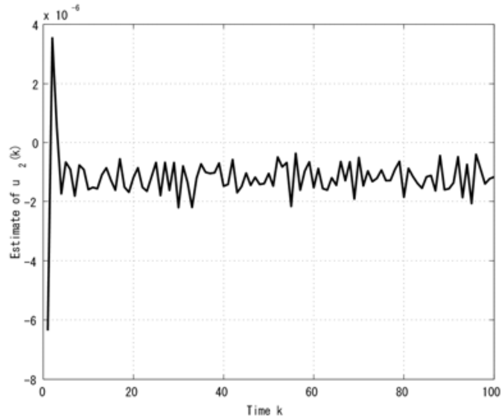

Fig. 5: Estimate $\hat{u}_2(k)$ of exogenous input $u_2(k)$ vs. k for white Gaussian observation noise $N(0,0.3^2)$, provided that $\eta(k)=10$, $\gamma=10$, $\tilde{R}=0.0001$ and $Q(k)=1$.

Table 1. Mean-square values of tracking errors $\eta(k)-\breve{z}(k)$, $\breve{z}(k)=\overleftrightarrow{C}(k)\overleftrightarrow{x}(k)$ and $\eta(k)-\hat{\breve{z}}(k)$, $\hat{\breve{z}}(k)=\hat{\overleftrightarrow{C}}\hat{\overleftrightarrow{x}}(k)$, $1\le k\le 1200$, by H-infinity tracking control algorithm of Theorem 1 plus robust RLS Wiener filter of Theorem 2 for $\gamma=10$ and $\gamma=0.01$, provided that $\eta(k)=10$, $\tilde{R}=0.0001$ and $Q(k)=1$.

White Gaussian observation noise	$\gamma=10$		$\gamma=0.01$	
	MSV of tracking errors $\eta(k)-\breve{z}(k)$	MSV of tracking errors $\eta(k)-\hat{\breve{z}}(k)$	MSV of tracking errors $\eta(k)-\breve{z}(k)$	MSV of tracking errors $\eta(k)-\hat{\breve{z}}(k)$
$N(0,0.1^2)$	0.2469	0.1376	0.2473	0.1382
$N(0,0.3^2)$	0.2126	0.1168	0.2108	0.1162
$N(0,0.5^2)$	0.1765	0.0968	0.1755	0.0961
$N(0,1)$	0.1042	0.0572	0.1049	0.0573
$N(0,5^2)$	0.0100	0.0019	0.0099	0.0019

$N(0,5^2)$. In the computation of the robust Kalman filter, [24], the program uses the values $\epsilon_p=100$, $H_1=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $H_2=[1 \quad 1]$ and $E=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. From Table 3, the combination of the H-infinity tracking controller of Theorem 1 and the robust Kalman filter, [24], does not track the desired value at all or diverges.

EXAMPLE 2

Consider linear discrete-time systems for the observation and multi-input state equations of the F16 aircraft, [30]. The angle of attack $\alpha(k)$, the rate of pitch $q(k)$, and the elevator angle of deflection $\delta_c(k)$ constitute the state vector $x(k)=[\alpha(k) \quad q(k) \quad \delta_c(k)]^T$, $x_1(k)=\alpha(k)$, $x_2(k)=$

Table 2. Mean-square values of tracking errors $\eta(k)-\check{z}(k)$, $\check{z}(k)=\overleftrightarrow{C}(k)\overleftrightarrow{x}(k)$ and $\eta(k)-\hat{\check{z}}(k)$, $\hat{\check{z}}(k)=\hat{\overleftrightarrow{C}}\hat{\overleftrightarrow{x}}(k)$, $1\le k\le 1200$, by H-infinity tracking control algorithm of Theorem 1 plus RLS Wiener filter, [26], for $\gamma=10$ and $\gamma=0.01$, provided that $\eta(k)=10$, $\tilde{R}=0.0001$ and $Q(k)=1$.

White Gaussian observation noise	$\gamma=10$		$\gamma=0.01$	
	MSV of tracking errors $\eta(k)-\check{z}(k)$	MSV of tracking errors $\eta(k)-\hat{\check{z}}(k)$	MSV of tracking errors $\eta(k)-\check{z}(k)$	MSV of tracking errors $\eta(k)-\hat{\check{z}}(k)$
$N(0,0.1^2)$	2.0878e+003	282.4114	49.9650	51.4861
$N(0,0.3^2)$	671.1428	90.6103	49.9649	51.3064
$N(0,0.5^2)$	321.3791	42.9632	49.9647	51.2206
$N(0,1)$	58.7219	8.1504	49.9649	50.9183
$N(0,5^2)$	4.1633	0.8318	49.9650	50.4742

Table 3. Mean-square values of tracking errors $\eta(k)-\check{z}(k)$, $\check{z}(k)=\overleftrightarrow{C}(k)\overleftrightarrow{x}(k)$ and $\eta(k)-\hat{\check{z}}(k)$, $\hat{\check{z}}(k)=\hat{\overleftrightarrow{C}}\hat{\overleftrightarrow{x}}(k)$, $1\le k\le 1200$, by H-infinity tracking control algorithm of Theorem 1 plus robust Kalman filter, [24], for $\gamma=10$ and $\gamma=0.01$, provided that $\eta(k)=10$, $\tilde{R}=0.0001$ and $Q(k)=1$.

White Gaussian observation noise	$\gamma=10$		$\gamma=0.01$	
	MSV of tracking errors $\eta(k)-\check{z}(k)$	MSV of tracking errors $\eta(k)-\hat{\check{z}}(k)$	MSV of tracking errors $\eta(k)-\check{z}(k)$	MSV of tracking errors $\eta(k)-\hat{\check{z}}(k)$
$N(0,0.1^2)$	4.8511e+017	3.7728e+018	Divergence	Divergence
$N(0,0.3^2)$	49.3066	9.5687	Divergence	Divergence
$N(0,0.5^2)$	46.2886	9.3179	Divergence	Divergence
$N(0,1)$	46.1339	9.4984	Divergence	Divergence
$N(0,5^2)$	46.0462	9.3938	8.4403e+235	7.0283e+238

$q(k)$, $x_3(k)=\delta_c(k)$. The system output, $z(k)$, corresponds to the angle of attack $\alpha(k)$. In (36), $u_1(k)$ is the control input and $u_2(k)$ is the exogenous input. The desired value of $\alpha(k)$ is set to $\eta(k)=0.1\,[rad]$. The LTI system in (36) is observable and controllable.

$$
\begin{aligned}
&y(k)=z(k)+v(k), z(k)=Cx(k),\\
&C=[1\quad 0\quad 0], x(k)=\begin{bmatrix}x_1(k)\\x_2(k)\\x_3(k)\end{bmatrix},\\
&x(k+1)=\mathrm{A}x(k)+Gu(k)+\Gamma w(k),\\
&u(k)=\begin{bmatrix}u_1(k)\\u_2(k)\end{bmatrix},\\
&\mathrm{A}=\begin{bmatrix}0.906488 & 0.0816012 & 0.0005\\ 0.0741349 & 0.90121 & -0.0007083\\ 0 & 0 & 0.132655\end{bmatrix},\\
&\mathrm{G}=\begin{bmatrix}-0.00150808 & 0.00951892\\ -0.0096 & 0.00038373\\ 0.867345 & 0\end{bmatrix},\\
&\Gamma=\begin{bmatrix}0\\0\\1\end{bmatrix}, E[v(k)v(s)]=V\delta_K(k-s),\\
&E[w(k)w(s)]=0.3^2\delta_K(k-s)
\end{aligned}
\tag{36}
$$

The order of the AR model in (20) is $N=10$. We assume that the degraded observed value $\check{y}(k)$ and the degraded signal $\check{z}(k)$ are generated by the following observation and state equations.

$$
\begin{aligned}
&\check{y}(k)=\check{z}(k)+v(k), \check{z}(k)=\overleftrightarrow{C}(k)\overleftrightarrow{x}(k),\\
&\overleftrightarrow{x}(k+1)=\overleftrightarrow{A}(k)\overleftrightarrow{x}(k)+Gu(k)+\\
&\Gamma w(k), \overleftrightarrow{x}(k)=\begin{bmatrix}\overleftrightarrow{x}_1(k)\\\overleftrightarrow{x}_2(k)\\\overleftrightarrow{x}_3(k)\end{bmatrix},\\
&\check{A}(k)=A+\Delta\mathrm{A}(k),\\
&\overleftrightarrow{C}(k)=C+\Delta\mathrm{C}(k),\\
&\Delta\mathrm{A}(k)=\begin{bmatrix}0 & 0 & 0\\ 0 & 0.05*rand & 0\\ 0 & 0 & 0\end{bmatrix},\\
&\Delta\mathrm{C}(k)=[0.03*rand\quad 0\quad 0],
\end{aligned}
\tag{37}
$$

The robust RLS Wiener filter in Theorem 2 computes the filtering estimate $\hat{\overleftrightarrow{x}}(k)$ in (28) to obtain the estimate $\hat{u}(k)$ of $u(k)$ in (15). A total of 351 datasets are used to calculate the expected values $E[\overleftrightarrow{x}(k+1)\overleftrightarrow{x}^T(k)]$, $E[\overleftrightarrow{x}(k)\overleftrightarrow{x}^T(k)]$, and $E[\check{y}(k)\overleftrightarrow{x}^T(k)]$ for $\hat{\overleftrightarrow{A}}$ and $\hat{\overleftrightarrow{C}}$, respectively. Fig. 6 illustrates the degraded signal $\check{z}(k)=\overleftrightarrow{C}(k)\overleftrightarrow{x}(k)$ and its filtering estimate $\hat{\check{z}}(k)=\hat{\overleftrightarrow{C}}\hat{\overleftrightarrow{x}}(k)$ vs. k for the white Gaussian observation noise $N(0,0.3^2)$, provided that the desired value $\eta(k)=0.1[rad]$, $\gamma=10$, $\tilde{R}=0.0001$ and $Q(k)=1$. From Fig. 6, the sequence of the filtering estimates $\hat{\check{z}}(k)$ is closer to the desired value of 0.1 $[rad]$ than the degraded signal $\check{z}(k)$. Fig. 7 illustrates the estimate $\hat{u}_1(k)$ of the control input $u_1(k)$ vs. k for the white Gaussian observation noise $N(0,0.3^2)$, provided that $\eta(k)=0.1\,[rad]$, $\gamma=10$, $\tilde{R}=0.0001$ and $Q(k)=1$. Fig. 8 illustrates the estimate $\hat{u}_2(k)$ of the exogenous input $u_2(k)$ vs. k for the white Gaussian observation noise $N(0,0.3^2)$, provided that $\eta(k)=0.1\,[rad]$, $\gamma=10$,

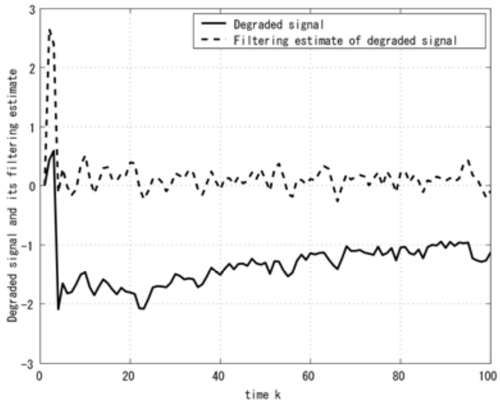

Fig. 6: Degraded signal $\check{z}(k) = \overleftrightarrow{C}(k)\overleftrightarrow{x}(k)$ and its filtering estimate $\hat{\check{z}}(k) = \hat{\overleftrightarrow{C}}\hat{\overleftrightarrow{x}}(k)$ vs. k for white Gaussian observation noise $N(0, 0.3^2)$, provided that the desired value $\eta(k) = 0.1\ [rad]$, $\gamma = 10$, $\tilde{R} = 0.0001$ and $Q(k) = 1$.

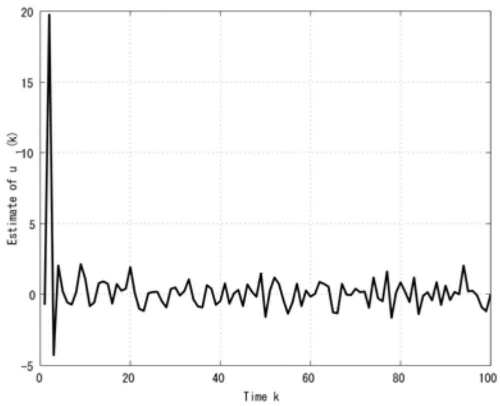

Fig. 7: Estimate $\hat{u}_1(k)$ of control input $u_1(k)$ vs. k for white Gaussian observation noise $N(0, 0.3^2)$, provided that $\eta(k) = 0.1\ [rad]$, $\gamma = 10$, $\tilde{R} = 0.0001$ and $Q(k) = 1$.

$\tilde{R} = 0.0001$ and $Q(k) = 1$. From Fig. 7 and Fig. 8, it can be seen that the amplitude of the exogenous input $\hat{u}_2(k)$ sequence is very small compared with that of the control input $\hat{u}_1(k)$ sequence. Table 4 shows the MSVs of the tracking errors $\eta(k) - \check{z}(k)$, $\check{z}(k) = \overleftrightarrow{C}(k)\overleftrightarrow{x}(k)$ and $\eta(k) - \hat{\check{z}}(k)$, $\hat{\check{z}}(k) = \hat{\overleftrightarrow{C}}\hat{\overleftrightarrow{x}}(k)$, $1 \leq k \leq 1200$, by the H-infinity tracking controller of Theorem 1 and the robust RLS Wiener filter of Theorem 2 for $\gamma = 10$ and $\gamma = 0.05$, provided that $\eta(k) = 0.1\ [rad]$, $\tilde{R} = 0.0001$ and $Q(k) = 1$. Here, the observation noise is subject to $N(0, 0.1^2)$, $N(0, 0.3^2)$, $N(0, 0.5^2)$, $N(0,1)$ and $N(0, 5^2)$. The MSV of the tracking errors $\eta(k) - \hat{\check{z}}(k)$ is smaller than that of the tracking errors $\eta(k) - \check{z}(k)$ for each observation noise. This indicates that the filtering estimate $\hat{\check{z}}(k)$ tracks the desired value more

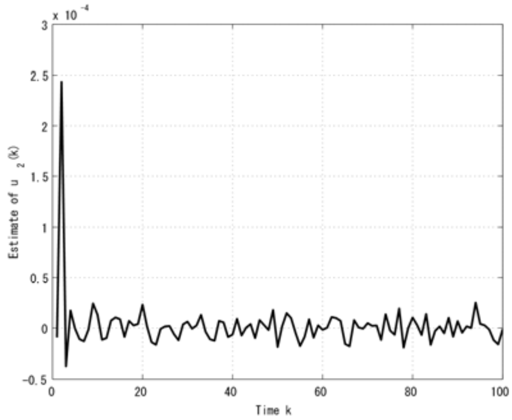

Fig. 8: Estimate $\hat{u}_2(k)$ of exogenous input $u_2(k)$ vs. k for white Gaussian observation noise $N(0, 0.3^2)$, provided that $\eta(k) = 0.1\ [rad]$, $\gamma = 10$, $\tilde{R} = 0.0001$ and $Q(k) = 1$.

Table 4. Mean-square values of tracking errors $\eta(k) - \check{z}(k)$, $\check{z}(k) = \overleftrightarrow{C}(k)\overleftrightarrow{x}(k)$ and $\eta(k) - \hat{\check{z}}(k)$, $\hat{\check{z}}(k) = \hat{\overleftrightarrow{C}}\hat{\overleftrightarrow{x}}(k)$, $1 \leq k \leq 1200$, by H-infinity tracking control algorithm of Theorem 1 plus robust RLS Wiener filter of Theorem 2 for $\gamma = 10$ and $\gamma = 0.05$, provided that $\eta(k) = 0.1\ [rad]$, $\tilde{R} = 0.0001$ and $Q(k) = 1$.

White Gaussian observation noise	$\gamma = 10$		$\gamma = 0.05$	
	MSV of tracking errors $\eta(k) - \check{z}(k)$	MSV of tracking errors $\eta(k) - \hat{\check{z}}(k)$	MSV of tracking errors $\eta(k) - \check{z}(k)$	MSV of tracking errors $\eta(k) - \hat{\check{z}}(k)$
$N(0, 0.1^2)$	0.1161	0.0095	0.1028	0.0097
$N(0, 0.3^2)$	0.1318	0.0166	0.1246	0.0164
$N(0, 0.5^2)$	0.1426	0.0175	0.1524	0.0182
$N(0,1)$	0.1684	0.0186	0.1383	0.0181
$N(0, 5^2)$	0.3270	0.0159	0.2600	0.0148

accurately than $\check{z}(k)$. For both $\gamma = 10$ and $\gamma = 0.05$, the MSV of the tracking errors $\eta(k) - \hat{\check{z}}(k)$ for each observed noise is almost identical. As the variance of the white Gaussian observation noise increases, the MSV of the tracking errors $\eta(k) - \hat{\check{z}}(k)$ tends to increase gradually in both $\gamma = 10$ and $\gamma = 0.05$ for $N(0, 0.1^2)$, $N(0, 0.3^2)$ and $N(0, 0.5^2)$. Concerning Fig 6, from Table 4, for the white Gaussian observation noise $N(0, 0.3^2)$, the MSV of the tracking errors $\eta(k) - \hat{\check{z}}(k)$ is 0.0166. The MSV of the tracking errors $\eta(k) - \check{z}(k)$ is 0.1318. This indicates that the proposed H-infinity tracking control algorithm improves tracking accuracy by using the robust RLS Wiener filter in Theorem 2.

5 Conclusion

This paper developed the H-infinity tracking control technique combined with the robust RLS Wiener filter for linear discrete-time stochastic systems with uncertainties. For linear discrete-time stochastic systems (2) with uncertainties, based on the separation principle of control and estimation, $u(k)$ satisfies (12) along with (10) and (11). The filtering estimate $\hat{\overleftrightarrow{x}}(k)$ of $\overleftrightarrow{x}(k)$ is updated from $\hat{\overleftrightarrow{x}}(k-1)$ by (28) with the information of the estimate $\hat{u}(k-1)$ of $u(k-1)$, the degraded observed value $\breve{y}(k)$ and the filtering estimate $\hat{\breve{x}}(k-1)$ of the degraded state $\breve{x}(k-1)$. The estimate $\hat{u}(k)$ of $u(k)$ in (15) uses the filtering estimate $\hat{\overleftrightarrow{x}}(k)$ by the robust RLS Wiener filter.

Numerical simulation examples have demonstrated the characteristics of tracking control using the H-infinity tracking controller of Theorem 1 and the robust RLS Wiener filter of Theorem 2 in linear discrete-time stochastic systems with uncertainties. Tables 1 and 2 show that the tracking controller of Theorem 1 with the robust RLS Wiener filter of Theorem 2 is superior in tracking control accuracy to the tracking controller of Theorem 1 with the RLS Wiener filter for the white Gaussian observation noise $N(0, 0.1^2)$, $N(0, 0.3^2)$, $N(0, 0.5^2)$, $N(0,1)$ and $N(0, 5^2)$. In addition, from Table 3, the MSVs of the tracking errors $\eta(k) - \hat{\breve{z}}(k)$ by the H-infinity tracking controller of Theorem 1 with the robust Kalman filter show that the tracking technique either fails to track the desired value at all or diverges for the observation noise. In the example for the F16 aircraft, Table 4 shows that the MSV of the tracking errors $\eta(k) - \hat{\breve{z}}(k)$ is less than that of the tracking errors $\eta(k) - \breve{z}(k)$ for each observation noise. This indicates that the filtering estimate $\hat{\breve{z}}(k)$ tracks the desired value more accurately than $\breve{z}(k)$.

In particular, as the uncertainties in the system and observation matrices increase, the accuracy of the estimates for $\hat{\overleftrightarrow{A}}$ and $\hat{\overleftrightarrow{C}}$ is numerically required. In EXAMPLE 1, in the calculation of $\hat{\overleftrightarrow{C}}$, $E[\breve{y}(k)\overleftrightarrow{x}^T(k)]$ is approximated by $\frac{1}{2000}\sum_{k=1}^{2000} \breve{y}(k)\overleftrightarrow{x}^T(k)$ instead of $\frac{1}{350}\sum_{k=1}^{350} \breve{y}(k)\overleftrightarrow{x}^T(k)$.

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The author has no conflict of interest to declare.