Quantized Compressive Sensing Measurement Based on Improved Subspace Pursuit Algorithm

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Abstract: - Recent research results in compressive sensing have shown that sparse signals can be recovered from a small number of random measurements. Whether quantized compressive measurements can provide an efficient representation of sparse signals in information-theoretic needs discuss. In this paper, the distortion rate functions are used as a tool to research the quantizing compressive sensing measurements bring about average distortion rate. Both uniform quantization and non-uniform quantization were considered, for quantized measurements, the improved subspace pursuit was adapted to accommodate quantization error based on the concept of consistency, and experimental results show that the improved algorithm significantly reduces the reconstruction distortion when compared to standard compressive sensing techniques.

Key-Words: - Compressive Sensing; Rate Distortion Function; Subspace Pursuit

1 Introduction

Compressive sensing [1-2] is a linear sampling method that converts the unknown input signal in high-dimensional space into a small enough signal. In general, with the measured value of the dimension reduction, it is hard to just recover an unknown signal. However, when the input signal is sparse enough, accurate recovery is possible. This paper assumes that the unknown signal is K-sparse, namely, there is at most K non-zero value. A direct method is to search all possible signals, and then find the one which is the most consistent with linear measurement and the sparsest. This method only needs M-2K random measured values, but finding the sparsest signal representation is a NP-difficult issue. On the other hand, Donoho and Candes have proved that in the case of more measurement, sparse signal reconstruction is a NP-difficult polynomial time problem, which shall be solved through converting the reconstruction problem into a linear programming problem with tracking and other methods.

In most practical applications, it is necessary to use quantifying mechanisms for compress sensing information, but quantifying will result in distortion of the data obtained. Therefore, in the case of quantization error, using distortion theory to analyze compressed sensing performance is useful. When the quantization error is bounded and known in advance, Reference [3] gives the upper bound of the subspace tracking distortion (SP) reconstruction. In terms of the magnitude of the transform coefficients decay exponentially bounded compressible, Reference [4] gives an upper bound of reconstruction distortion caused by uniform quantization. The same quantization is applied strictly in Reference [5] in K-Item sparse signal and indicates a large part of unused quantized field. The methods described above focus on the worst case, or a simple 1 - bit quantization.

In contrast with the worst case, this dissertation studies the average degree of distortion of quantifying the measured value. Firstly, when the measurement matrix and sparse signals both follow a specific probability model, it can be inferred the exact rate-distortion function [6-9] for scalar quantization, including uniform quantization and non-uniform quantization. The rate-distortion function based on the measured value further describes the distortion of the signal reconstruction in restructuring algorithm. Secondly, this dissertation improved two standard compressed sensing reconstruction algorithms to adapt to the quantization error. Simulation results show that compared with the classic compressed sensing reconstruction mechanism which does not count quantization error, the improved algorithm has better performance. Improved reconstruction algorithm is mainly based on a reconstruction ideology.

The section 2 describes compressive sensing of quantitative rate-distortion analysis; section 3 is improved SP reconstruction algorithm being used to quantify compressive sensing, simulation experiment as section 4 shown.

2 Compressive sensing of quantitative rate-distortion analysis

Here we regard signal measurement as a lossy compression process, which is a question of whether random measurement value can show the sparse signal effectively or not.

2.1 Compressive Sensing (CS)

Compressed sensing is accomplished by linear projection to encode an N dimension signal X as dimension measurement vector. Namely it is:

$$\mathbf{Y} = \mathbf{\Phi} \mathbf{X} \qquad (1)$$

Among them, is measurement matrix. Assuming x is strict k-sparse, the reconstitution should be described as using given Y and Φ to recover x.

Base tracking method transfers the question of reconstitution to the question of minimum ℓ_1 norm.

$$\min \|X\|_1 \quad \text{s.t.} \quad \mathbf{Y} = \mathbf{\Phi} \mathbf{X} \qquad (2)$$

Which, $\|X\|_1 = \sum_{i=1}^{N} |X_i|$ describes the norm ℓ_1 of vector X. This is a Convex Optimization Problem that could be solved by linear programming. The sufficient condition for the algorithm can accurately reconstruct the signal is based on restricted isometric property (RIP). The definition shows below:

Definition 1 (Limit isometric property) for any K=1, 2… the isometric constant δ_K of matrix $\Phi \in \boldsymbol{R}^{M \times N}$ meets function below:

$$\delta_K = \inf\left\{\delta \left| (1-\delta)\|X\|_2^2 \le \|\Phi X\|_2^2 \le (1+\delta)\|X\|_2^2 \right.\right\} \qquad (3)$$

X is any K-sparse vector. Once the isometric constant parameter meet the condition $\delta_{c_1 K} \le c_0, c_1 \in R^+, c_0 \in (0,1)$, it can be defined that matrix Φ meets RIP. That is to say, the reconstruction algorithm can accurately recover K-sparse signal.

Compressive sensing [10-12] contains the measurement and reconstruction of the original signal. Then, the compressed measurements will be quantified and the original signal will be also reconstructed through the quantitative measurements. At the same time, the quantization distortion of measured value in the process and the resulting reconstruction distortion will be analyzed.

2.2 The scalar quantization of measured values

Assuming $C \subset \boldsymbol{R}$ is a finite discrete set, it can be called codebook. Quantizer is the mapping from R to codebook C, it satisfies the condition that while $Y \in R_\omega$, $\hat{Y} = \omega \in C$. ω is quantization level and Q_ω is related quantized interval. Performance of quantizer can be described with rate-distortion function. Let distortion measure be square error distortion (minimum mean squared error, MSE). As for a random signal source $Y \in \boldsymbol{R}$, the degree of distortion of quantizer is $D_q = E\left[\|Y - \hat{Y}\|_2^2\right]$. As for a given codebook C, the best quantization function to make distortion measure minimum is below:

$$\hat{Y}^* = \arg\min_{\omega \in C} \|Y - \omega\|_2^2 \qquad (4)$$

Therefore, relevant quantized interval should be:

$$Q_\omega = \left\{ Y \subset \boldsymbol{R} \left| \|Y - \omega\|_2^2 \le \|Y - \omega'\|_2^2, \forall \omega' \in C \right.\right\} \qquad (5)$$

Distortion factor of codebook can be:

$$D(C) = E\left[\|Y - \hat{Y}^*\|_2^2\right] \qquad (6)$$

$R := \frac{1}{m}\log_2|C|$ is the code rate of codebook C. Rate-distortion function of given code rate R can be:

$$D^*(R) = \inf_{C: \frac{1}{m}\log_2|C| \le R} D(C) \qquad (7)$$

Quantization of the measurement value could be assumed that using same quantizer to respectively quantify each coordinate of Y. relevant rate-distortion function shows below:

$$D_{SQ}^*(R) = \inf_{C_{SQ}: \log_2|C_{SQ}| \le R} E_Y\left[\sum_{i=1}^{m} \left|Y_i - \hat{Y}_i^*\right|^2\right] \qquad (8)$$

Assuming that quantization level is $\omega_i \in C, i = 1,2,\cdots 2^R$, relevant quantized interval is $Q_\omega = [t_{i-1}, t_i]$, the best scalar quantization meet conditions below:

If the best quantizer has quantization level ω_{i-1} and ω_i, quantized interval threshold value that minimized distortion is:

$$t_i = \frac{1}{2}(\omega_i + \omega_{i-1}) \tag{9}$$

If section threshold value of best quantizer are t_{i-1} and t_i, quantization level that make distortion minimum is:

$$\omega_i = E[Y], Y \in [t_{i-1}, t_i] \tag{10}$$

Lloyd algorithm that design quantizer codebook is based on two requirements above. Lloyd algorithm starts from an initial codebook and calculate threshold value in iteration using function (13). Then update codebook by using function (14). Although Lloyds algorithm might not found the global optimum quantized interval, it gives local optimum codebook.

Assuming quantificat codebook C has been generated and be fixed while it was measured. Progressive performance of rate-distortion of quantized measurement value will be analyzed. Assuming $X \in \boldsymbol{R}^N$ is strict K-sparse signal, its non-zero term follow independent identically distributed Gaussian random variable. Let $\Phi = \frac{1}{\sqrt{M}} A \in \boldsymbol{R}^{M \times N}$, each term of A is Gaussian random matrix with independent identically distributed mean value 0 and variance 1.

2.3 Quantization distortion analysis of measurement values

Let $T = \{1 \le i \le N : X_i \ne 0\}$ be support set of x, that is to say, when $i \in T$, $X_i \ne 0$ and when $i \notin T$, $X_i = 0$. As for all $1 \le i \le M$ and $T \subset \{1, \cdots, N\}$ meets $|T| = K$, it is easy to find that average and variance are respectively 0 and K. According to central-limit theorem, when $(K, M, N) \to \infty$, convergence in distribution of $\frac{1}{\sqrt{K}} \sum_{j \in T} A_{i,j} X_j$ will converged to Gaussian distribution. Therefore, convergence in distribution of $\sqrt{\frac{M}{K}} Y_i$ is also on Gaussian distribution.

Using a scalar quantizer with 2^R electrical level to quantize random variable, it can be observed:

$$\begin{aligned} \frac{1}{K} \mathrm{E}\left[\left\|Y - \hat{Y}\right\|_2^2\right] &= \frac{1}{M} \sum_{i=1}^{m} \mathrm{E}\left[\left(\sqrt{\frac{M}{K}} Y_i - \sqrt{\frac{M}{K}} \hat{Y}_i\right)^2\right] \\ &= \mathrm{E}\left[\left(\sqrt{\frac{M}{K}} Y_i - \sqrt{\frac{M}{K}} \hat{Y}_i\right)^2\right] \end{aligned} \tag{11}$$

The last function is distortion factor of quantized $\sqrt{\frac{M}{K}} Y_i$. Inhomogeneous scalar quantizing distortion function of Gaussian random variable is:

$$\lim_{R \to \infty} 2^{2R} D_g^*(R) = \frac{\pi\sqrt{3}}{2} \delta^2 \tag{12}$$

δ^2 is variance of Gaussian signal source. So advance gradually rate-distortion limiting performance of well-distributed quantized observed value is:

$$\lim_{R \to \infty} \lim_{(K,m,N) \to \infty} \frac{2^{2R}}{K} D_{SQ}^*(R) = \lim_{R \to \infty} 2^{2R} D_g^*(R) = \frac{\pi\sqrt{3}}{2} \tag{13}$$

Let scalar quantizer with codebook C_u, $|C_u| = 2^R$, quantize random variable $\sqrt{\frac{M}{K}} Y_i$. Rate-distortion function in uniform scalar quantizer of Gaussian random variable is:

$$\lim_{R \to \infty} \frac{2^{2R}}{R} D_{u,g}^*(R) = \frac{4}{3} \delta^2 \ln 2 \tag{14}$$

So advance gradually rate-distortion limiting performance of well-distributed quantized observed value is:

$$\lim_{R \to \infty} \lim_{(K,m,N) \to \infty} \frac{2^{2R}}{KR} D_{u,SQ}^*(R) = \lim_{R \to \infty} \frac{2^{2R}}{R} D_{u,g}^*(R) = \frac{4}{3} \ln 2 \tag{15}$$

Unit of quantized code rate is bit/ Number of measurement.

It can be seen that when quantized code rate is big enough, distortion factor of optimum non-uniform quantizing is just 1/R of optimum uniform quantizing.

2.4 Reconstitution signal distortion analysis of quantization measurement values

Reconstruction distortion is primarily due to the distortion measure. Based on the conclusion of compressed sensing measurements quantization distortion previously, the reconstitution caused by the quantization error of the distortion led further qualitative analysis to investigate two specific reconstruction algorithms: base tracking algorithm (BP) and subspace tracking algorithm (SP). Consider the following equation to quantify the measured values are given as the following:

$$\hat{Y} = q(Y) = \Phi X + E \quad (16)$$

$E \in \boldsymbol{R}^M$ is quantization error. Let $\hat{x}$ be the signal that reconstruct from quantized observed value $\hat{Y}$. So upper bound of reconstruction distortion shows below:

$$\left\| X - \hat{X} \right\|_2^2 \le c^2 \left\| E \right\|_2^2 \quad (17)$$

Constant c displays different value for different reconstruction algorithm. Bound constant of BP algorithm is:

$$c_{bp} = \frac{4}{\sqrt{3(1-\delta_{4K})} - \sqrt{1+\delta_{4K}}} \quad (18)$$

Bound constant of SP algorithm is:

$$c_{sp} = \frac{1+\delta_{3K}+\delta_{3K}^2}{\delta_{3K}(1-\delta_{3K})} \quad (19)$$

So upper bound of reconstruction distortion derivation shows below. Assuming support set T of sparse signal $\hat{x}$ can be accurately reconstructed, reconstruction signal $\hat{X}$ would be:

$$\hat{X} = \left(\Phi_T^* \Phi_T\right)^{-1} \Phi_T^* \hat{Y} \quad (20)$$

The singular value of matrix $\left(\Phi_T^* \Phi_T\right)^{-1} \Phi_T^*$ is $\sqrt{1-\delta_K} \big/ (1+\delta_K)$. So lower bound of reconstruction distortion shows below:

$$\left\| X - \hat{X} \right\|_2^2 \ge \left(\frac{\sqrt{1-\delta_K}}{1+\delta_K} \right)^2 \left\| Y - \hat{Y} \right\|_2^2 = \frac{1-\delta_K}{(1+\delta_K)^2} \left\| E \right\|_2^2 \quad (21)$$

To make the expression more concise, $c_{lb} = \sqrt{1-\delta_K} \big/ (1+\delta_K)$.

Combined conclusion (17), (19), (22) with (26), Gradual boundaries of reconstruction distortion can be found. When signal is strict sparse and measurement matrix is Gaussian random matrix, bound of scalar quantized reconstruction distortion is:

$$c_{lb}^2 \frac{\pi\sqrt{3}}{2} \le \lim_{R\to\infty} \lim_{(K,m,N)\to\infty} \frac{2^{2R}}{K} E\left[\left\| X - \hat{X} \right\|_2^2 \right] \le \begin{cases} c_{sp}^2 \frac{\pi\sqrt{3}}{2}, \text{ for SP algorithem} \\ c_{bp}^2 \frac{\pi\sqrt{3}}{2}, \text{ for BP algorithem} \end{cases} \quad (22)$$

It is noteworthy that the distortion of the reconstructed boundary (27) given, in some cases, is not critical. Experiments show that the upper bound of the reconstruction distortion often given in excessive.

3 Improved SP reconstruction algorithm being used to quantify compressive sensing

Most reconstruction algorithm of compressive sensing can be directly used in the quantization, and regard the quantization error as noise. But in this case, the quantizer information cannot be used. The key to improve the standard reconstruction algorithm here is to make full use of the structure of quantizer, and to reconstruct consistent single.

Consistency: suppose $f: X \to Y$, $x \in X$ and $y = f(x)$. If $f(\hat{x}) = y$, then $\hat{x} \in X$ is called the consistent estimation of x coming from y. The algorithm consistent estimation is called Consistent Restructuring Algorithm.

$\hat{Y}$ is quantized measurement vector, so relevant quantized interval can be ensured easily. Let $\hat{Y}_i \in \boldsymbol{Q}_{k_i}$, among them, $\hat{Y}_i$ is the ith element of $\hat{Y}$, $\boldsymbol{Q}_{k_i}$ is relevant quantized interval. $\boldsymbol{Q}$ is Descartes direct product set of $\boldsymbol{Q}_{k_i}$, vector $Y \in \boldsymbol{Q}$ if and only if $Y \in \boldsymbol{R}^M$ and

$Y_i \in \boldsymbol{Q}_{k_i}, i = 1,2,\cdots,M$. As for uniform quantizing, $\left|\hat{Y} - Y\right| \le \Delta/2$. So it can be described as:

$$\Phi\hat{X} \le \frac{1}{2}\Delta + \hat{Y} \text{ and } \Phi\hat{X} \ge -\frac{1}{2}\Delta + \hat{Y} \tag{23}$$

The equation is consisting of elements, so it can be written as:

$$\begin{bmatrix} \Phi \\ -\Phi \end{bmatrix} \hat{X} \le \begin{bmatrix} \frac{1}{2}\Delta + \hat{Y} \\ \frac{1}{2}\Delta - \hat{Y} \end{bmatrix} \tag{24}$$

As for non-uniform quantizing, $\Phi\hat{X} \in \boldsymbol{Q}$. So standard BP algorithm transfers to:

$$\min\|x\|_1 \ \ s.t.\ \Phi X \in \boldsymbol{Q} \tag{25}$$

In order to improve SP algorithm to satisfy quantized compressed sensing, at first, geometric interpretation of projection operation in SP algorithm should be given. Assuming index set $T \subset \{1,\cdots,N\}$, and $|T| = K$. Φ_T is the matrix which consist by column vector of Φ of index T, assuming row Φ_T is nonsingular, that is column linear independent. $span(\Phi_T)$ is sub-space generated by row Φ_T. If $\Phi_T^*\Phi_T$ is reversible, for random given $\hat{Y} \in \boldsymbol{R}^M$, the definition of projection from $\hat{Y}$ to $span(\Phi_T)$ is:

$$Y_p = proj\left(\hat{Y}, \Phi_T\right) = \Phi_T(\Phi_T^*\Phi_T)^{-1}\Phi_T^*\hat{Y} \tag{26}$$

Φ_T^* is conjugate transposition of Φ_T. Relevant projection residual vector Y and projection coefficient vector X_p are defined as:

$$Y_r = resid\left(\hat{Y}, \Phi_T\right) = \hat{Y} - Y_p \tag{27}$$

and

$$X_p = pcoeff\left(\hat{Y}, \Phi_T\right) = (\Phi_T^*\Phi_T)^{-1}\Phi_T^*\hat{Y} \tag{28}$$

Given projection operation in improved SP algorithm is equivalent to solve optimization problem.

$$\min_{X \in \boldsymbol{R}^{|T|}} \left\|\hat{Y} - \Phi_T X\right\|_2^2 \tag{29}$$

Algorithm process shows below:

Input: K, Φ, $\hat{Y}$

Initialization:

$T^0 = \{\text{Indexes of the K largest figures in } \Phi^* Y\}$ and $Y_r^0 = resid\left(\hat{Y}, \Phi_{T^0}\right)$

Iteration: procedure of l th Iteration shows below:

1) $\tilde{T}^l = T^{l-1} \cup \{\text{Indexes of the K largest figures in } \Phi^* Y_r^{l-1}\}$;

2) $X_p = \arg\min_{X_p \in R^{|T|}} \left\|Y - \Phi_{\tilde{T}^l} X_p\right\|$;

3) $T^l = \{\text{Indexes of the K largest figures in } X_p\}$;

4) $Y_r^l = \min\left\|Y - \Phi_T X_T\right\|_2^2$;

5) If $\left\|y_r^l\right\|_2 > \left\|y_r^{l-1}\right\|_2$, then $T^l = T^{l-1}$ and exit the iteration。

Output: vector $\hat{X}$ meets $\hat{X}_{\{1,\cdots,N\}-T^l} = 0$ and $\hat{X}_{T^l} == pcoeff\left(Y, \Phi_{T^l}\right)$

Step 2 and Step 4 reconstruction algorithm uses a consistent ideology, inequality using elements is a quadratic programming in process of reconstruction, quadratic programming that is consistent with estimates of the feasible set of collections.

The computational complexity of the improved SP algorithm is given below. Projection operation in the improved algorithm calculates the projection coefficients with quadratic programming. Quadratic programming can be solved in different ways, for example, the ellipsoid method and interior point method. The both methods calculate the computational complexity of polynomial, but the interior point method is faster in computing speed in practical operation than ellipsoid method. When considering the scalar quantity mechanism, computational complexity of the quadratic optimization problem is $O(K^2M^{3/2})$. Therefore, the computational complexity of improved projection operation is polynomial time. The upper limit for the number of iterations is $O(\log K)$, and the upper limit for total computational complexity of the improved algorithm is $O(M \log K(N + K^2M^{1/2}))$. Since the projection step is performed twice in iteration, and the reconstruction distortion increases exponentially with the decrease in the number of iterations, the improved SP algorithm can calculate effectively.

4 Simulation Experiment

In this dissertation, a large number of simulation experiments have been made about reconstruction algorithms and different types of quantizer and are composed. For the given parameters N and K, measurement matrix Φ is generated by an independent and identically distributed Gaussian random population. Signal with potential K in support set T is uniformly and randomly selected from $\{1,\cdots,N\}$, the corresponding support set are generated by the standards of independent and identically distributed Gaussian distribution population, the rest are set to 0.

In order to test different quantization and reconstruction algorithm, x and Φ were randomly generated a thousand times. Each time, we calculate the measured value Y and its quantized value $\hat{Y}$ and the corresponding reconstructed signal $\hat{X}$. In the simulation, we test a set of parameters: N =256, K =6， M =64, quantization code rate changing from 2 bits to 6 bits.

In figure 1, it is compared that the degree of distortion in measurements of the uniform quantizer and non-uniform quantizer. When quantization code rate increases, the gap between uniform and non-uniform quantization distortion increases this is consistent with the results given in the formula. In figure 2, it is compared that the reconstruction distortion degree of SP and BP algorithms. When the given quantization code rate increases, the reconstruction distortion gap between BP and SP has increased. In figure 3, it is compared that the uniform quantization and non-uniform quantization in both cases with improved BP and SP reconstruction distortion algorithm, the non-uniform quantization is also superior to the uniform quantization, and the improved SP algorithm has better performance than BP algorithm. In figure 4, it is compared that the uniform quantization distortion of the two improved reconstruction algorithm with their original algorithm, the decrease of distortion increases as the rate increase. Since SP algorithm is far less complicated than BP algorithm, the improved SP algorithm has obvious advantages in signal reconstruction after quantizing the compressed measurements.

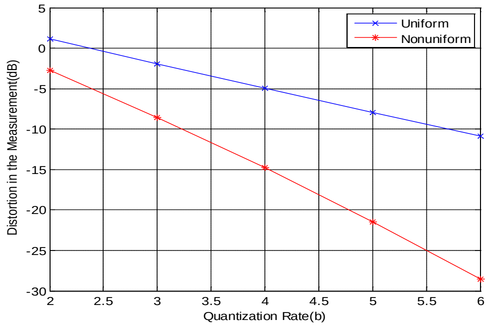

Figure 1 quantization error of measurement value

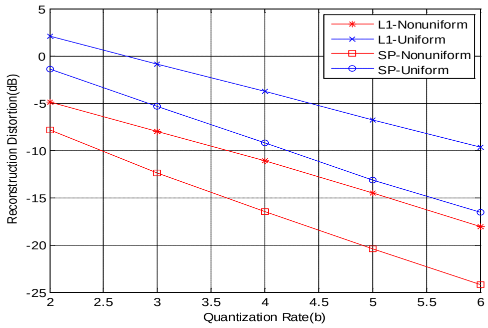

Figure 2 algorithm reconstruction error

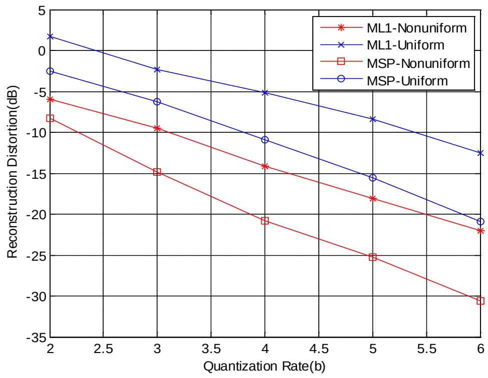

Figure 3 Improved algorithm reconstruction error

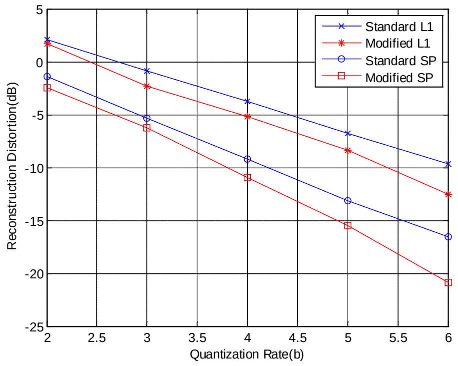

Figure 4 Comparison of uniform quantized improved algorithm reconstruction error

5 Conclusion

This dissertation gives a theoretical limit of compressed sensing to quantify the rate-distortion performance, and proves that quantifying the compression measured value will cause a lot of distortion. With the information of quantizer and the ideas of consistent improvement, we re-modeled two typical compressed sensing reconstruction algorithms. Simulation results show that, compared with the classic error of not considering the quantization compressed sensing reconstruction mechanisms, improving the algorithm has better performance, the improvement of subspace tracking algorithm has a low calculation.

In the future, we will research how to decrease computational complexity of the improved SP algorithm.

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