

On stochastic differential equations generating non-Gaussian continuous Markov processes

VLADIMIR LYANDRES

School of Electrical and Computer Engineering,
 Ben-Gurion University of the Negev
 P.O. Box 653, Beer-Sheva, 84105
 ISRAEL

Abstract: - Continuous Markov processes widely used as a tool for modeling random phenomena in numerous applications, can be defined as solutions of generally nonlinear stochastic differential equations (SDEs) with certain drift and diffusion coefficients which together governs the process' probability density and correlation functions. Usually it is assumed that the diffusion coefficient does not depend on the process' current value. For presentation of non-Gaussian real processes this assumption becomes undesirable, leads generally to complexity of the correlation function estimation. We consider its analysis for the process with particular pairs of the drift and diffusion coefficients providing the given stationary probability distribution of the considered process.

Key-Words: Signal Processing, Continuous Markov process; Stochastic differential equation; Fokker-Planck equation.

Received: February 15, 2021. Revised: July 19, 2021. Accepted: July 28, 2021. Published: August 12, 2021.

1. Introduction

There is a wide range of phenomena in economics (stock prices) [1, 2], biology (population, epidemics) [3, 4], radio physics [5] that can be modeled as diffusion (continuous-time) Markov processes, which in turn, are represented by solutions of certain first-order stochastic differential equations (SDEs) characterized by the drift $K_1(x)$ and diffusion $K_2(x)$ coefficients. For some of them, those coefficients depend on their current values, and for such processes their generating SDE is written as [6]

$$\dot{x} = K_1(x) + \sqrt{K_2(x)} \xi(t) \quad (1)$$

where $\xi(t)$ is White Gaussian Noise (WGN) with correlation function

$$\langle \xi(t)\xi(t+\tau) \rangle = \delta(\tau). \quad (2)$$

Our purpose is to find an approximate method of calculation the correlation function for the continuous Markov process with the given stationary probability density and arbitrary diffusion coefficient. The paper is organized as follows: some preliminary results are presented in Section 2. Exact solution, i.e. spectrum of eigenvalues and expression for eigenfunctions of

the Fokker-Planck equation (FKE) corresponding to SDE (1), known only for some particular pairs of functions $K_1(x)$ and $K_2(x)$, is considered in Section 3.

2. Preliminary results

Using the Stratonovich definition of stochastic integral [6] we can express the drift coefficient $K_1(x)$ as a function of the process $x(t)$ stationary density $f_{st}(x)$ and $K_2(x)$

$$K_1(x) = f_{st}(x) - \frac{1}{4} K_2(x) \quad (3)$$

and the density $f_{st}(x)$ depends on $K_1(x)$ and $K_2(x)$ as

$$f_{st}(x) = \frac{C}{\sqrt{K_2(x)}} \exp \left[2 \int_{-\infty}^x \frac{K_1(z)}{K_2(z)} dz \right]. \quad (4)$$

For a given $f_{st}(x)$, solving (4) with respect to $K_1(x)$ we have

$$\kappa_1(x) = \frac{1}{2} \left\{ \frac{\kappa_2(x) \frac{d}{dx} \ln f_{st}(x) +}{\sqrt{\kappa_2(x)} \frac{d}{dx} [\sqrt{\kappa_2(x)}]} \right\}. \quad (5)$$

According to (4) the prescribed stationary density $f_{st}(x)$ can be obtained for SDE with any diffusion coefficient.

To find the correlation function of a stationary continuous Markov process the parabolic type Fokker-Planck equation (FKE) for the probability density $f[x, t]$ of the process has to be solved. One of the methods of the FKE solution [4, 6] is based on the presentation of this density as

$$f[x, t] = Y(x)T(t). \quad (6)$$

Substituting (6) in the FKE [6]

$$\begin{aligned} \frac{\partial}{\partial t} f(x, t) &= -\frac{\partial}{\partial x} [K_1(x)f(x, t)] + \\ \frac{1}{2} \frac{\partial^2}{\partial x^2} [K_2(x)f(x, t)] &= 0 \end{aligned} \quad (7)$$

we obtain the following transformation

$$\begin{aligned} \frac{1}{T(t)} \frac{\partial}{\partial t} T(t) &= \\ -\frac{\partial}{\partial x} [K_1(x)Y(x)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [K_2(x)Y(x)] & \quad (\\ \frac{\hspace{10em}}{Y(x)} &= 0 \end{aligned}$$

8)

which, in turn, is equivalent to the system of two ordinary differential equations

$$\frac{1}{T(t)} \frac{d}{dt} T(t) = -\lambda, \quad \lambda \geq 0, \quad (9)$$

$$\begin{aligned} \frac{1}{2} \frac{d^2}{dx^2} [K_2(x)Y(x)] - \\ \frac{d}{dx} [K_1(x)Y(x)] + \lambda Y(x) &= 0 \end{aligned}, \quad (10)$$

and the solution of the FKE may be written as

$$f(x, t) = \sum_{n=0}^{\infty} Y_n(x_n, \lambda_n) \exp(-\lambda_n t). \quad (11)$$

Here λ_n and $Y_n(x)$ are real eigenvalues and eigenfunctions of (10) corresponding to the boundary conditions

$$Y_n(0) = Y_n(\infty) = 0 \quad (12)$$

or

$$Y_n(-\infty) = Y_n(\infty) = 0, \quad (13)$$

depending on the process range.

The eigenvalue $\lambda_0 = 0$ is related to the stationary solution of the FKE $f_{st}(x)$. If the initial value of the process is fixed, i.e.

$$p(x, t=0) = \delta(x-x_0), \quad (14)$$

the transitional density of the process is written in the following form [5]

$$\begin{aligned} \pi(x, t / x_0, t_0) &= \\ \sum_{n=0}^{\infty} \frac{Y_n(x)Y_n(x_0)}{f_{st}(x)} \exp[-\lambda_n(t-t_0)] & \quad (15) \end{aligned}$$

while the correlation function is expressed as

$$B(\tau) = \sum_{n=0}^{\infty} h_n^2 \exp[-\lambda_n|\tau|], \quad (16)$$

where

$$h_n = \int_{-\infty}^{\infty} x Y_n(x) f_{st}(x). \quad (17)$$

3. Correlation function of the continuous Markov process with particular pair of the drift and diffusion coefficients

As is clear from (16) the correlation function of the continuous Markov process is monotonically decreasing but its convergence to an approximately exponential function depends on the set of functions $\{Y_n(x)\}$ and on the spectrum of eigenvalues $\{\lambda_n\}$ of the equation (10) which can be rewritten as

$$\left[\frac{\frac{d^2 Y_n(x)}{dx^2} + \frac{d \ln f_{st}(x)}{dx} \frac{d Y_n(x)}{dx} + \frac{2 \lambda_n}{K_2(x)} \frac{d \ln K_2(x)}{dx} \frac{d \ln f_{st}(x)}{dx} - \frac{d^2 \ln f_{st}(x)}{dx^2} \right] Y_n(x) = 0 \quad (18)$$

The closed form solution of (18) may be obtained only for special pair of $K_2(x)$ and $f_{st}(x)$. In the case of the process with constant diffusion it is available for the Gaussian or the Nakagami-m density [7] (generalizing Rayleigh, Weibull, Maxwell, the single-sided normal and the multidimensional vector modulus distributions)

$$f_{st}(x) = \begin{cases} 0, & -\infty < x < 0 \\ \frac{2m^m}{\Gamma(m)} \frac{x^{2m-1}}{\sigma^{2m}} \exp\left(-\frac{mx^2}{\sigma^2}\right), & x \geq 0, m \geq 0.5 \end{cases} \quad (19)$$

For the Gaussian process generated by SDE with linear drift and constant diffusion all eigenvalues except the first are equal to zero and the correlation function is exactly exponential [5]. For densities of the group (19), the solution of (18) may be found in [7]

$$Y_n(x) = 2 \sqrt{\frac{n!}{\Gamma(m+n)\Gamma(m)}} \frac{x^{2m-1}}{\sigma^{2m}} \exp\left(-\frac{mx^2}{\sigma^2}\right) L_n^{(m-1)}\left(\frac{m}{\sigma^2}\right) \quad (20)$$

$$\lambda_n = \frac{2N_0m}{\sigma^2} n, \quad n = 0, 1, 2, \dots, \quad (21)$$

where $L_n^{(m-1)}(z)$ is the generalized Laguerre polynomial [9]. The correlation function of such processes is represented by the row [8]

$$B(\tau) = \frac{\sigma^2 \Gamma^2(m+0.5)}{4m\pi\Gamma(m)} \sum_{n=1}^{\infty} \frac{\Gamma^2(n-0.5)}{\Gamma(m+n)n!} \exp\left(-\frac{2N_0m}{\sigma^2} n|\tau|\right) \quad (22)$$

In the case of Gamma distribution [10] generalizing Pearson, Erlang, Laplace, exponential power PDFs

$$f_{st}(x) = \begin{cases} 0, & -\infty < x < 0 \\ \frac{x^\alpha}{\beta^{\alpha+1}} \frac{x^{2m-1}}{\Gamma(\alpha+1)} \exp\left(-\frac{x}{\beta}\right), & x \geq 0, \alpha > -1, \beta > 0 \end{cases} \quad (23)$$

and constant diffusion coefficient, eigenfunctions of equation (18) cannot be expressed with the help of known functions, but this can be done if

$$K_2(x) = N_0 x \quad (24)$$

In this case its eigenfunctions are

$$Y_n(x) = \sqrt{\frac{n!}{\Gamma(\alpha+1)\Gamma(\alpha+1+n)}} x^\alpha \exp\left(-\frac{x}{\beta}\right) L_n^{(\alpha)}\left(\frac{x}{\beta}\right) \quad (25)$$

and the only nonzero eigenvalue is

$$\lambda_1 = \frac{N_0}{2\beta} \quad (26)$$

while the correlation function of the Gamma distributed Markov process with linear diffusion coefficient is

$$B(\tau) = \beta^2(\alpha+1) \exp\left(-\frac{N_0}{2\beta} |\tau|\right), \quad (27)$$

which is identical to that of the Gaussian Markov process [6].

4. Conclusion

Continuous Markov process may be presented as a solution of a generally nonlinear stochastic differential equation. The process' properties, including stationary probability distribution and correlation function, depend on the drift and diffusion forming its operator. The required process' density may be obtained with the help of their different pairs, while the correlation function appears as an infinite sum of exponents with the decays to be found from the corresponding Fokker-Planck equation with certain boundary conditions. We have considered analysis of the correlation function of the generated continuous Markov process.

References

- [1] Hull W. J., *Futures and other Derivatives*, Pearson Education India, 2006.
- [2] Nicolato E., Venardos E., Option pricing in stochastic volatility models of e Ornstein-Uhlenbeck type, *Mathematical Finance*, 13 (4) (2003) 445-466.

- [3] Smouse P. E., Focardi S., Moorcroft P. Kie J. G., Forester J. D., Morales J. M., Stochastic modelling of animal movement, *Philosophical Transactions of the Royal Society B: Biological Sciences*, 365 (1550) 2201-2211.
- [4] Barucha-Reid A. T., *Elements of the Theory of Markov Processes and their Applications*, McGraw Hills, 1960.
- [5] Lyandres V., Model of deep fading, *Information and Control Systems*, 1 (2018) 123-127.
- [6] Stratonovich. R. L., *Topics in the Theory of Random Noise*, Vol. 1, Gordon and Breach, 1967.
- [7] Nakagami M., The m-distribution – A general formula of intensity of rapid fading, *Methods in Radio Wave Propagation*, Pergamon, 1960.
- [8] Primak S., Kontorovich V., Lyandres V, *Stochastic Methods and their Applications to Communications. Stochastic Differential Equations Approach*, Wiley, 2004.
- [9] Gradshtein L., Ryzhik I., *Tables of Integrals, Series, and Products*, Academic Press, 2014.
- [10] Zayezdny A., Tabak D., Wulich D., *Engineering Applications of Stochastic Processes: Theory, Problems and Solutions*, Research Studies Press, Tauton, Somerset, 1989.

Creative Commons Attribution License 4.0 (Attribution 4.0 International, CC BY 4.0)

This article is published under the terms of the Creative Commons Attribution License 4.0

https://creativecommons.org/licenses/by/4.0/deed.en_US