Recursive Least-Squares Wiener Consensus Filter and Fixed-Point Smoother in Distributed Sensor Networks

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Abstract: - Distributed Kalman filter (DKF) is classified into the information fusion Kalman filter (IFKF), i. e. the centralized Kalman filter (CKF), and the Kalman consensus filter (KCF) in distributed sensor networks. The KCF has the advantage to improve the estimate of the state at the sensor node uniformly by incorporating the information of the observations and the filtering estimates at the neighbor nodes. In the first devised KCF, a user adjusts the consensus gain. This paper designs the recursive least-squares (RLS) Wiener consensus filter and fixed-point smoother that do not need to be adjusted in linear discrete-time stochastic systems. In addition to the observation equation at the sensor node, an observation equation is introduced excessively. Here, the new observation is the sum of the filtering estimates of the signals at the neighbor nodes of the sensor node. Thus, it is interpreted that the RLS Wiener consensus estimators incorporate the information of the observations at the neighbor nodes indirectly because the observations are used in the calculations of the filtering estimates. A numerical simulation example shows that the proposed RLS Wiener consensus filter and fixed-point smoother are superior in estimation accuracy to the RLS Wiener estimators.

Key-Words: - RLS Wiener consensus filter, RLS Wiener consensus fixed-point smoother, distributed sensor networks, Kalman consensus filter, information fusion filter.

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1 Introduction

Over the last decade or more, the Kalman consensus filter (KCF) has been studied extensively in linear discrete-time or continuous-time systems, e.g., [1]-[9]. Casbeer and Beard [10] present an information consensus filter (ICF) in distributed sensor networks. Li, Caimou, and Haoji, [11], study the KCF and the ICF, where they propose a new optimization procedure to update the consensus weights of the ICF. Wu et al., [4], propose the KCF by introducing the consensus gain, as shown in (7) and (8) in the paper, for linear continuous-time systems. AminiOmam, Torkamani-Azar, and Ghorashi, [12], propose a generalized Kalman consensus filter for nonlinear discrete-time systems, and its stability on the asymptotical convergence is proved based on the Lyapunov method. Chen et al., [13], propose the distributed state estimator in discrete-time nonlinear systems and present the distributed cubature information filtering algorithm.

Olfati-Saber [14]-[16] is the first proposer of the KCF. Referring to Olfati-Saber [14]-[16], Takaba [17] explains in Japanese the distributed Kalman filter (DKF). The DKF is classified into the information fusion Kalman filter (IFKF), i. e. the centralized Kalman filter (CKF), and the KCF in distributed sensor networks. The KCF has the advantage to improve the estimate of the state at the sensor node uniformly by incorporating the information of the observations and the filtering estimates at the neighbor nodes. In the calculation of the filtering estimate at the sensor node, the KCF, [16], uses the one-step-ahead prediction estimates of the states at the neighbor nodes of the sensor node in addition to the observed value at the sensor node. In the first devised KCF, a user adjusts the consensus gain. This paper designs the recursive least-squares (RLS) Wiener consensus filter and fixed-point smoother that do not need to be adjusted in linear discrete-time stochastic systems. In addition to the observation equation at the sensor node, a new observation equation is introduced excessively. Here, the new observation is the sum of the filtering estimates of the signals at the neighbor nodes of the sensor node. Thus, it is interpreted that the RLS Wiener consensus estimators incorporate the information of the observations at the neighbor nodes indirectly because these observations are used in the calculations of the filtering estimates at the neighbor nodes.

Section 2 introduces the least-squares consensus fixed-point smoothing problem. Section 3 presents the RLS Wiener consensus filtering and fixed-point smoothing algorithms. Section 4 presents the recursive algorithm for the estimation error variance function of the RLS Wiener consensus fixed-point smoother. Also, the asymptotic stability condition of the RLS Wiener consensus filter and the existence of the RLS Wiener consensus fixed-point smoother are shown. A numerical simulation example is shown in section 5 to demonstrate the estimation characteristic of the RLS Wiener consensus filter and fixed-point smoother. From the numerical simulation example, the proposed RLS Wiener consensus filter and fixed-point smoother are superior in estimation accuracy to the RLS Wiener filter and fixed-point smoother respectively.

2 Least-squares consensus fixed-point smoothing problem

Consider the state equation for the state vector $x(k)$ in linear discrete-time stochastic systems

$$x(k+1) = \Phi x(k) + \Gamma w(k), \qquad (1)$$

where $x(k)$ is the state vector with n components at time k, Φ is the system matrix, Γ is the input matrix and $w(k)$ is the zero-mean input noise. For the sensor nodes, $i = 1,2,\cdots,N$, each sensor node has the observation equation

$$y_i(k) = C_i x(k) + v_i(k), z_i(k) = C_i x(k), \qquad (2)$$

where $y_i(k)$ is the m-dimensional observed value at the sensor node i, $z_i(k)$ is the signal at the sensor node i, C_i is the m by n observation matrix at the sensor node i and $v_i(k)$ is the zero-mean observation noise at the sensor node i. The auto-covariance functions of $w(k)$ and $v_i(k)$ are given by

$$\begin{aligned} &E[w(k)w^T(s)] = Q\delta_K(k-s), Q > 0,\\ &E[w(k)] = 0,\\ &E\left[v_i(k)\left(v_j\right)^T(s)\right] = R_{ij}\delta_K(k-s), \end{aligned} \qquad (3)$$

where $\delta_K(\cdot)$ denotes the Kronecker δ function. According to the observation equation (2), Olfati-Saber [16] shows the Kalman consensus filter as follows.

Filtering estimate of the state $x(k)$ at the sensor node i: $\hat{x}_i(k \mid k)$

$$\begin{aligned} &\hat{x}_i(k \mid k) = \hat{x}_i(k \mid k-1)\\ &+K_i(k)\left(y_i(k) - C_i\hat{x}_i(k \mid k-1)\right) +\\ &+U_i(k)\sum_{j\in N_i}(\hat{x}_j(k \mid k-1)\\ &-\hat{x}_i(k \mid k-1)),\\ &\hat{x}_i(k \mid k-1) = \Phi\hat{x}_i(k-1 \mid k-1),\\ &\hat{x}_j(k \mid k-1) = \Phi\hat{x}_j(k-1 \mid k-1) \end{aligned} \qquad (4)$$

$K_i(k)$ and $U_i(k)$ are the Kalman gain and the consensus gain respectively. The equations for the Kalman gain and the consensus gain are shown in the paper. In (4), the filtering estimate at the sensor node i uses the observed value $y_i(k)$ at the sensor node i together with the one-step-ahead prediction estimates $\hat{x}_j(k \mid k-1)$ of $x(k-1)$ at the neighbor nodes $j \in N_i$ of the sensor node i. Here, it should be noted that the Kalman filter calculates the estimates $\hat{x}_j(k \mid k-1)$ recursively with the observed values $y_j(k-1)$.

Referring to Olfati-Saber [14]-[16], Takaba [17] summarizes the Kalman consensus filter as follows. The observation $\check{z}_i(k)$ at the sensor node i is given by

$$\begin{aligned} &\check{z}_i(k) = H_i x(k) + d_i(k),\\ &H_i = \begin{bmatrix} C_i \\ C_{i_1} \\ \vdots \\ C_{i_{N_i}} \end{bmatrix}, d_i(k) = \begin{bmatrix} v_i(k) \\ v_{i_1}(k) \\ \vdots \\ v_{i_{N_i}}(k) \end{bmatrix}. \end{aligned} \qquad (5)$$

From (2) and (5), $\check{z}_i(k)$ consists of the observed value $y_i(k)$ at the sensor node i and its neighbor observed values $y_{i_1}(k), y_{i_2}(k),\cdots, y_{i_{N_i}}(k)$ at time k. $d_i(k)$ has the auto-covariance function.

$$R_i = E[d_i(k)(d_i)^T(k)] = \begin{bmatrix} R_{ii} & R_{i,i_1} & \cdots & R_{i,i_{N_i}} \\ R_{i_1,i} & R_{i_1,i_1} & \cdots & R_{i_1,i_{N_i}} \\ \vdots & \vdots & \ddots & \vdots \\ R_{i_{N_i},i} & R_{i_{N_i},i_1} & \cdots & R_{i_{N_i},i_{N_i}} \end{bmatrix} \quad (6)$$

The Kalman consensus filter calculates the filtering estimate $\hat{x}_i(k \mid k)$, at the sensor node i, of the state $x(k)$ recursively by (7) - (10).

$$\begin{aligned} &\hat{x}_i(k \mid k) = \hat{x}_i(k \mid k-1) \\ &+K_i(k)\big(\breve{z}_i(k) - H_i\hat{x}_i(k \mid k-1)\big) \\ &+\varepsilon P_i(k|k-1) \\ &\times \sum_{j\in N_i}(\hat{x}_j(k \mid k-1) - \hat{x}_i(k \mid k-1)), \\ &\hat{x}_i(k \mid k-1) = \Phi\hat{x}_i(k-1 \mid k-1), \\ &\hat{x}_j(k \mid k-1) = \Phi\hat{x}_j(k-1 \mid k-1) \end{aligned} \quad (7)$$

Kalman gain:

$$\begin{aligned} &K_i(k) = P_i(k|k-1)(H_i)^T \\ &\times (R_i + C_iP_i(k|k-1)(C_i)^T)^T \end{aligned} \quad (8)$$

Riccati equation:

$$\begin{aligned} &P_i(k \mid k-1) \\ &= \Phi P_i(k-1 \mid k-1)(\Phi)^T + Q \end{aligned} \quad (9)$$

$$\begin{aligned} &P_i(k|k) \\ &= (I_n - K_i(k)H_i)\,P_i(k|k-1) \\ &\times (I_n - K_i(k)H_i)^T \\ &+K_i(k)R_i(K_i(k))^T \end{aligned} \quad (10)$$

Here, ε is a positive parameter determined by a user. So, the Kalman consensus filter is suboptimal. In (7), N_i denotes the neighbor nodes of the sensor node i in the distributed sensor networks. At the sensor node i, in estimating the state $x(k)$, the Kalman consensus filter uses the observations at the sensor node i and the observations at the neighbor nodes of the sensor node i together with the one-step-ahead prediction estimates $\hat{x}_j(k \mid k-1)$ at the neighbor nodes $j \in N_i$ of the sensor node i. Taking into consideration of the Kalman consensus filtering algorithm, we newly introduce the augmented observation equation as follows.

$$\begin{aligned} &\bar{Y}_i(k) = \bar{H}_i x(k) + \bar{V}_i(k), \\ &\bar{Y}_i(k) = \begin{bmatrix} y_i(k) \\ \sum_{j\in N_i} C_j\,\hat{x}_j(k|k) \end{bmatrix}, \\ &\bar{H}_i = \begin{bmatrix} C_i \\ \sum_{j\in N_i} C_j \end{bmatrix}, \\ &\bar{V}_i(k) = \begin{bmatrix} v_i(k) \\ \tilde{v}(k) \end{bmatrix} \end{aligned} \quad (11)$$

$y_i(k)$ is the observed value at the sensor node i. $\sum_{j\in N_i} C_j\,\hat{x}_j(k|k)$ denotes the sum of the filtering estimates $\hat{z}_j(k|k)$ of the signal $z_j(k) = C_jx(k)$ in the neighbor nodes $j \in N_i$ of the sensor node i. In this paper, the RLS Wiener filter calculates the filtering estimates $\hat{x}_j(k|k)$ of the state $x(k)$, for the neighbor nodes j, $j \in N_i$, of the sensor node i, with the observed values $y_j(k)$ recursively. Since the filtering estimate $\hat{x}_j(k|k)$ is calculated with the information of the observed values $y_j(k)$, the RLS Wiener consensus estimators in this paper do not include the observed values from the neighbor nodes in the observation equation (11). $\tilde{v}(k)$ represents the sum of the filtering errors of the signals at the neighbor nodes $j \in N_i$ of the sensor node i.

$$\begin{aligned} &\tilde{v}(k) = C_{i_1}\left(\hat{x}_{i_1}(k \mid k) - x(k)\right) \\ &+C_{i_2}\left(\hat{x}_{i_2}(k \mid k) - x(k)\right)\cdots \\ &+C_{i_{N_i}}\left(\hat{x}_{i_{N_i}}(k \mid k) - x(k)\right) \\ &= \sum_{j=i_1}^{i_{N_i}}(\hat{z}_j\,(k \mid k) - z_j(k)) \end{aligned} \quad (12)$$

It is seen that the processes $\hat{x}_{i_1}(k \mid k) - x(k)$, $\hat{x}_{i_2}(k \mid k) - x(k)$, $\cdots$, $\hat{x}_{i_{N_i}}(k \mid k) - x(k)$ are mutually uncorrelated. Let the auto-covariance function $K(k,s)$ of the state $x(k)$ have the semi-degenerate kernel form

$$\begin{aligned} &K(k,s) = E[x(k)x^T(s)] \\ &= \begin{cases} A(k)B^T(s), & 0 \le s \le k, \\ B(s)A^T(k), & 0 \le k \le s, \end{cases} \\ &A(k) = \Phi^k, \quad B^T(s) = \Phi^{-s}K(s,s), \end{aligned} \quad (13)$$

in wide-sense stationary stochastic systems [18]. The auto-covariance function $\tilde{R}(k)$ of $\tilde{v}(k)$ is given by

$$E[\tilde{v}(k)\tilde{v}^T(s)] = \tilde{R}(k)\delta_K(k-s),$$
$$\tilde{R}(k) = \sum_{j=i_1}^{i_{N_i}} C_j\,(K(k,k) -\hat{P}_j(k|k))\,(C_j)^T), \tag{14}$$
$$\hat{P}_j(k\,|\,k) = E[\hat{x}_j(k\,|\,k)(\hat{x}_j(k\,|\,k))^T],$$
$$j = i_1, i_2, \cdots, i_{N_i}.$$

Hence, the auto-covariance function of $\bar{V}_i(k)$ is given by

$$E[\bar{V}_i(k)(\bar{V}_i(s))^T] = \bar{R}_i(k)\delta_K(k-s),$$
$$\bar{R}_i(k) = \begin{bmatrix} R_{ii} & 0 \\ 0 & \tilde{R}(k) \end{bmatrix}. \tag{15}$$

Now, the consensus estimation problem is reduced to estimate the state $x(k)$ with the augmented observation $\bar{Y}_i(k)$ of (11).

Let the fixed-point smoothing estimate $\hat{x}_i(k|k+L)$ of $x(k)$, at the sensor node i, be expressed by

$$\hat{x}_i(k|L) = \sum_{j=1}^{L} h_i\,(k,j|L)\bar{Y}_i(j) \tag{16}$$

as a linear transformation of the observed values $\bar{Y}_i(j)$, $1 \le j \le L$. In (16), $h_i(k,j|L)$ is called the impulse response function. We consider the fixed-point smoothing problem, which minimizes the mean-square value (MSV)

$$J = E[||x(k) - \hat{x}_i(k|L)||^2] \tag{17}$$

of the fixed-point smoothing error at the sensor node i. From an orthogonal projection lemma, [18],

$$x(k) - \sum_{j=1}^{L} h_i\,(k,j|L)y(j) \perp y(s), \quad 0 \le k,s \le L, \tag{18}$$

the impulse response function, at the sensor node i, satisfies the Wiener-Hopf equation

$$[x(k)(\bar{Y}_i(s))^T] = \sum_{j=1}^{L} h^i\,(k,j\,|\,L)\bar{K}_i(j,s), \tag{19}$$
$$\bar{K}_i(k,s) = E[\bar{Y}_i(k)(\bar{Y}_i(s))^T].$$

In (18), ' $\perp$ ' denotes the notation of the orthogonality. $\bar{K}_i(k,s)$ is the auto-covariance function of the augmented observed value $\bar{Y}_i(k)$. Substituting (11), (13), and (15) into (19), we obtain the equation for the optimal impulse response function $h_i(k,s\,|\,L)$ at the sensor node i.

$$h_i(k,s\,|\,L)\bar{R}_i(s) = K(k,s)(\bar{H}_i)^T -\sum_{j=1}^{L} h_i\,(k,j\,|\,L)\bar{H}_i K(j,s)(\bar{H}_i)^T, \tag{20}$$
$$0 \le k,s \le L$$

Starting with (20), the RLS Wiener estimation algorithms are derived based on the invariant imbedding method. Section 3 proposes the RLS Wiener consensus filtering and fixed-point smoothing algorithms.

3 RLS Wiener consensus filtering and fixed-point smoothing algorithms

Starting with (20), which the optimal impulse response function $h_i(k,s\,|\,L)$ satisfies, based on the preliminary formulations of the least-squares consensus estimation problem, Theorem 1 presents the RLS Wiener consensus filtering and fixed-point smoothing algorithms.

Theorem 1 Let the state equation for the state $x(k)$ be given by (1). Let the observation equation at the sensor node i with the consensus of the neighbor nodes $j \in N_i$ of the sensor node i be given by (11). The auto-covariance function of the observation noise is given by (15). Then the RLS Wiener consensus filtering and fixed-point smoothing algorithms consist of (21)-(29) in the linear discrete-time wide-sense stationary stochastic system.

Fixed-point smoothing estimate of the signal $z_i(k)$ at the sensor node i: $\hat{z}_i(k|L)$

$$\hat{z}_i(k\,|\,L) = C_i\hat{x}_i(k\,|\,L), i = 1,2,\cdots,N_i \tag{21}$$

Fixed-point smoothing estimate of the state $x(k)$ at the sensor node i: $\hat{x}_i(k|L)$

$$\hat{x}_i(k\,|\,L) = \hat{x}_i(k\,|\,L-1) +h_i(k,L\,|\,L)(\bar{Y}_i(L) -\bar{H}_i\Phi\hat{x}_i(L-1\,|\,L-1)) \tag{22}$$

Smoother gain at the sensor node i: $h_i(k,L\,|\,L)$

$$h_i(k, L \mid L) = (K(k,k)(\Phi^T)^{L-k}(\bar{H}_i)^T$$
$$-q_i(k \mid L-1)\Phi^T (\bar{H}_i)^T)$$
$$\times (\bar{R}_i(L) + \bar{H}_i K(L,L)$$
$$\times -\bar{H}_i \Phi S_i(L-1)\Phi^T)(\bar{H}_i)^T)^{-1} \quad (23)$$

$$q_i(k \mid L) = q_i(k \mid L-1)\Phi^T$$
$$+h_i(k, L \mid L)(\bar{H}_i K(L,L)$$
$$-\bar{H}_i \Phi S_i(L-1)\Phi^T),$$
$$q_i(k \mid k) = S_i(k) \quad (24)$$

Filter gain at the sensor node i: $G_i(L,L)$

$$G_i(L,L)$$
$$= (K(L,L) - \Phi S_i(L-1)\Phi^T)(\bar{H}_i)^T$$
$$\times (\bar{R}_i(L) + \bar{H}_i(K(L,L)$$
$$-\Phi S_i(L-1)\Phi^T)$$
$$\times (\bar{H}_i)^T)^{-1} \quad (25)$$

Filtering estimate of the signal $z_i(k)$ at the sensor node i: $\hat{z}_i(k|k)$

$$\hat{z}_i(k|k) = C_i \hat{x}_i(k \mid k) \quad (26)$$

Filtering estimate of the state $x(L)$ at the sensor node i: $\hat{x}_i(L \mid L)$

$$\hat{x}_i(L|L) = \Phi \hat{x}_i(L-1 \mid L-1)$$
$$+G_i(L,L)(\bar{Y}_i(L)$$
$$-\bar{H}_i \Phi \hat{x}_i(L-1|L-1)),$$
$$\hat{x}_i(0 \mid 0) = 0 \quad (27)$$

The variance of the filtering estimate $\hat{x}_i(L \mid L)$ at the sensor node i: $S_i(L)$

$$S_i(L) = \Phi S_i(L-1)\Phi^T$$
$$+G_i(L,L)(\bar{H}_i K(L,L)$$
$$-\bar{H}_i \Phi S_i(L-1)\Phi^T),$$
$$S_i(0) = 0 \quad (28)$$

Here, the variance of the observation noise $\bar{V}_i(L)$ is given by

$$\bar{R}_i(L) = \begin{bmatrix} R_{ii} & 0 \\ 0 & \tilde{R}(L) \end{bmatrix},$$
$$\tilde{R}(L) = \sum_{j=i_1}^{i_{N_i}} C_j\,(K(L,L)$$
$$-\hat{P}_j(L|L))\,(C_j)^T)$$
$$= \sum_{j=i_1}^{i} C_j\,(K(L,L) - S_j(L))\,(C_j)^T). \quad (29)$$

Proof of Theorem 1 is deferred to the Appendix. Section 4 proposes the algorithm for the RLS Wiener consensus fixed-point smoothing error variance function. Also, the asymptotic stability condition of the RLS Wiener consensus filter and the existence of the RLS Wiener consensus fixed-point smoother are shown.

4 RLS Wiener consensus fixed-point smoothing error variance function

The RLS Wiener consensus fixed-point smoothing error variance function is defined by

$$\tilde{P}_i(k \mid L) = E[(x(k) - \hat{x}_i(k \mid L))(x(k) - \hat{x}_i(k \mid L))^T]. \quad (30)$$

From (22) and the relationship $S_i(L) = E[\hat{x}_i(L \mid L)(\hat{x}_i(L \mid L))^T]$, (30) is developed as

$$\tilde{P}_i(k \mid L)$$
$$= K(k,k) - E[\hat{x}_i(k \mid L)(\hat{x}_i(k \mid L))^T]$$
$$= K(k,k) - E[(\hat{x}_i(k \mid L-1)$$
$$+h_i(k, L \mid L)(\bar{Y}_i(L)$$
$$-\bar{H}_i \Phi \hat{x}_i(L-1 \mid L-1))$$
$$\times (\hat{x}_i(k \mid L-1) + h_i(k, L \mid L)(\bar{Y}_i(L)$$
$$-\bar{H}_i \Phi \hat{x}_i(L-1 \mid L-1))^T]$$
$$= K(k,k)$$
$$- E[\hat{x}_i(k \mid L-1)(\hat{x}_i(k \mid L-1))^T]$$
$$-h_i(k, L \mid L)E[(\bar{Y}^i(L)$$
$$-\bar{H}_i \Phi \hat{x}_i(L-1 \mid L-1))(\bar{Y}_i(L)$$
$$-\bar{H}_i \Phi \hat{x}_i(L-1 \mid L-1))^T]$$
$$\times (h_i(k, L \mid L))^T$$
$$= \tilde{P}_i(k \mid L-1) - h_i(k, L \mid L)$$
$$\times (\bar{R}_i(L) + \bar{H}_i(K(L,L)$$
$$-\Phi S_i(L-1)\Phi^T)(\bar{H}_i)^T)$$
$$\times (h_i(k, L \mid L))^T. \quad (31)$$

Here, $h_i(k, L \mid L)$ is calculated by (23)-(25) and (28) recursively. $S_i(L)$ is calculated by (25) and (28) recursively.

Also, the RLS Wiener consensus fixed-point smoothing error variance function $\tilde{P}_i(k \mid L)$ is written as $\tilde{P}_i(k \mid L) = K(k,k) - E[\hat{x}_i(k \mid L)(\hat{x}_i(k \mid L))^T]$. $E[\hat{x}_i(k \mid L)(\hat{x}_i(k \mid L))^T]$ represents the variance of the fixed-point smoothing estimate $\hat{x}_i(k \mid L)$ at the sensor node i. $\tilde{P}_i(k \mid L)$ and $E[\hat{x}_i(k \mid L)(\hat{x}_i(k \mid L))^T]$ are positive-semidefinite matrices. From this fact, the variance of the fixed-point smoothing estimate $E[\hat{x}_i(k \mid L)(\hat{x}_i(k \mid L))^T]$ is upper bounded by the variance of the state $x(k)$ and lower bounded by the zero matrix as

$$0 \le E[\hat{x}_i(k \mid L)(\hat{x}_i(k \mid L))^T] \le K(k,k).$$

This shows the existence of the fixed-point smoothing estimate $\hat{x}_i(k \mid L)$.

The asymptotic stability of the filtering equation (27) is assured by the condition that $\Phi - G_i(L,L)\bar{H}_i\Phi$ is a stable matrix. Namely, for the stability of the filtering equation (27), all the eigenvalues of $\Phi - G_i(L,L)\bar{H}_i\Phi$ must lie within the unit circle.

5 A numerical simulation example

Let us consider the state equation

$$\begin{aligned} &x(k+1) = \Phi x(k) + \Gamma w(k), \\ &\Phi = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}, \\ &x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \\ &a_{11} = 0.85, a_{12} = -0.2, a_{21} = 0.2, \\ &a_{22} = 0.76, \\ &\Gamma_1 = 0.952, \Gamma_2 = 0.2. \end{aligned} \tag{32}$$

The auto-covariance function of the input noise $\omega(k)$, with mean zero, is given by

$$E[\omega(k)\omega(s)] = \sigma^2 \delta_K(k-s), \sigma^2 = 0.5^2.$$

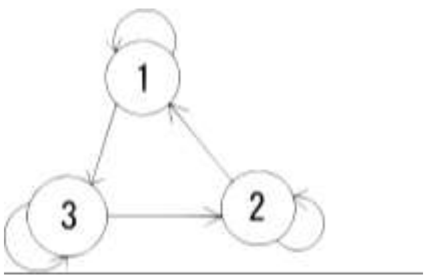

Fig.1 Directed graph of topological structure for the distributed sensor networks with three sensor nodes.

Fig.1 illustrates the directed graph of the topological structure for the distributed sensor networks with three sensor nodes. Its adjacency matrix is given by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

The observation equations at the sensor nodes are given as follows.

Observation equation at the sensor node 1:

$$\begin{aligned} &y_1(k) = C_1 x(k) + v_1(k), \\ &z_1(k) = C_1 x(k), \\ &C_1 = [0.95 \quad -0.4] \end{aligned} \tag{33}$$

Observation equation at the sensor node 2:

$$\begin{aligned} &y_2(k) = C_2 x(k) + v_2(k), \\ &z_2(k) = C_2 x(k), \\ &C_2 = [1 \quad 0.5] \end{aligned} \tag{34}$$

Observation equation at the sensor node 3:

$$\begin{aligned} &y_3(k) = C_3 x(k) + v_3(k), \\ &z_3(k) = C_3 x(k), \\ &C_3 = [0.5 \quad 1] \end{aligned} \tag{35}$$

Here, the variance R_{ii} of the observation noises $v_i(k)$, $i = 1,2,3$, are the same. Substituting $\bar{H}_i$, Φ, $K(k,k)$, $\bar{Y}_i(L)$ and $\bar{R}_i(L)$ into the RLS Wiener consensus fixed-point and filtering algorithms of Theorem 1, we calculate the fixed-point smoothing estimate $\hat{z}_i(k|L)$ and the filtering estimate $\hat{z}_i(k|k)$ of the signal $z_i(k)$, $i = 1,2,3$, recursively.

Fig.2 illustrates the signal $z_1(k)$, the filtering estimate $\hat{z}_1(k|k)$ and the fixed-point smoothing estimate $\hat{z}_1(k|k+5)$ for the observation noise $N(0, 0.5^2)$ at sensor node 1 under the consensus with neighbor node 2. Fig.3 illustrates the mean-square values of the filtering and fixed-point smoothing errors $z_1(k) - \hat{z}_1(k \mid k + Lag)$, $0 \le Lag \le 5$, of the signal $z_1(k)$ at the sensor node 1 vs. Lag, for the white Gaussian observation noises $N(0, 0.3^2)$, $N(0, 0.5^2)$ and $N(0, 0.7^2)$, by the RLS Wiener consensus estimators, under the consensus of the sensor node 1 with the neighbor node 2, and the RLS Wiener estimators. Fig.4 illustrates the mean-square values of the filtering and fixed-point smoothing errors $z_2(k) - \hat{z}_2(k \mid k + Lag)$, $0 \le Lag \le 5$, of the signal $z_2(k)$ at the sensor node 2 vs. Lag, for the white Gaussian observation noises $N(0, 0.3^2)$, $N(0, 0.5^2)$ and $N(0, 0.7^2)$, by the RLS Wiener consensus estimators, under the consensus of the sensor node 2 with the neighbor node 3, and the RLS Wiener estimators. Fig.5 illustrates the mean-square values of the filtering and fixed-point smoothing errors $z_3(k) - \hat{z}_3(k \mid k + Lag)$, $0 \le Lag \le 5$, of the signal $z_3(k)$ at the sensor node 3 vs. Lag, for the white Gaussian observation noises $N(0, 0.3^2)$, $N(0, 0.5^2)$ and $N(0, 0.7^2)$, by the RLS Wiener consensus estimators, under the consensus of the sensor node 3 with the neighbor node 1, and the RLS Wiener estimators. From Fig. 3, Fig.4, and Fig.5, it is seen that the estimation accuracies of the RLS Wiener consensus filter and fixed-point smoother are superior to those of the RLS Wiener filter and fixed-point smoother respectively for each observation noise. Here, the MSVs of the filtering and fixed-point smoothing errors are calculated by $\sum_{k=1}^{2000}(z_i(k) - \hat{z}_i(k|k + Lag))^2/2000, 0 \le Lag \le 5$, for the RLS Wiener consensus estimators and the RLS Wiener estimators.

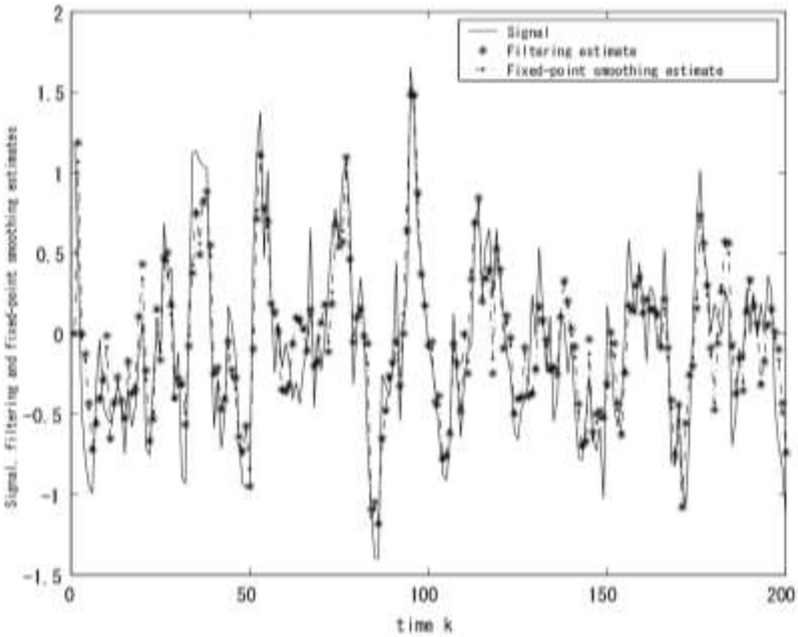

Fig.2 Signal $z_1(k)$, filtering estimate $\hat{z}_1(k|k)$ and fixed-point smoothing estimate $\hat{z}_1(k|k + 5)$ for the observation noise $N(0, 0.5^2)$ at sensor node 1 under consensus with neighbor node 2.

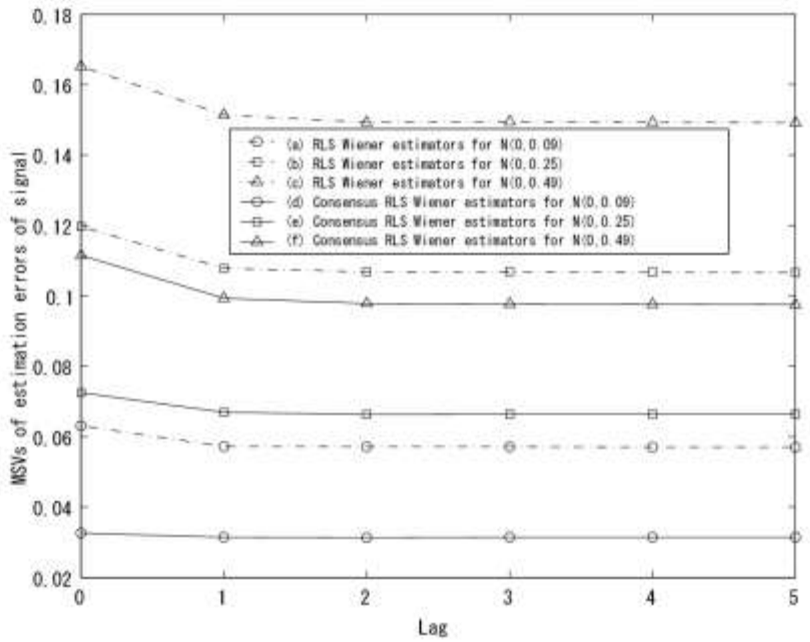

Fig.3 Mean-square values of the filtering and fixed-point smoothing errors $z_1(k) - \hat{z}_1(k \mid k + Lag), 0 \leq Lag \leq 5$, of the signal $z_1(k)$ at the sensor node 1 vs. Lag, for the white Gaussian observation noises $N(0, 0.3^2)$, $N(0, 0.5^2)$ and $N(0, 0.7^2)$, by the RLS Wiener consensus estimators under the consensus of the sensor node 1 with the neighbor node 2 and the RLS Wiener estimators.

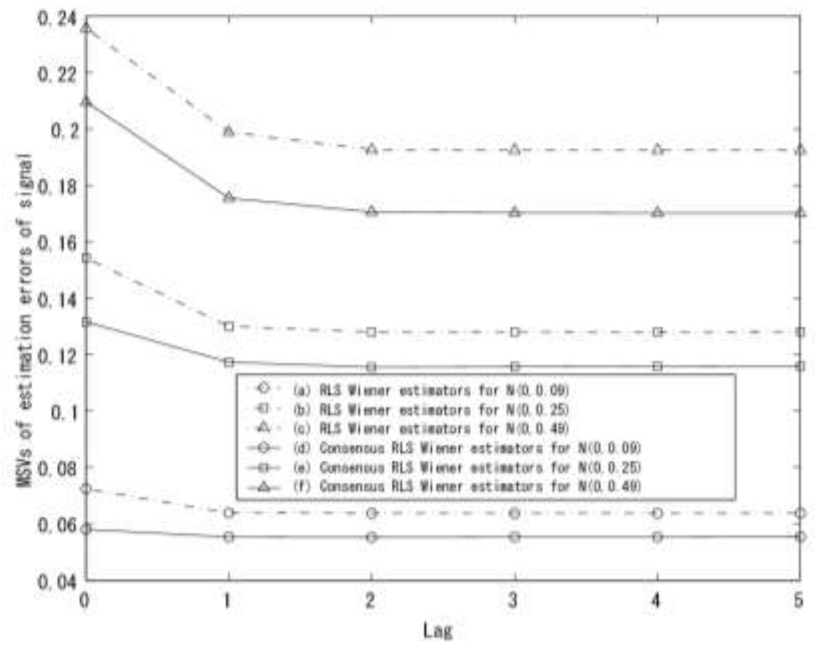

Fig.4 Mean-square values of the filtering and fixed-point smoothing errors $z_2(k) - \hat{z}_2(k \mid k + Lag)$, $0 \leq Lag \leq 5$, of the signal $z_2(k)$ at the sensor node 2 vs. Lag, for the white Gaussian observation noises $N(0, 0.3^2)$, $N(0, 0.5^2)$ and $N(0, 0.7^2)$, by the RLS Wiener consensus estimators under the consensus of the sensor node 2 with the neighbor node 3 and the RLS Wiener estimators.

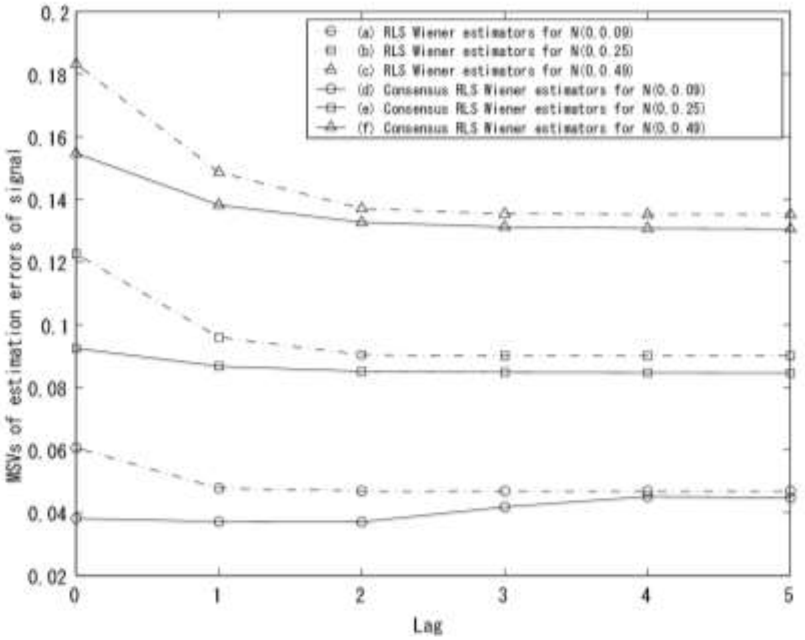

Fig.5 Mean-square values of the filtering and fixed-point smoothing errors $z_3(k) - \hat{z}_3(k \mid k + Lag)$, $0 \le Lag \le 5$, of the signal $z_3(k)$ at the sensor node 3 vs. Lag, for the white Gaussian observation noises $N(0, 0.3^2)$, $N(0, 0.5^2)$ and $N(0, 0.7^2)$, by the RLS Wiener consensus estimators under the consensus of the sensor node 3 with the neighbor node 1 and the RLS Wiener estimators.

6 Conclusion

This paper has originally developed the RLS Wiener consensus filter and fixed-point smoother in linear discrete-time stochastic systems. The new points of this paper are to incorporate the sum of the filtering estimates of the signals at the neighbor nodes as the observed value in the observation equation as shown in the augmented observation equation (11) and derive the RLS Wiener consensus estimators.

From the numerical simulation results in section 5, the estimation accuracies of the RLS Wiener consensus filter and the fixed-point smoother are superior to those of the RLS Wiener filter and fixed-point smoother respectively for each observation noises

A future task is to apply the robust RLS Wiener filter to linear distributed sensor networks with degraded observations generated by state-space model and observation equation with uncertain parameters.

Appendix A: Proof of Theorem 1

From (20), the impulse response function $h(k, s|L)$ satisfies

$$h_i(k, s \mid L)\bar{R}_i(s) = K(k, s)(\bar{H}_i)^T - \sum_{j=1}^{L} h_i(k, j \mid L)\bar{H}_i K(j, s)(\bar{H}_i)^T, \quad 0 \le k, s \le L. \tag{A-1}$$

Subtracting $h_i(k, s \mid L - 1)\bar{R}_i(s)$ from $h_i(k, s \mid L)\bar{R}_i(s)$, we have

$$\begin{aligned}(h_i(k, s \mid L) - h_i(k, s \mid L - 1))\bar{R}_i(s) &= -h_i(k, L \mid L)\bar{H}_i K(L, s)(\bar{H}_i)^T \\ &- \sum_{j=1}^{L-1} (h_i(k, j \mid L) - h_i(k, j \mid L - 1))\bar{H}_i K(j, s) \times (\bar{H}_i)^T.\end{aligned} \tag{A-2}$$

Introducing

$$J_i(L,s)\bar{R}_i(s) = B^T(s)(\bar{H}_i)^T - \sum_{j=1}^{L} J_i(L,j)\bar{H}_i K(j,s)(\bar{H}_i)^T, \quad \text{(A-3)}$$

we obtain

$$h_i(k,s \mid L) - h_i(k,s \mid L-1) = -h_i(k,L \mid L)\bar{H}_i A(L) J_i(L-1,s). \quad \text{(A-4)}$$

Subtracting $J_i(L-1,s)\bar{R}_i(s)$ from $J_i(L,s)\bar{R}_i(s)$, we have

$$(J_i(L,s) - J_i(L-1,s))\bar{R}_i(s) = -J_i(L,L)\bar{H}_i K(L,s)(\bar{H}_i)^T - \sum_{i=1}^{L} (J_i(L,j) - J_i(L-1,j))\bar{H}_i K(j,s)(\bar{H}_i)^T. \quad \text{(A-5)}$$

From (A-3) and (A-5), we obtain

$$J_i(L,s) - J_i(L-1,s) = -J_i(L,L)\bar{H}_i A(L) J_i(L-1,s). \quad \text{(A-6)}$$

From (A-3), $J_i(L,L)$ satisfies

$$J_i(L,L)\bar{R}_i(L) = B^T(L)(\bar{H}_i)^T - \sum_{j=1}^{L} J_i(L,j)\bar{H}_i K(j,L)(\bar{H}_i)^T = B^T(L)(\bar{H}_i)^T - \sum_{j=1}^{L} J_i(L,j)\bar{H}_i B(j) A^T(L)(\bar{H}_i)^T. \quad \text{(A-7)}$$

Introducing

$$r_i(L) = \sum_{j=1}^{L} J_i(L,j)\bar{H}_i B(j), \quad \text{(A-8)}$$

we obtain

$$J_i(L,L)\bar{R}_i(L) = B^T(L)(\bar{H}_i)^T - r_i(L) A^T(L)(\bar{H}_i)^T. \quad \text{(A-9)}$$

Subtracting $r_i(L-1)$ from $r_i(L)$ and using (A-6), we obtain

$$r_i(L) - r_i(L-1) = J_i(L,L)\bar{H}_i B(L) + \sum_{j=1}^{L-1} \big(J_i(L,j) - J_i(L-1,j)\big)\bar{H}_i B(j) = J_i(L,L)(\bar{H}_i B(L) - \bar{H}_i A(L) r_i(L-1)), \quad r_i(0) = 0. \quad \text{(A-10)}$$

Let us introduce the function

$$S_i(L) = \Phi^L r_i(L) (\Phi^T)^L. \quad \text{(A-11)}$$

From (A-10), we obtain

$$S_i(L) = \Phi S_i(L-1)\Phi^T + G_i(L,L)(\bar{H}_i K(L,L) - \bar{H}_i \Phi S_i(L-1)\Phi^T). \quad \text{(A-12)}$$

Here,

$$G_i(L,L) = \Phi^L J_i(L,L). \quad \text{(A-13)}$$

From (A-9), we have

$$G_i(L,L) = (K(L,L)(\bar{H}_i)^T - S_i(L)(\bar{H}_i)^T)(\bar{R}_i(L))^{-1}. \quad \text{(A-14)}$$

From (A-12) and (A-14), we obtain (25).

From (20), $h_i(k,L \mid L)$ satisfies

$$h_i(k,L \mid L)\bar{R}_i(s) = K(k,L)(\bar{H}_i)^T - \sum_{j=1}^{L} h_i(k,j \mid L)\bar{H}_i K(j,L)(\bar{H}_i)^T = K(k,k)(\Phi^T)^{L-k}(\bar{H}_i)^T - p_i(k|L)(\Phi^T)^L(\bar{H}_i)^T, \quad \text{(A-15)}$$

where

$$p_i(k|L) = \sum_{j=1}^{L} h_i(k,j \mid L)\bar{H}_i B(j). \quad \text{(A-16)}$$

Subtracting $p_i(k|L-1)$ from $p_i(k|L)$ and using (A-4) with (A-8), we have

$$\begin{aligned}&p_i(k\,|\,L)=p_i(k\,|\,L-1)\\&+h_i(k,L\,|\,L)\bar{H}_iB(L)\\&+\sum_{j=1}^{L-1}(h_i\,(k,j\,|\,L)\\&-h_i(k,j\,|\,L-1))\bar{H}_iB(j)\\&=p_i(k\,|\,L-1)\\&+h_i(k,L\,|\,L)(\bar{H}_iB(L)\\&-\bar{H}_iA(L)r_i(L-1)).\end{aligned} \tag{A-17}$$

Introducing

$$q_i(k\,|\,L)=p_i(k\,|\,L)(\Phi^T)^L, \tag{A-18}$$

from (A-17) and (A-18), we obtain

$$\begin{aligned}&q_i(k\,|\,L)=q_i(k\,|\,L-1)\\&+h_i(k,L\,|\,L)(\bar{H}_iB(L)\\&-\bar{H}_iA(L)r_i(L-1))(\Phi^T)^L\\&=q_i(k\,|\,L-1)\\&+h_i(k,L\,|\,L)(\bar{H}_iK(L,L)\\&-\bar{H}_i\Phi S_i(L-1)\Phi^T).\end{aligned} \tag{A-19}$$

In (A-1), putting $L=k$, we have

$$\begin{aligned}&h_i(k,s\,|\,k)\bar{R}_i(s)=K(k,s)(\bar{H}_i)^T\\&-\sum_{j=1}^{k}h_i\,(k,j\,|\,k)\bar{H}_iK(j,s)(\bar{H}_i)^T,\\&0\le s\le k.\end{aligned} \tag{A-20}$$

From (A-3), it is clear that

$$h_i(k,s\,|\,k)=\Phi^kJ_i(k,s). \tag{A-21}$$

From (A-18), we have

$$q_i(k\,|\,k)=p_i(k\,|\,k)(\Phi^T)^k. \tag{A-22}$$

Putting $L=k$ in (A-16), from (A-21), we have

$$\begin{aligned}&p_i(k\,|\,k)=\sum_{j=1}^{k}h_i\,(k,j\,|\,k)\bar{H}_iB(j)\\&=\Phi^k\sum_{j=1}^{k}J_i\,(k,j)\bar{H}_iB(j)\\&=\Phi^kr_i(k).\end{aligned} \tag{A-23}$$

From (A-11) and (A-22), we have

$$q_i(k\,|\,k)=S_i(k). \tag{A-24}$$

From (A-15) and (A-18), we obtain

$$\begin{aligned}&h_i(k,L\,|\,L)\bar{R}_i(s)\\&=K(k,k)(\Phi^T)^{L-k}(\bar{H}_i)^T\\&-q_i(k|L)(\bar{H}_i)^T.\end{aligned} \tag{A-25}$$

Substituting (A-19) into (A-25), after some manipulations, we obtain (23).

Now, from (16), the filtering estimate $\hat{x}_i(L|L)$ is given by

$$\hat{x}_i(L\,|\,L)=\sum_{j=1}^{L}h_i\,(L,j\,|\,L)\bar{Y}_i(j). \tag{A-26}$$

Let us introduce the function

$$e_i(L\,|\,L)=\sum_{j=1}^{L}J_i\,(L,j)\bar{Y}_i(j).A-27 \tag{A-27}$$

From (A-21), we get

$$\hat{x}_i(L\,|\,L)=\Phi^Le_i(L\,|\,L). \tag{A-28}$$

Subtracting the equation obtained by putting $L\to L-1$ in (A-27) from (A-27), we have

$$\begin{aligned}&e_i(L\,|\,L)-e_i(L-1\,|\,L-1)\\&=J_i(L,L)\bar{Y}_i(j)\\&+\sum_{j=1}^{L}(J_i(L,j)-J_i(L-1,j))\bar{Y}_i(j)\\&=J_i(L,L)\bar{Y}_i(j)\\&-J_i(L,L)\bar{H}_iA(L)e_i(L-1\,|\,L-1).\end{aligned} \tag{A-29}$$

Substituting (A-29) into (A-28), using (A-13), we obtain (27).

The fixed-point smoothing estimate $\hat{x}_i(k|L)$ is given by (16). Subtracting $\hat{x}_i(k|L-1)$ from $\hat{x}_i(k|L)$, and using (A-4), we have

$$\begin{aligned}&\hat{x}_i(k\,|\,L)-\hat{x}_i(k\,|\,L-1)\\&=h_i(k,L\,|\,L)\bar{Y}_i(L)\\&+\sum_{j=1}^{L-1}(h_i\,(k,j\,|\,L)\\&-h_i(k,j\,|\,L-1))\bar{Y}_i(j)\\&=h_i(k,L\,|\,L)(\bar{Y}_i(L)\\&-\bar{H}_i\Phi\hat{x}_i(k\,|\,L-1)).\end{aligned} \tag{A-30}$$

The initial condition of the fixed-point smoothing estimate $\hat{x}_i(k\,|\,L)$ at $L = k$ is the filtering estimate $\hat{x}_i(k\,|\,k)$.

(Q.E.D.)

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