

The Cramer-Rao Bound for 3-D Frequencies in a Colored Gaussian Noise

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Abstract: - Estimation of model parameters (3-D frequencies), based on the high resolution spectral analysis methods known by their performances and their precision such as 3-D ESPRIT, remains a problem which is essential in the modeling of the signals by a sum of 3-D complexes exponential (3-D SCE model) embedded in an additive gaussian noise. Indeed, good results are obtained when the noise is white and by using the Second Order Statistics (autocorrelations), but if it becomes colored, the results are degraded which forces us to remedy this problem, to think about the Higher Order Statistics (cumulants). To verify the efficiency of estimators of 3-D frequency, we calculate the asymptotic Cramer-Rao Bound (CRB).

Key-Words: - Spectral Analysis, High Resolution, 3-D ESPRIT, Second Order Statistics, Higher Order Statistics, Fourth Order Cumulant, Cramer-Rao Bound.

1 Introduction

The modeling of signals embedded in noise occupies a very important place in the areas of research these last years. It's a technique used in several fields and applications such as telecommunications, treatment of antenna and image processing.

Spectral analysis methods can be classified into two categories: scanning methods and high-resolution methods named also analytical methods or subspace approach. The first category trays to restore spectral information by the mean of a functional depending of a frequency vector; these methods are also known as pseudo-spectrum. The second family includes the methods that exploit the inherent matrix structure in 3-D SCE model. These methods contain a phase of estimating the triplets frequencies contained in the model; we cite the MEMP method [1] or its extensions in 3-D case [2] [3], the ACMP method [4] and ESPRIT method [5] and their extension for 2-D signals [6] [7] and in the 3-D case [8].

The Higher Order Statistics (HOS) [3] [9] [10] [11] are essentially used in complement with the Second Order Statistics (SOS). Indeed, they give a more complete description of data and their properties and they allow the resolution of insoluble problems particularly when the noise is colored. In

this work, we will apply this approach to the new 3-D ESPRIT method developed in [8].

Generally the Cramer-Rao bound [12][13][14] allows to fix a lower limit to the precision which it can be to reach in the estimator of one or more parameters. The calculation of this bound in signal processing is often very interesting. Indeed, in theoretical problems, an estimator that reaches the Cramer-Rao bound is therefore known as efficient.

This paper is organized as follows: section 2 presents the 3-D SCE model. In section 3, fourth order cumulant are developed then cumulants of the new 3-D ESPRIT method are calculated. In section 4 the Cramer-Rao bound are developed. The simulation results and comparison are presented in section 5. Finally, the work ends with a conclusion and perspectives.

2 Problem Formulation

Let us consider that every voxel $y(m, n, t)$ of block of the observed image $\{y(m, n, t)\}$ corresponds to the sum of two terms:

$$y(m, n, t) = x(m, n, t) + b(m, n, t) \quad (1)$$

With $1 \leq m \leq M$, $1 \leq n \leq N$ and $1 \leq t \leq T$
The useful signal $x(m, n, t)$ is modeled as follows:

$$x(m,n,t) = \sum_{i=1}^K a_i \exp(j2\pi(f_{1i}m + f_{2i}n + f_{3i}t) + j\varphi_i) \quad (2)$$

The K components of the signal are defined by the frequency triplets $\{f_{1i}, f_{2i}, f_{3i}\}$, the amplitudes $\{a_i\}$ and the phases $\{\varphi_i\}$.

The component $b(m, n, t)$ is the gaussian additive colored noise.

The problem with we del here is how to estimate the K 3-D frequencies $\{f_{1i}, f_{2i}, f_{3i}\}$ and to estimate their CBR.

3 Cumulant based new 3-D ESPRIT method

3.1 Fourth Order Cumulant for 3-D data

In this paragraph, the indices are noted:

$$m = (m_1, m_2, m_3), f = (f_1, f_2, f_3), h = (h_1, h_2, h_3), \alpha = (\alpha_1, \alpha_2, \alpha_3)$$

and $m + h = (m_1 + h_1, m_2 + h_2, m_3 + h_3)$

To develop the techniques for high resolution estimation from the higher order statistics, we use the fourth cumulant. With the previous notation, the fourth cumulants of the observation are defined by:

$$C_{4y}(h, k, l) = Cum[y(m), y^*(m+h), y(m+k), y^*(m+l)] \quad (3)$$

and the fourth-order cumulant is given by:

$$Cum[y(i), y(j), y(k), y(l)] = E[y(i)y(j)y(k)y(l)] - E[y(i)y(j)]E[y(k)y(l)] - E[y(i)y(k)]E[y(j)y(l)] - E[y(i)y(l)]E[y(j)y(k)] \quad (4)$$

Where (*) denotes complex conjugate operator, and E is the expectation operator

By taking into account the model of the noisy observations given in (1), the fourth cumulant of the observations are given by:

$$C_{4y}(h, k, l) = C_{4x}(h, k, l) + C_{4b}(h, k, l) \quad (5)$$

However the higher order statistics than two for gaussian signals do not bring information: the second order statistics are zero. Thus, the fourth cumulant of the observations coincide theoretically with non-noisy observations:

$$C_{4y}(h, k, l) = C_{4x}(h, k, l) \quad (6)$$

It is clear that with the higher order statistics than 2, we can theoretically eliminate the noise. Moreover, the fourth-order cumulants of the considered 3-D harmonic process verifies the following relation:

$$C_{4y}(h, k, l) = - \sum_{i=1}^K a_i^4 \exp[j2\pi(-h+k+l)f_i^T] \quad (7)$$

In the following, we consider only the diagonal slice called the fourth cumulant defined as follows:

$$C_{4y}(h) = C_{4y}(h, h, h), h = (h_1, h_2, h_3) \quad (8)$$

The two equations (6) and (7) allows us to write:

$$C_{4y}(h) = - \sum_{i=1}^K a_i^4 \exp[j2\pi(h_1 f_{1i} + h_2 f_{2i} + h_3 f_{3i})] \quad (9)$$

The equation (9) shows that the diagonal of the fourth cumulants contains all the useful information to estimate the frequencies of the harmonic model and it is the basis of the extension of the high resolution techniques.

3.2 Estimation of the frequencies in the third dimension

To estimate the frequency we use the special average assuming the ergodicity of the process y.

Consider a $P \times Q \times L$ block for estimating the C_{4y} .

To estimate the frequencies of the third dimension $\{f_{3i}\}_{i=1}^K$, we consider the $P \times P$ triply Teoplitz matrix (Teoplitz block-block Teoplitz matrix) given by:

$$C_{y,1} = \begin{bmatrix} C_0 & C_{-1} & \dots & C_{-(P-1)} \\ C_1 & C_0 & \dots & C_{-(P-2)} \\ \cdot & \cdot & \cdot & \cdot \\ C_{P-1} & C_{P-2} & \dots & C_{10} \end{bmatrix} \quad (10)$$

For each $p \in [-(P-1), (P-1)]$, C_p is the $Q \times Q$ matrix having a Toeplitz-block-Toeplitz structure such as:

$$C_p = \begin{bmatrix} c_p^0 & c_p^{-1} & \dots & c_p^{-(Q-1)} \\ c_p^1 & c_p^0 & \dots & c_p^{-(Q-2)} \\ \cdot & \cdot & \dots & \cdot \\ c_p^{Q-1} & c_p^{Q-2} & \dots & c_p^0 \end{bmatrix} \quad (11)$$

and each block C_p^q , where $q \in [-(Q-1), (Q-1)]$ is an $L \times L$ Teoplitz matrix given by:

$$C_p^q = \begin{bmatrix} c_{4y}(p, q, 0) & c_{4y}(p, q, -1) & \dots & c_{4y}(p, q, -(L-1)) \\ c_{4y}(p, q, 1) & c_{4y}(p, q, 0) & \dots & c_{4y}(p, q, -(L-2)) \\ \vdots & \vdots & \ddots & \vdots \\ c_{4y}(p, q, L-1) & c_{4y}(p, q, L-2) & \dots & c_{4y}(p, q, 0) \end{bmatrix} \quad (12)$$

The matrix of the fourth cumulant can be written as follows:

$$C_{y,1} = S_{[PQL, K]}^1 \Psi S_{[PQL, K]}^{1H} \quad (13)$$

Where:

Ψ is the $K \times K$ diagonal matrix given by:

$$\Psi = \text{diag}(-a_i^4) \quad (14)$$

$$1 \leq i \leq K$$

H is the Hermitian operator

$S_{[PQL,K]}^1$ is the $PQL \times K$ 3-D Vandermonde matrix having the following form:

$$S_{[PQL,K]}^1 = [s_{31,L} \otimes s_{21,Q} \otimes s_{11,P}, \dots, s_{3K,L} \otimes s_{2K,Q} \otimes s_{1K,P}] \quad (15)$$

Where :

$$s_{mn,G} = [1 \quad \exp(j2\pi f_{mn}) \quad \dots \quad \exp(j2\pi f_{mn}(G-1))]^T \quad (16)$$

For $m = 1,2,3$, $n = 1,2,\dots,K$ and \otimes is the Kronecker product.

The 3-D Vandermonde matrix $S_{[PQL,K]}^1$ can be written according to the 2-D Vandermonde matrix $S_{[PQ,K]}^1$:

$$S_{[PQL,K]}^1 = \begin{bmatrix} S_{[PQ,K]}^1 \\ S_{[PQ,K]}^1 \Phi_3 \\ \vdots \\ S_{[PQ,K]}^1 \Phi_3^{L-1} \end{bmatrix} \quad (17)$$

$$S_{[PQ,K]}^1 = \begin{bmatrix} S_{[P,K]}^1 \\ S_{[P,K]}^1 \Phi_2 \\ \vdots \\ S_{[P,K]}^1 \Phi_2^{Q-1} \end{bmatrix} \quad (18)$$

Where $S_{[P,K]}^1$ is the 1-D Vandermonde.

We denote by $S_{[G,K]}^m$ the 1-D Vandermonde matrix associated with the frequencies $\{f_{mi}\}_{i=1}^K$ defined by:

$$S_{[G,K]}^m = \begin{bmatrix} 1 & \dots & 1 \\ \exp(j2\pi f_{m1}) & \dots & \exp(j2\pi f_{mK}) \\ \vdots & \dots & \vdots \\ \exp(j2\pi f_{m1}(G-1)) & \dots & \exp(j2\pi f_{mK}(G-1)) \end{bmatrix} \quad (19)$$

m is the spatial dimension, $m = 1,2,3$, and G is the size of the related window in the m dimension, $G \in \{P, Q, L\}$

The diagonal Φ_m matrix is given by:

$$\Phi_m = \text{diag}(\exp(j2\pi f_{mi})) \quad (20)$$

$$1 \leq i \leq K$$

Consider the eigenvalue decomposition of the matrix $C_{y,1}$:

$$C_{y,1} = UDU^H \quad (21)$$

With

$$U = [u_1, \dots, u_K, u_{K+1}, \dots, u_{PQL}] \text{ and } D = \text{diag}(\lambda_i) \quad (22)$$

$$1 \leq i \leq PQL$$

Where the obtained eigenvalue λ_i are real and ordered in a decreasing order as follows:

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_K \geq \lambda_{K+1} = \dots = \lambda_{PQL} = 0 \quad (23)$$

Thus, the signal subspace is spanned by the columns of the matrix $S_{[PQL,K]}^1$ and by the eigenvectors of $C_{y,1}$ associated with the K nonzero eigenvalues noted U_{S_1} . This means that there exist a $K \times K$ invertible matrix Θ_1 satisfying the relation:

$$U_{S_1} = S_{[PQL,K]}^1 \Theta_1 \quad (24)$$

The 3-D Vandermonde matrix $S_{[PQL,K]}^1$ can be partitioned in two different ways as in the following equation:

$$S_{[PQL,K]}^1 = \begin{bmatrix} S_{[PQ,K]}^1 \\ \text{-----} \\ EM1 \end{bmatrix} \begin{matrix} \Downarrow PQ \\ \\ \Downarrow PQ \end{matrix} = \begin{bmatrix} EM1 \Phi_3^{-1} \\ \text{-----} \\ S_{[PQ,K]}^1 \Phi_3^{L-1} \end{bmatrix} \begin{matrix} \Downarrow PQ \\ \\ \Downarrow PQ \end{matrix} \quad (25)$$

Similarly, we consider the partitionnements for the accessible matrix U_{S_1}

$$U_{S_1} = \begin{bmatrix} xxx \\ \text{-----} \\ \overline{U_{S_1}} \end{bmatrix} \begin{matrix} \Downarrow PQ \\ \\ \Downarrow PQ \end{matrix} = \begin{bmatrix} \underline{U_{S_1}} \\ \text{-----} \\ xxx \end{bmatrix} \begin{matrix} \Downarrow PQ \\ \\ \Downarrow PQ \end{matrix} \quad (26)$$

$\overline{U_{S_1}}$ (respectively $\underline{U_{S_1}}$) is obtained from U_{S_1} by eliminating the PQ first rows (respectively the last PQ rows).

From equations (23), (24) and (25) we can write

$$\begin{cases} EM1 \Theta_1 = \overline{U_{S_1}} \\ EM1 \Phi_3^{-1} \Theta_1 = \underline{U_{S_1}} \end{cases} \quad (27)$$

$$\begin{cases} EM1 = \overline{U_{S_1}} \Theta_1^{-1} \\ \overline{U_{S_1}} \Theta_1^{-1} \Phi_3^{-1} \Theta_1 = \underline{U_{S_1}} \end{cases} \quad (28)$$

Hence $\Theta_1^{-1} \Phi_3^{-1} \Theta_1 = (\overline{U_{S_1}})^{\wedge} \underline{U_{S_1}} = F_3$

$(.)^{\ddagger}$ stands for the pseudo inverse operator. Therefore, the frequencies $\{f_{3i}\}$ contained in the matrix Φ_3 will be estimated from the eigenvalues of the matrix $F_3 = (\overline{Us_1})^{\ddagger} Us_1$ as follows:

$$f_{3i} = \frac{1}{2\pi} \text{Im}[\log(\lambda_i[F_3])] \quad (29)$$

3.3 Estimation of the frequencies in the first dimension

To estimate the frequencies associated to the first dimension, we introduce a new matrix of cumulant $C_{y,2}$ (TBBT) built like previously. Indeed, it involves existence of an invertible matrix Θ_2 of size $K \times K$ verifying the relation:

$$Us_2 = S_{[QLP,K]}^2 \Theta_2 \quad (30)$$

The 3-D Vandermonde $S_{[QLP,K]}^2$ is given by:

$$S_{[QLP,K]}^2 = \begin{bmatrix} S_{[QL,K]}^2 \\ S_{[QL,K]}^2 \Phi_1 \\ \vdots \\ S_{[QL,K]}^2 \Phi_1^{P-1} \end{bmatrix} \quad (31)$$

$$S_{[QL,K]}^2 = \begin{bmatrix} S_{[Q,K]}^2 \\ S_{[Q,K]}^2 \Phi_3 \\ \vdots \\ S_{[Q,K]}^2 \Phi_3^{L-1} \end{bmatrix} \quad (32)$$

With $S_{[QL,K]}^2$ and $S_{[Q,K]}^2$ are the 2-D and 1-D Vandermonde matrix respectively.

However the Kronecker product is not commutative, thus the matrix $S_{[PQL,K]}^1$ and $S_{[QLP,K]}^2$ are joined by the following relation:

$$S_{[QLP,K]}^2 = E_1^2 S_{[PQL,K]}^1 \quad (33)$$

With E_1^2 is the permutation matrix given by:

$$E_1^2 = \sum_{i=1}^P \sum_{j=1}^Q \sum_{k=1}^L E_{i,j}^{P,Q} \otimes E_{j,k}^{Q,L} \otimes E_{k,i}^{L,P} \quad (34)$$

and $E_{i,j}^{P,Q}$ is the elementary permutation matrix of size $P \times Q$ having the value 1 for the coordinates (i, j) and zeros elsewhere.

Similarly for the matrix Us_2 :

$$Us_1 = E_1^2 Us_2 \quad (35)$$

The frequencies $\{f_{1i}\}$ contained in the matrix Φ_1 will be estimated from the eigenvalues of the matrix $F_1 = \overline{Us_2}^{\ddagger} Us_2$ by:

$$f_{1i} = \frac{1}{2\pi} \text{Im}[\log(\lambda_i[F_1])] \quad (36)$$

3.4 Estimation of the frequencies in the second dimension

As previously, to estimate the frequencies of the second dimension, we build the matrix of cumulants $C_{y,3}$ (TBBT) with Θ_3 a matrix of size $K \times K$ verifying the relation:

$$Us_3 = S_{[QLP,K]}^3 \Theta_3 \quad (37)$$

The 3-D Vandermonde matrix $S_{[LPQ,K]}^3$ is given by:

$$S_{[LPQ,K]}^3 = \begin{bmatrix} S_{[LP,K]}^3 \\ S_{[LP,K]}^3 \Phi_2 \\ \vdots \\ S_{[LP,K]}^3 \Phi_2^{Q-1} \end{bmatrix} \quad (38)$$

$$S_{[LP,K]}^3 = \begin{bmatrix} S_{[L,K]}^3 \\ S_{[L,K]}^3 \Phi_1 \\ \vdots \\ S_{[L,K]}^3 \Phi_1^{P-1} \end{bmatrix} \quad (39)$$

With $S_{[LP,K]}^3$ and $S_{[L,K]}^3$ are the 2-D and 1-D Vandermonde matrix respectively.

Similarly, we have the following relations:

$$S_{[LPQ,K]}^3 = E_2^3 S_{[QLP,K]}^2 \quad (40)$$

$$Us_3 = E_2^3 Us_2 \quad (41)$$

With E_2^3 is the permutation matrix given by:

$$E_2^3 = \sum_{i=1}^Q \sum_{j=1}^L \sum_{k=1}^P E_{i,j}^{Q,L} \otimes E_{j,k}^{L,P} \otimes E_{k,i}^{P,Q} \quad (42)$$

Finally, the frequencies $\{f_{2i}\}$ contained in the matrix Φ_2 will be estimated from the eigenvalues of the matrix $F_2 = \overline{Us_3}^{\ddagger} Us_3$ by:

$$f_{2i} = \frac{1}{2\pi} \text{Im}[\log(\lambda_i[F_2])] \quad (43)$$

At this level, the frequencies of each dimension are estimated, so the step of the pairing or the formation of frequential triplets is required. Indeed, we consider a matrix Θ of size $K \times K$ satisfying:

$$\Theta_1 = \Theta_2 = \Theta_3 = \Theta \quad (44)$$

Then we build a matrix F from the matrices F_1 , F_2 and F_3 as:

$$F = \alpha_1 F_1 + \alpha_2 F_2 + (1 - (\alpha_1 + \alpha_2)) F_3 = \Theta^{-1} \Delta \Theta \quad (45)$$

Where α_1 and α_2 are a scalar.

In [8] we proposed a new 3-D ESPRIT method to estimate the frequential triplets. Indeed, we construct three permutation matrix P_1 , P_2 and P_3 as:

$$\begin{cases} P_1 = \Theta^{-1} \Theta_1 \\ P_2 = \Theta^{-1} \Theta_2 \\ P_3 = \Theta^{-1} \Theta_3 \end{cases} \quad (46)$$

Hence we have the following relations:

$$\begin{cases} \Phi'_1 = P_1^{-1} \Phi_1 P_1 \\ \Phi'_2 = P_2^{-1} \Phi_2 P_2 \\ \Phi'_3 = P_3^{-1} \Phi_3 P_3 \end{cases} \quad (47)$$

Thus the new matrices F_1 , F_2 and F_3 become:

$$\begin{cases} F'_1 = \Theta^{-1} \Phi_1^{-1} \Theta \\ F'_2 = \Theta^{-1} \Phi_2^{-1} \Theta \\ F'_3 = \Theta^{-1} \Phi_3^{-1} \Theta \end{cases} \quad (48)$$

And finally the frequencies $\{f_{1i}\}$, $\{f_{2i}\}$ and $\{f_{3i}\}$ contained in the matrices Φ'_1 , Φ'_2 and Φ'_3 will be estimated from the eigenvalues of the matrices F'_1 , F'_2 and F'_3 by:

$$\begin{cases} f'_{1i} = \frac{1}{2\pi} \text{Im}[\log(\lambda_i[F'_1])] \\ f'_{2i} = \frac{1}{2\pi} \text{Im}[\log(\lambda_i[F'_2])] \\ f'_{3i} = \frac{1}{2\pi} \text{Im}[\log(\lambda_i[F'_3])] \end{cases} \quad (49)$$

The steps of the new 3-D ESPRIT algorithm are summarized in table I.

TABLE I
STEPS OF THE NEW 3-D ESPRIT ALGORITHM

New 3-D ESPRIT algorithm	
Step 1:	Compute the signal subspace U_{s1} Eq (24), U_{s2} Eq (30) and U_{s3} Eq (37), by performing eigendecomposition of the sample cumulants matrix.
Step 2:	Build the tree matrices F_1 Eq (36), F_2 Eq (43) and F_3 Eq (29).
Step 3:	Diagonalization the linear combination $\alpha_1 F_1 + \alpha_2 F_2 + (1 - \alpha_1 - \alpha_2) F_3 = T^{-1} D T$ Eq (45).
Step 4:	Construct the tree matrices P_1 , P_2 and P_3 Eq (46).
Step 5:	Extract the frequency components from the angles of the tree matrices Φ'_1 , Φ'_2 and Φ'_3 Eq (49).

4 The Cramer-Rao Bound

In this section, we develop the analytical expression for the asymptotic Cramer-Rao bound for the vector of the parameters of the no noisy useful signal x [15]. For this we consider the following additional assumptions:

A1: the spectral density $S_b(f)$ of the additive noise is continuous and shows no localized maxima at frequencies $f_k, k = 1, \dots, K$.

A2: the parameters vectors x and b do not have any common element.

Under these conditions, we first prove that the exact CRB for an unbiased estimator $\hat{\theta} = [\hat{x}, \hat{b}]^T$ block diagonal matrix given

$$BCR(\hat{\theta}) = \begin{bmatrix} BCR(\hat{x}) & 0 \\ 0 & BCR(\hat{b}) \end{bmatrix} \quad (50)$$

The (k,l) th element of the associated vector with the BCR \mathbf{x} is given by the following relation:

$$[BCR(\mathbf{x})]_{kl}^{-1} = 2 \operatorname{Re} \left\{ \frac{\partial \mathbf{x}^H(\mathbf{x})}{\partial (\mathbf{x})_k} \frac{\partial \mathbf{x}(\mathbf{x})}{\partial (\mathbf{x})_l} \right\} \quad (51)$$

Where $\operatorname{Re}\{\cdot\}$ denotes the real part of the complex quantity in question and \mathbf{b} is the autocorrelation matrix of the noise. The vector of no noisy observations $\mathbf{x}(\mathbf{x})$ is:

$$\mathbf{x}(\mathbf{x}) = [x(0,0,0, \mathbf{x}), \dots, x(M-1,0,0, \mathbf{x}), \dots, x(0, N-1,0, \mathbf{x}), \dots, x(M-1, N-1,0, \mathbf{x}), \dots, x(M-1, N-1, T-1, \mathbf{x})]^T \quad (52)$$

For the considered problem, the asymptotic CRB is given by the following limit:

$$AsBCR(\mathbf{x}) = \lim_{J \rightarrow \infty} \mathbf{K}_J BCR(\mathbf{x}) \mathbf{K}_J \quad (53)$$

Where $J = MNT$ and \mathbf{K}_J is a normalization matrix, diagonal by block, of size $5K \times 5K$ defined by

$$\mathbf{K}_J = \mathbf{I}_K \otimes \mathbf{D} \quad (54)$$

With \mathbf{I}_K the K identity matrix $\mathbf{D} = \operatorname{diag}(\sqrt{J}, \sqrt{J}, M\sqrt{J}, N\sqrt{J}, T\sqrt{J})$ and \otimes denotes the Kronecker product.

By developing the derivative of the vector $\mathbf{x}(\mathbf{x})$, we demonstrate that the expression (47) can be written as follows:

$$BCR(\mathbf{x}) = \frac{1}{2} \left[\operatorname{Re} \left\{ \mathbf{G}^H \frac{\partial \mathbf{x}(\mathbf{x})}{\partial \mathbf{x}} \mathbf{G} \right\} \right]^{-1} \quad (55)$$

Where \mathbf{G} is a $J \times 5K$ matrix given by the concatenation of gradients vectors

$$\mathbf{g}(\mathbf{m}, \mathbf{x}) = \frac{\partial \mathbf{x}(m_1, m_2, m_3, \mathbf{x})}{\partial \mathbf{x}}$$

Where

$$\mathbf{g}(\mathbf{m}, \theta_x) = [\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_K]^T$$

$$\mathbf{g}_k = [1, \mathbf{j}c_k, \mathbf{j}c_k 2\pi m_1, \mathbf{j}c_k 2\pi m_2, \mathbf{j}c_k 2\pi m_3]^T e^{j(2\pi m_k^T + \theta_k)} \quad (56)$$

$$\mathbf{G} = [g(0,0,0, \theta_x), \dots, g(M-1,0,0, \theta_x), \dots, g(0, N-1,0, \theta_x), \dots, g(M-1, N-1,0, \theta_x), \dots, g(M-1, N-1, T-1, \theta_x)]^T \quad (57)$$

Using the two equations (47) and (51), the equation (49) become :

$$AsBCR(\mathbf{x}) = \frac{1}{2} \operatorname{Re} \left\{ \lim_{J \rightarrow \infty} \mathbf{K}_J \mathbf{G}^H \frac{\partial \mathbf{x}(\mathbf{x})}{\partial \mathbf{x}} \mathbf{G} \mathbf{K}_J \right\}^{-1} \quad (58)$$

The analytical expression of the matrix \mathbf{G}^H obtained from the equations (52) and (54), and the TBBT

structure of the autocorrelation matrix of additive noise \mathbf{b} , allows us to show that the matrix of asymptotic CRB is given by:

$$AsBCR(\mathbf{x}) = \begin{bmatrix} [AsBCR]_1 & & & & \\ & \ddots & & & \\ & & [AsBCR]_k & & \\ & & & \ddots & \\ & & & & [AsBCR]_K \end{bmatrix} \quad (59)$$

Where each block $[AsBCR]_k$ is given according to the spectral density of the process additive noise $b(\mathbf{m})$ as follows:

$$[AsBCR]_k = \frac{1}{2} \begin{bmatrix} S_b(w_k) & 0 & 0 & 0 & 0 \\ 0 & 10 \frac{S_b(w_k)}{c_k^2} & -6 \frac{S_b(w_k)}{c_k^2} & -6 \frac{S_b(w_k)}{c_k^2} & -6 \frac{S_b(w_k)}{c_k^2} \\ 0 & -6 \frac{S_b(w_k)}{c_k^2} & 12 \frac{S_b(w_k)}{c_k^2} & 0 & 0 \\ 0 & -6 \frac{S_b(w_k)}{c_k^2} & 0 & 12 \frac{S_b(w_k)}{c_k^2} & 0 \\ 0 & -6 \frac{S_b(w_k)}{c_k^2} & 0 & 0 & 12 \frac{S_b(w_k)}{c_k^2} \end{bmatrix} \quad (60)$$

Thus, for $k = 1, \dots, K$, the asymptotic expressions of the CRB $[\check{c}_k, \check{\phi}_k, \check{f}_{1k}, \check{f}_{2k}, \check{f}_{3k}]$ are the diagonal elements of the matrix $[AsBCR]_k$. It is noticed that the CRB relating to the frequencies and to the phase is inversely proportional to the local signal to noise ratio (SNR) $SNR_k = c_k^2 / S_b(f_k)$.

5 Experimental results

In this section, we present some numerical simulation examples. Our approach is tested on a 3-D SCE model. The data are generated according to the model of equation (1). We consider three waves i.e. $K=3$ with the amplitude $a_i = 200$, the 3-D frequencies are given in Table II. The data and the sizes of the cumulants matrix are respectively $(M, N, T) = (32, 32, 32)$, and $(P, Q, L) = (3, 3, 3)$. The colored gaussian noise is obtained by filtering the white gaussian noise by a 3-D filter AR [16] [17] [18]. We consider the value of signal to noise ratio $SNR = 20$ dB.

TABLE II

3-D FREQUENCIES USED IN SIMULATION EXAMPLE

	f_{1i}	f_{2i}	f_{3i}
1 st wave	0.1100	0.1400	0.1700
2 nd wave	0.2400	0.2300	0.2100
3 rd wave	0.4500	0.4800	0.4700

TABLE III

3-D FREQUENCIES ESTIMATED FOR SNR=20 dB

	F_{1i}		F_{2i}		F_{3i}	
	HOS	SOS	HOS	SOS	HOS	SOS
1 st wave	0.1102	0.1092	0.1399	0.1390	0.1701	0.1681
2 nd wave	0.2401	0.2386	0.2303	0.2270	0.2098	0.2090
3 th wave	0.4500	0.4492	0.4798	0.4779	0.4702	0.4689

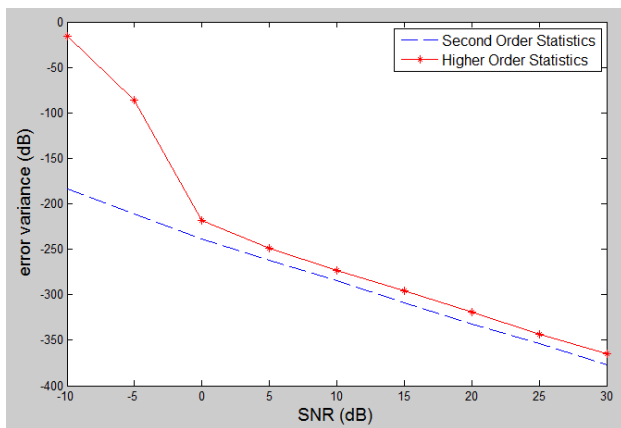


Fig. 1: Estimation-error variance versus the SNR.

New 3-D ESPRIT algorithm's computational complexity:

The main steps of the new 3-D ESPRIT algorithm are:

- (1) According to [19][20] singular value decomposition (SVD) of the cumulants matrix. The total number of floating point operations required for the computing the SVD of the $Z \times (M - P + 1)(N - Q + 1)(T - L + 1)$ Teoplitz block-block Teoplitz matrix C_{y1} is

$$N_{SVD} = 8Z^2((M - P + 1)(N - Q + 1)(T - L + 1) + 8Z/3)$$

where $Z = PQL$ is defined such that $PxQxL$ denoted the size of the observation window.

- (2) Build the tree matrices F_1 , F_2 and F_3 by applying selection matrices to the signal subspaces Us_1 , Us_2 and Us_3 . Applying the selection matrices requires no computation; it

only requires a set of memory accesses. However, memory accesses could be time consuming. Thus, we assume that a memory accesses is equal to a half multiplication [20]. To construct the matrices F_1 , F_2 and F_3 the computation load is approximately equal to $p(2Z - P - Q - L)$ for computing $\overline{Us_1}$, $\overline{Us_2}$, $\overline{Us_3}$ and $16p^2(2Z - P - Q - L)$ for computing the tree pseudo-inverses, and finally $8p^2(2Z - P - Q - L)$ for computing the tree inner products of complex matrices $F_i = \overline{Us_i} \hat{A} \overline{Us_i}$ $i = 1, 2, 3$. The global cost is approximately

$$N_{build} = (2Z - P - Q - L)(24p^2 + p),$$

where p is the number of frequencies.

- (3) Diagonalizing the linear combination of F_1 , F_2 and F_3 i.e.

$$\alpha_1 F_1 + \alpha_2 F_2 + (1 - \alpha_1 - \alpha_2) F_3 = T^{-1} D T$$

costs about $80p^3$, applying the transformation T to the tree matrices costs $16p^3$ for the inversion of $p \times p$ matrix T and four $p \times p$ matrix multiplications $23p^3$. Hence, the total number of operations amount to $N_{diag} = 128p^3$.

- (4) Construct of the tree matrices P_1 , P_2 and P_3 only required a set of memory accesses. Thus, we assume that a memory accesses is equal to a half multiplication [20]. Hence, the total number of operations amount to $N_{const} = 3(K^3 + K^2(2K - 1))$.

The total number of floating point operations needed for the new 3-D ESPRIT algorithm is obtained by summing the above components. Thus, the numbers represent number of flops per $M \times N \times T$ data block.

$$N_{ESPRIT} = 8Z^2((M - P + 1)(N - Q + 1)(T - L + 1) + 8Z/3) + (2Z - P - Q - L)(24p^2 + p) + 128p^3 + 3(K^3 + K^2(2K - 1))$$

or asymptotically the order $\mathcal{O}(Z^2 MNT)$.

6 Conclusion

In this paper, we showed the interest of higher order statistics compared to the second order statistics in 3-D frequencies. Indeed, the estimation of the 3-D frequencies by the new 3-D ESPRIT method when the signal is embedded in a colored gaussian noise and by using the cumulants gives good performances that by using the autocorrelations. The development of theoretical expressions of the asymptotic Cramer-Rao bound of the parameters model, in particular frequency 3-D, is presented.

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